LEE-YANG ZEROS FOR DHL AND 2D RATIONAL DYNAMICS,

I. FOLIATION OF THE PHYSICAL CYLINDER.

PAVEL BLEHER, MIKHAIL LYUBICH AND ROLAND ROEDER

Abstract. In a classical work of the 1950's, Lee and Yang proved that the zeros of the partition functions of a ferromagnetic Ising model always lie on the unit circle. Distribution of these zeros is physically important as it controls phase transitions in the model. We study this distribution for the Migdal-Kadanoff Diamond Hierarchical Lattice (DHL). In this case, it can be described in terms of the dynamics of an explicit rational function $\mathcal{R}$ in two variables (the renormalization transformation). We prove that $\mathcal{R}$ is partially hyperbolic on an invariant cylinder $\mathcal{C}$. The Lee-Yang zeros are organized in a transverse measure for the central-stable foliation of $\mathcal{R}|\,\mathcal{C}$. Their distribution is absolutely continuous. Its density is C^∞ (and non-vanishing) below the critical temperature. Above the critical temperature, it is C^∞ on a open dense subset, but it vanishes on the complementary set of positive measure.

Contents

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1. Introduction

1.1. **Phenomenology of Lee-Yang zeros.** The Ising model is designed to describe magnetic matter and, in particular, to explain the appearance of spontaneous magnetization in ferromagnets and transitions between ferromagnetic and paramagnetic phases as the temperature T varies.

The matter in a certain scale is represented by a graph Γ. Let $\mathcal{V}$ and $\mathcal{E}$ stand respectively for the set of its vertices (representing atoms) and edges (representing magnetic bonds between the atoms).

A magnetic state of the matter is represented by a *spin configuration* $\sigma : \mathcal{V} \to \{\pm 1\}$ on Γ. The spin $\sigma(v)$ represents a magnetic momentum of an atom $v \in \mathcal{V}$. The total magnetic momentum of the configuration is equal to

$$\mathrm{M}(\sigma) = \sum_{v \in \mathcal{V}} \sigma(v). \tag{1.1}$$

Each configuration σ has energy $H(\sigma)$ depending on the interactions $J(v, w)$ between the atoms and the external magnetic field $h(v)$. In the simplest isotropic case, J and h are constants, and the Hamiltonian assumes the form:

$$\mathrm{H}(\sigma) = -J \sum_{\{v,w\} \in \mathcal{E}} \sigma(v)\sigma(w) - h\mathrm{M}(\sigma), \tag{1.2}$$

where the first sum accounts for the energy of interactions between the atoms while the second one accounts to the energy of interactions of the matter with the external field.

By the Gibbs Principle, the spin configurations are distributed according to the Gibbs measure that assigns to configuration σ a probability proportional to its *Gibbs weight* $W(\sigma) = \exp(-H(\sigma)/T)$, where T is the temperature. Various observable magnetic quantities (e.g., magnetization M) are calculated by averaging of the corresponding functionals (e.g., $\mathrm{M}(\sigma)$) over the Gibbs distribution.

The total Gibbs weight $\mathsf{Z} = \sum W(\sigma)$ is called the *partition function*. It is a Laurent polynomial in two variables (z, t), where $z = e^{-h/T}$ is a "field-like" variable and $\mathrm{t} = e^{-J/T}$ is "temperature-like".[1] For a fixed t, the complex zeros of $\mathsf{Z}(z, \mathrm{t})$ in z are called the *Lee-Yang zeros*. Their role comes from the fact that some important observable quantities can be calculated as electrostatic-like potentials of the equally charged particles located at the Lee-Yang zeros. (For instance, the free energy is equal to the logarithmic potential of such a family of particles.)

A celebrated theorem of Lee and Yang [YL, LY] asserts that for the ferromagnetic[2] Ising model on any graph, for any real temperature $T > 0$, *the Lee-Yang zeros lie on the unit circle* $\mathbb{T}$ in the complex plane (corresponding to purely imaginary magnetic field $h = -iT\phi$).[3]

Magnetic matter in various scales can be modeled by a hierarchy of graphs Γ_n of increasing size (corresponding to finer and finer scales of matter). For suitable

[1] We will often refer to them as just "field" and "temperature".

[2] i.e., with $J > 0$, which favors the same orientation of neighboring spins

[3] We will take a liberty to use either z-coordinate or the angular coordinate $\phi = \arg z \in \mathbb{R}/2\pi\mathbb{Z}$ on $\mathbb{T}$ without a comment.

models, the Lee-Yang zeros of the partition functions Z_n will have an asymptotic distribution $d\mu_{\rm t} = \rho_{\rm t}(\phi)d\phi/2\pi$ on the unit circle. This distribution supports singularities of the magnetic observables (or rather, their thermodynamical limits), and hence it captures phase transitions in the model. For instance, Lee and Yang showed that the spontaneous magnetization of the matter (as the external field vanishes) is equal to $\rho_{\rm t}(0)$. So, the matter is ferromagnetic (meaning that it exhibits non-zero spontaneous magnetization) at temperature t if and only if $\rho_{\rm t}(0) > 0$.

The Lee-Yang zeros for the 1D Ising model with periodic boundary conditions (corresponding to the hierarchy of cyclic lattices $\Gamma_n = \mathbb{Z}/n\mathbb{Z}$) can be explicitly calculated using the transfer matrix technique (see e.g., [Ba]):

(1.3)
$$z_k^{\pm} = e^{i\phi_k^{\pm}}, \quad \phi_k^{\pm} = \pm \arccos\left[\sqrt{1-{\rm t}^4}\,\cos\left(\frac{\pi(k+1/2)}{n}\right)\right]; \qquad k = 0, 1, \ldots, n-1;$$

see Appendix E. Their asymptotic distribution is supported on two symmetric intervals, $I^+ = [\phi^*, \pi - \phi^*]$ and $I^- = -I^+$, where $0 \le \phi^* \le \frac{\pi}{2}$ satisfies $\sin\phi^* = {\rm t}^2$, and its density is equal to

$$\rho_{\rm t}(\phi) = \frac{|\sin\phi|}{2\pi\sqrt{\sin^2\phi - {\rm t}^2}}. \tag{1.4}$$

We see that for positive temperature, the support $I^+ \cup I^-$ does not contain point $\phi = 0$, and so the matter is paramagnetic and there are no phase transitions. As $T \to 0$ the gap between I^+ and I^- closes up and the Lee-Yang zeros get equidistributed on the unit circle (so, in this model, the matter becomes ferromagnetic only at the zero-temperature limit). Note that $\rho_{\rm t}$ is real-analytic on $I^{\pm}$ and has power-like singularities with exponent $(-1/2)$ at the end-points.

For the ferromagnetic Ising model on lattices $\mathbb{Z}^d$ with $d \ge 2$, a similar picture is believed to be true for high temperatures (above some critical temperature $T_c > 0$), while below T_c the Lee-Yang distributions are conjectured to have full support with positive density. This scenario would lead to a second-order phase transition: a ferromagnet for $T < T_c$ turns into a paramagnet for $T > T_c$. However, these conjectures are hard to prove rigorously as no exact formulas for the Lee-Yang zeros are available.

For the two-dimensional lattice, the phase transitions can be rigorously justified by means of the Onsager exact solution[4], see [Ba]. In all dimensions $d > 1$, it was proven that for high temperatures, the Lee-Yang zeros do not accumulate on the point $\phi = 0$ (no spontaneous magnetization) [GMR, R1], while for low temperatures, they have positive density at $\phi = 0$ (the spontaneous magnetization is observed) [P, Gr]. However, unlike the one-dimensional Ising model, for sufficiently low temperatures, $\rho_{\rm t}(\phi)$ is not real-analytic at $\phi = 0$ [Isa], see Remark 2.2.

In a recent breakthrough, it was proven in [BBCKK] that the Lee-Yang zeros $\phi_k^n({\rm t}) \in \mathbb{T}$ for the $\mathbb{Z}^d$ Ising model with periodic boundary conditions, $\Gamma_n = \mathbb{Z}^d/(n\mathbb{Z})^d$, can be calculated at sufficiently low temperature t as

$$\phi_k^n({\rm t}) = g_{\rm t}\left(\frac{\pi\left(k+\frac{1}{2}\right)}{n^d}\right) + O(\lambda^{-n}) \quad k = 0, 1, \ldots, 2n^d - 1 \tag{1.5}$$

[4]Note that Onsager's solution requires $h = 0$, so it only gives some information about the Lee-Yang distributions at $\phi = 0$, not a global description.

where $\lambda > 1$ and $g_t(\phi)$ is a C^2-diffeomorphism of the circle smoothly depending on t. In particular, for sufficiently low temperatures, the limiting density $\rho_t(\phi)$ is C^2.

The classical renormalization theory [WK] predicts the critical exponents for thermodynamic quantities of the Ising model on the $\mathbb{Z}^d$ lattice near the critical point $(h,t) = (0,t_c)$ in terms of eigenvalues of the renormalization transformation. In particular, if $t = t_c$ and h is varied near 0, it predicts that the magnetization satisfies $M \sim h^{1/\delta}$, where $\delta = 15$ for $d = 2$, $\delta = 4.790...$ for $d = 3$, and $\delta = 3$ for $d \geq 4$. Using an expression relating the limiting distribution of Lee-Yang zeros to M (Proposition 2.2), this predicts for the $\mathbb{Z}^d$ lattice at $t = t_c$ that the limiting distribution of the Lee-Yang zeros vanishes at $\phi = 0$ with exponent $1/\delta$ for the values of δ listed above.

At high temperatures, a quantum field theory interpretation gives a prediction of the power exponents of the densities ρ_t near the end-points of $I^{\pm}$, see Fisher [F1] and Cardy [Car]. For instance, for $d = 2$ the exponent is predicted to be $(-1/6)$, while for $d > 6$ it is predicted to be $1/2$.

Study of the Lee-Yang zeros is an active direction of research in contemporary statistical mechanics, see [MSh], [BB], [R4] and references therein for recent developments.

1.2. **Diamond hierarchical model.** The Ising model on hierarchical lattices was introduced by Berker and Ostlund [BO] and further studied by Bleher & Žalys [BZ1, BZ2, BZ3] and Kaufman & Griffiths [KG1].

Let Γ be an oriented graph with two vertices marked and ordered. The corresponding *hierarchical lattice* is a sequence of graphs Γ_n with two marked and ordered vertices such that Γ_0 is an interval, $\Gamma_1 = \Gamma$, and Γ_{n+1} is obtained from Γ_n by replacing each edge of Γ_n with Γ so that the marked vertices of Γ match with the vertices of Γ_n and their order matches with the orientation of the corresponding edges of Γ_n. We then mark two vertices in Γ_{n+1} so that they match with the two marked vertices of Γ_n.

For instance, the *diamond hierarchical lattice* (DHL) illustrated on Figure 1.1 corresponds to the diamond graph Γ.[5] Our paper is fully devoted to this lattice.

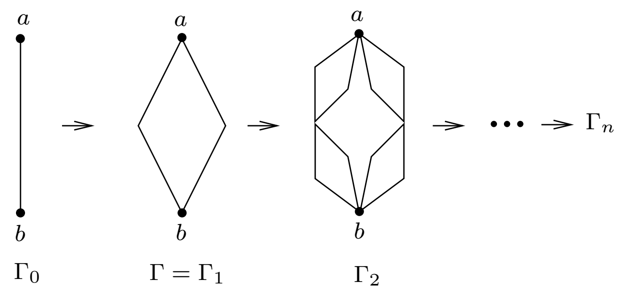

FIGURE 1.1. Diamond hierarchical lattice (DHL).

[5]In fact, in this case, we do not need to orient Γ and order the marked vertices since the diamond is symmetric with respect to a reflection interchanging the marked vertices.

Remark 1.1. The definition of the total magnetic momentum that we will use for the DHL will be slightly different from (1.1) (see (2.1) and Appendix E.4 for a motivation). Also, we will use $t := \mathrm{t}^2 = e^{-2J/T}$ for the temperature-like variable as it makes formulas nicer.

It was shown in [BZ3] that the thermodynamic limit exists[6] for the Ising Model on the DHL and that for each temperature $t \in I := [0,1]$ the Lee-Yang zeros are dense on the unit circle.

Existence of the thermodynamic limit implies that for each $t \in I$ there is a measure μ_t on the unit circle describing the asymptotic distribution of Lee-Yang zeros[7] at temperature t. In this paper, we will describe these asymptotic distributions for the DHL. They are illustrated in Figure 1.2. It shows the cylinder $\mathcal{C} = \mathbb{T} \times I$ in the angular coordinate $\phi \in [0, 2\pi]$ on the circle $\mathbb{T}$. We will prove that for $t \in [0,1)$ the Lee-Yang distributions are absolutely continuous with respect to Lebesgue measure on the unit circle, so we may write $d\mu_t = \rho_t(\phi)\, d\phi/2\pi$. In the blue (dark) region the density ρ_t is a positive C^∞ function, while in the orange (light) region it vanishes. We can see blue (dark) "tongues" going from the bottom to the top of the cylinder and orange (light) "hairs" sticking from the top. The tongues fill the cylinder densely. However, the hairs fill a set of positive area – in fact, of almost full area near the top. This creates a false impression that everything is orange (light) near the top of Figure 1.2. One can also see that the lowest temperature is reached by hairs for zero field h ($\phi = 0$): this is the critical temperature t_c that separates the ferromagnetic and paramagnetic phases. The critical temperature $t_c = 0.296...$ is the unique real solution to $t^3 + t^2 + 3t - 1 = 0$.

Here is a precise statement:

Main Theorem (physical version). *For any temperature $t \in [0,1)$ the limiting distribution $\mu_t(\phi)$ of the Lee-Yang zeros exists and it is absolutely continuous with respect to the Lebesgue measure on $\mathbb{T}_t \equiv \mathbb{T} \times \{t\}$: $d\mu_t = \rho_t(\phi)\, d\phi$. It has the following properties:*

1. *For $0 \le t < t_c$, the density $\rho_t(\phi)$ is a positive C^∞ function on the circle $\mathbb{T}_t$. Moreover, μ_0 is the Lebesgue measure on $\mathbb{T}_0$ (i.e., $\rho_0(\phi) \equiv 1$).*
2. *For $t = t_c$, the density $\rho_t(\phi)$ is a positive C^∞ function on $\mathbb{T}_{t_c} \setminus \{0, \pi\}$ with a power singularity at $\phi = 0, \pi$:*
$$\rho_t(\phi) \asymp |\phi|^\sigma \text{ near } 0, \quad \rho_t(\phi) \asymp |\phi - \pi|^\sigma \text{ near } \pi,$$
with exponent $\sigma = 0.0643... \in (0,1)$. This critical exponent satisfies $\sigma = \frac{\log 4}{\log \lambda^u} - 1$, where $\lambda^u = \frac{4}{t_c^2+1}$ is the "horizontal" eigenvalue of the derivative of the Migdal-Kadanoff RG Equations (1.7), see below, at the renormalization fixed point $(\phi, t) = (0, t_c)$.
3. *For $t_c < t < 1$, the density ρ_t vanishes on a nowhere dense set $K_t \supset \{0, \pi\}$ of positive Lebesgue measure. Moreover, the Lebesgue measure of K_t tends to 2π as $t \to 1$. On each component of the complementary set $O_t = \mathbb{T}_t \smallsetminus K_t$, the density ρ_t is C^∞.*
4. *For $t = 1$, the distribution $d\mu_t$ becomes purely atomic: it is supported on a countable dense subset of $\mathbb{T}_1$.*

[6] See §2.4 for a precise definition of "thermodynamic limit" and §2.6 for their proof.

[7] See Prop. 2.2 from §2.4.

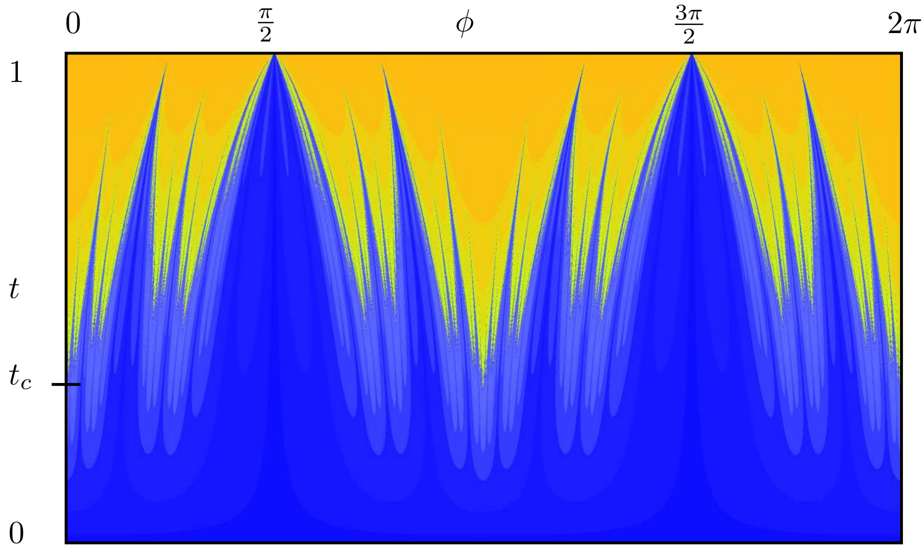

FIGURE 1.2. Distribution of Lee-Yang zeros and RG dynamics. Blue (dark) is the region where the Lee-Yang distributions have positive C^∞ density. Dynamically, it is the basin of attraction of the bottom.

Moreover, there is a family of homeomorphisms $g_t : \mathbb{T}_0 \to \mathbb{T}_t$, $t \in [0, 1)$, *such that* $g_t(\phi)$ *is smooth in* $t \in [0, 1)$ *for any* $\phi \in \mathbb{T}$, $h_0 = \mathrm{id}$, *and* $\rho_t = (g_t^{-1})'(\phi)$ *a.e. on* $\mathbb{T}$. *For* $t < t_c$, *the family* $g_t(\phi)$ *is* C^∞ *in two variables.*

We see, in particular, that $\rho_t(0) > 0$ below t_c and it vanishes above t_c, so we observe at t_c the ferromagnetic-paramagnetic phase transition. However, unlike the scenario described above for the standard $\mathbb{Z}^d$ lattices, the Lee-Yang zeros do accumulate on $\phi = 0$ even in the paramagnetic phase $t > t_c$. So, though for $t > t_c$, the magnetization $M(\phi, t)$ vanishes at $\phi = 0$, it is not analytic nearby.

The distributions $\mu_t(\phi)$ described above for the $\mathbb{Z}^d$ and the DHL are examples of *global distributions*. One can obtain *tangent distributions* as follows. We fix an arbitrary point $\tilde{\phi}$ inside the support of $\mu_t(\phi)$ as a reference point and rescale the zeros near $\tilde{\phi}$, by the affine map

$$\phi \mapsto \frac{L_n}{2\pi}\, \rho_{\mathrm{t}}(\tilde{\phi}) \cdot (\phi - \tilde{\phi}^n), \tag{1.6}$$

where L_n is the total number of Lee-Yang zeros at level n and $\tilde{\phi}^n$ is the one that is closest to $\tilde{\phi}$. We say that the Lee-Yang zeros are *locally rigid at* $\tilde{\phi}$ if the rescaled zeros converge locally uniformly to the $\mathbb{Z}$ lattice, as $n \to \infty$. (Similar phenomena appear in other areas of mathematical physics. See, for example, [ALS].)

It follows directly from (1.3) that the Lee-Yang zeros for the $\mathbb{Z}^1$ lattice are locally rigid everywhere. For the $\mathbb{Z}^d$ lattice (with periodic boundary conditions), local rigidity follows at sufficiently low temperatures from (1.5), since $L_n = 2n^d$.

Because there are $2 \cdot 4^n$ Lee-Yang zeros at level n for the DHL, an expression of the form (1.5) is not sufficient to show their local rigidity. Instead, we will show that the LY zeros $\phi_k^n(t) \in O_t$ can be expressed as $g_t^n(\phi_k^n(0))$, where the g_t^n are diffeomorphisms locally C^1 converging to the maps g_t. This is sufficient for the local rigidity, see Proposition 13.5.

Below we will re-interpret the above results in terms of the renorm-group.

1.3. **Migdal-Kadanoff RG equations.** There is a general physical principle that the values of physical quantities depend on the scale where the measurement is taken. The corresponding quantities are called *renormalized*, and the (semi-)group of transformations relating them at various scales is called *renorm-group (RG)*. However, it is usually hard to justify rigorously existence of RG, let alone to find exact formulas for RG transformations. The beauty of hierarchical models is that all this can actually be accomplished.

In [M1], [M2], Migdal suggested approximations to RG for the classical Ising model on $\mathbb{Z}^d$. They were further developed by Kadanoff [K], and became known as the *Migdal-Kadanoff approximate RG equations*. It was then noticed by Berker and Ostlund [BO] that these equations become exact for suitable hierarchical Ising models (see also [BZ1] and [KG1]). In particular, the DHL corresponds to the 2D lattice $\mathbb{Z}^2$. The Migdal-Kadanoff RG equations in this case assume the form:

$$(1.7)\qquad (z_{n+1}, t_{n+1}) = \left(\frac{z_n^2 + t_n^2}{z_n^{-2} + t_n^2},\ \frac{z_n^2 + z_n^{-2} + 2}{z_n^2 + z_n^{-2} + t_n^2 + t_n^{-2}} \right) := \mathcal{R}(z_n, t_n).$$

where z_n and t_n are the renormalized field-like and temperature-like variables on Γ_n. The map $\mathcal{R}$ that relates these quantities is also called the *renormalization transformation*.

The Lee-Yang zeros for Γ_n are solutions of the algebraic equation $\mathsf{Z}_n(z,t) = 0$, so they form a real algebraic curve $\mathcal{S}_n$ on the cylinder $\mathcal{C}$ (*the Lee-Yang locus of level* n), see Figure 1.3. Equation (1.7) shows that $\mathcal{S}_n$ is the pullback of $\mathcal{S}_0$ under the n-fold iterate of $\mathcal{R}$, i.e., $\mathcal{S}_n = (\mathcal{R}^n)^* \mathcal{S}_0$. In this way, the problem of asymptotical distribution of the Lee-Yang zeros is turned into a dynamical one.

1.4. **Renormalization dynamics on the cylinder.** The first observation is that the cylinder $\mathcal{C}$ is $\mathcal{R}$-invariant. Next, its bottom $\mathcal{B}$ is $\mathcal{R}$-invariant as well, and $\mathcal{R}$ restricts to $z \mapsto z^4$ on $\mathcal{B}$. Moreover, $\mathcal{B}$ is superattracting, so there is an open basin $\mathcal{W}^s(\mathcal{B})$ where the orbits converge to $\mathcal{B}$: this is exactly the blue region on Figure 1.2.

The top $\mathcal{T}$ of $\mathcal{C}$ is also invariant except for two indeterminacy points $\alpha_\pm = (\pm\pi/2, 1)$ that "blow up" to a curve $\mathcal{G}$ going across the cylinder (see Figure 3.3 below). Because of this phenomenon, the degree of $\mathcal{R}$ on the top drops to 2, (namely, $\mathcal{R} : z \mapsto z^2$ on $\mathcal{T}$), and its basin $\mathcal{W}^s(\mathcal{T})$ (roughly, the orange region on Figure 1.2) is not open, but rather a "Cantor bouquet" of hairs sticking from $\mathcal{T}$.

Despite this, $\mathcal{R}$ acts in a surprisingly nice way on the proper curves (i.e., curves connecting the bottom to the top) – namely, a proper curve in $\mathcal{C}$ crossing $\mathcal{G}$ only once lifts to four proper curves, compare Figure 1.3. In this sense, the action of $\mathcal{R}$ on proper curves has degree four.

Our main dynamical result asserts that $\mathcal{R}$ is *partially hyperbolic* on the cylinder $\mathcal{C}_1 := \mathcal{C} \smallsetminus \mathcal{T}$. This means that $\mathcal{R}$ admits an invariant horizontal tangent cone field $\mathcal{K}^h(x) \subset T_x\mathcal{C}$ such that the horizontal tangent vectors $v \in \mathcal{K}^h(x)$ get exponentially stretched under iterates of $\mathcal{R}$.

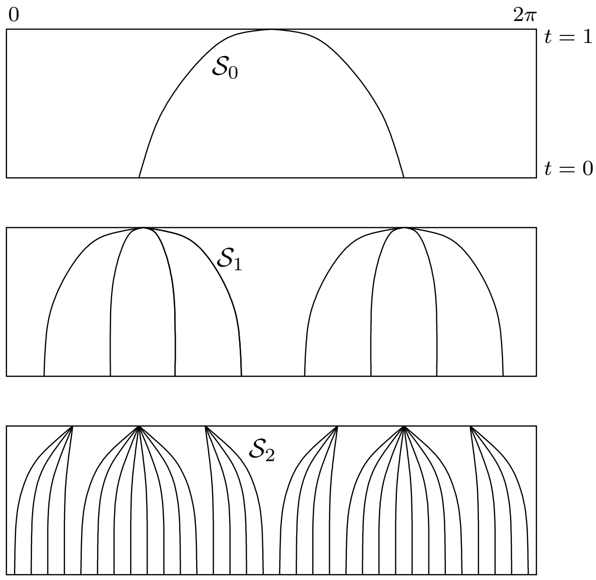

FIGURE 1.3. The level n Lee-Yang zeros $\mathcal{S}_n$ for $n = 0$, 1, and 2.

Let us also consider the complementary vertical cone field $\mathcal{K}^v(x) = T_x\mathcal{C} \smallsetminus \mathcal{K}^h(x)$. A smooth curve $\gamma(t)$ in $\mathcal{C}_1$ going though this cone field is called *vertical.* A *vertical foliation* on $\mathcal{C}_1$ is a foliation whose leaves are proper vertical curves.

Given a vertical foliation $\mathcal{F}$, the *holonomy* transformations $g_t : \mathcal{B} \to \mathbb{T} \times \{t\}$, $t \in [0, 1)$, are defined by the property that x and $g_t(x)$ belong to the same leaf of $\mathcal{F}$.

A *central foliation* for $\mathcal{R}$ is an invariant vertical foliation.

Recall that a measurable map $g : \mathbb{T} \to \mathbb{T}$ is called *absolutely continuous* if preimages $g^{-1}(X)$ of null-sets $X \subset \mathbb{T}$ are null-sets (where a *null-set* means a set of zero Lebesgue measure). Note that this is not a symmetric notion: it may happen that a homeomorphism g is absolutely continuous while the inverse one, g^{-1}, is not (and this is what actually happens below).

Main Theorem (dynamical version). *The renormalization transformation $\mathcal{R}$ is partially hyperbolic on $\mathcal{C}_1$, and it has a unique central foliation $\mathcal{F}^c$. This foliation is C^∞ on $\mathcal{W}^s(\mathcal{B})$ but is not absolutely continuous on $\mathcal{W}^s(\mathcal{T})$.*[8]

Given any proper vertical curve γ on $\mathcal{C}$, the pullback $(\mathcal{R}^n)^*\gamma$ comprises 4^n proper vertical curves, and $(\mathcal{R}^n)^*\gamma \to \mathcal{F}^c$ exponentially fast (away from the top).

The basin $\mathcal{W}^s(\mathcal{B})$ is open and dense in $\mathcal{C}$. The basin $\mathcal{W}(\mathcal{T})$ has positive area, with density 1 at the top. The union of the two basins has full area in $\mathcal{C}$.

Our proof of this geometric result is based upon counting arguments (making use of Bezout's Theorem). We call this method the "Enumerative Dynamics".

A *transverse invariant measure* μ for $\mathcal{F}$ is a family of measures μ_t, $t \in [0, 1)$, such that $\mu_t = (g_t)_*(\mu_0)$. It is uniquely determined by μ_0. The Lee-Yang distributions μ_t form a transverse invariant measure for $\mathcal{F}^c$ equal to the Lebesgue measure on $\mathcal{B}$.

[8] Meaning that restrictions of the homeomorphism g_t^{-1} to $\mathcal{W}^s(\mathcal{B})$ are C^∞ but their restrictions to $\mathcal{W}^s(\mathcal{T})$ are not absolutely continuous.

This fact makes a connection between the physical and dynamical versions of the Main Theorem that will allow us to derive (easily) the former from the latter.

Let us also mention that the dynamical picture described in the last part of the Main Theorem gives one more illustration of the "intertwined basins" phenomenon studied by Kan, Yorke et al [Kan, AYYK], and more recently by Bonifant and Milnor [BM], Ilyashenko [I], and Ilyashenko, Kleptsyn, and Saltykov [IKS].

In the upcoming Part II of this work [BLR2], we study the global structure of the renormalization transformation and zeros of the partition function (*Lee-Yang-Fisher zeros*) in the complex projective space $\mathbb{CP}^2$. The distribution of the zeros is interpreted as the dynamical *(1,1)-current* of $\mathcal{R}$, while the free energy itself is the *potential* of this current. In this way the classical Lee-Yang-Fisher Theory gets tightly linked to the contemporary Dynamical Pluripotential Theory.

We remark that connections between study of the Lee-Yang-Fisher zeros on more general lattices and the dynamics of rational maps in higher dimensions is discussed by De Simoi and Marmi in [DeSMa] and studied numerically by De Simoi in [DeS].

We also mention that it has been proved by Kaschner and the third author that $\mathcal{F}^c$ is not a real analytic foliation in the neighborhood of any point of $\mathcal{C}$. This leads an analog of Isakov's Theorem [Isa] for the DHL: There is a dense set of points $(t, \phi) \in \mathcal{C}$ so that the Lee-Yang zeros have non-analytic density in a neighborhood of ϕ within $\mathbb{T}_t$. We refer the reader to [KR, Thm. 5.2 and Cor. 5.3].

Several interesting questions remain open, arising from both physical and dynamical motivations. They are listed in Appendix F.

1.5. **Structure of the paper.** Let us now outline the structure of the paper indicating ideas of the proofs. Since this paper is naturally placed on the borderline of three fields (statistical mechanics, dynamics, and complex geometry) we have attempted to make exposition reader-friendly for a non-expert in any one of these fields, by motivating the problems and supplying needed background and basic references.

We begin in §2 with relevant background material in statistical mechanics: description of the Ising model on graphs, formulation of the Lee-Yang Theorem, and comments on physical significance of the Lee-Yang zeros. In particular, we supply explicit formulas for the free energy and spontaneous magnetization in terms of the asymptotic distributions of these zeroes. Then we pass to the diamond hierarchical model and derive the Migdal-Kadanoff Renorm-Group (RG) Equations. They lead to the renormalization transformation $\mathcal{R}$.

In §3 we describe the structure of $\mathcal{R}$ on the invariant cylinder $\mathcal{C}$. It is strongly influenced by the presence of two indeterminacy points $\alpha_\pm = (\pm i, 1)$ on the top $\mathcal{T}$ that blow up to the curve $\mathcal{G}$. (On Figure 1.2, these points are clearly seen as the tips of the two main tongues of the blue region.) Because of them, $\mathcal{R}$ does not evenly cover the cylinder: the region below $\mathcal{G}$ is covered four times while its complement is covered only twice. However, we show that $\mathcal{R}$ acts properly with degree four on the space of proper vertical curves. We will derive from here by a counting argument the Lee-Yang Theorem for the DHL (§5).

In §4 we follow up this discussion with a description of the global features of $\mathcal{R}$ on the complex projective space $\mathbb{CP}^2$: its critical and indeterminacy loci, superattracting fixed points and their separatrices. (In fact, here it is more convenient to deal with the map R that comes directly from the Migdal-Kadanoff RG Equations, without passing to the "physical" (z,t)-coordinates. This map is semi-conjugate to $\mathcal{R}$ by a degree two rational change of variable $\mathbb{CP}^2 \to \mathbb{CP}^2$.)

In §6 we prove that $\mathcal{R}$ admits a horizontal invariant cone field $\mathcal{K}^{ah}(x)$ on $\mathcal{C}$. We construct it explicitly by taking the principal Lee-Yang locus $\mathcal{S} \equiv \mathcal{S}_0$ (which comprises two vertical segments) and translating it around the cylinder. It gives us two transverse vertical foliations on $\mathcal{C}$. Then we define $\mathcal{K}^{ah}(x)$ as the horizontal cone tangent to these foliations at x. Using Bezout's Theorem, we check invariance of this cone field. Unfortunately, this cone field degenerates at the top. We partially fix this problem by modifying $\mathcal{K}^{ah}(x)$ near the top in such a way that the new field $\mathcal{K}^h(x)$ degenerates only at the indeterminacy points $\alpha_\pm$.

In §7 we prove that $\mathcal{R}|\,\mathcal{C}$ admits a *dominated splitting.* This means that there exists a "vertical" tangent line field $\mathcal{L}^c(x)$ and constants $C > 0$, $\lambda > 1$ such that

$$\|D\mathcal{R}^n(x)\, w\| \le C\lambda^{-n}\|DR^n(x)\, v\| \quad \text{for any} \quad x \in \mathcal{C} \smallsetminus \mathcal{U},\ w \in \mathcal{L}^c(x),\ v \in \mathcal{K}^h(x) \tag{1.8}$$

(where $\mathcal{U}$ is a neighborhood of the indeterminacy points), so the "horizontal" vectors get stretched exponentially faster than the "vertical" ones.[9] Integrating the line field $\mathcal{L}^c(x)$, we obtain an invariant family of smooth vertical curves filling in the whole cylinder. However, at this stage of the discussion we do not know yet that the integration is unique, so the integral curves may not form a foliation.

In §8 we prove our main dynamical result that the map $\mathcal{R}|\,\mathcal{C}$ is *horizontally expanding* (and thus partially hyperbolic). This means that under iterates the horizontal vectors get stretched exponentially fast:

$$\|D\mathcal{R}^n(x)\, v\| \ge c\lambda^n\|v\|, \qquad x \in \mathcal{C},\ v \in \mathcal{K}^h(x), \tag{1.9}$$

where $c > 0$, $\lambda > 1$. To establish this property, we consider a central projection π in $\mathbb{CP}^2$ onto the line at infinity. By a counting argument, we show that $\pi \circ R^n$ restricted to the horizontal sections of the solid cylinder is a Blaschke product B_n (in appropriate natural coordinates) vanishing at the origin to order 2^{n+2}. Such a B_n expands the circle metric at least by 2^{n+2}, which gives us (1.9) with $\lambda = 2$. We then provide a second proof of this expanding property that exploits the combinatorics of the DHL partition functions and a variant of the Lee-Yang Theorem that we call the *Lee-Yang Theorem with Boundary Conditions.*

In §9 we discuss the basin $\mathcal{W}^s(\mathcal{B})$ of the bottom $\mathcal{B}$ (the blue region of Figure 1.2). We explain where the tongues observed on this picture come from and prove that $\mathcal{W}^s(\mathcal{B})$ supports a C^∞ foliation, the stable foliation of $\mathcal{B}$.

In §10, we turn our attention to the top $\mathcal{T}$ of $\mathcal{C}$. We prove that its basin $\mathcal{W}^s(\mathcal{T})$ contains a "Cantor bouquet" of curves of positive measure. Moreover, the density of its slices by horizontal circles $\mathbb{T} \times \{t\}$ goes to 1 as $t \to 1$.

We then derive in §11, applying standard distortion techniques to horizontal curves, that almost any orbit on the cylinder converges either to the bottom $\mathcal{B}$ or to the top $\mathcal{T}$ (see [BloL] for the one-dimensional prototype of this method).

In the next section, §12, we use horizontal expansion to prove *unique* integrability of the invariant vertical line field $\mathcal{L}^c$ yielding the desired invariant central foliation $\mathcal{F}^c$. We then collect in §13 consequences about regularity of the Lee-Yang distributions (as formulated above) and calculate various critical exponents.

In the last section, §14, we analyze smoothness of periodic leaves that terminate at periodic points on the top. Such leaves are real analytic near the bottom and the

[9] This property is also referred to as *projective hyperbolicity* of $\mathcal{R}$.

top, but we show that they must loose analyticity somewhere in the middle (in fact, generically they can have only finite smoothness). This is another manifestation of the phase transitions in this model.

We finish with several Appendices. In Appendix A we collect needed background in complex geometry: rational maps, indeterminacy points and their blow-ups, degrees and divisors.

In Appendix B we supply some calculations on the cylinder $\mathcal{C}$, particularly, near its top $\mathcal{T}$ and the indeterminacy points $\alpha_\pm$ (performing their "blow-ups" in various coordinates).

In Appendix C we construct an extension of $\mathcal{K}^h(x)$ that is invariant on an appropriate complex neighborhood of $\mathcal{C} \smallsetminus \{\alpha_\pm\}$. It gives us a supply of complex horizontal curves that are used in §11 to obtain the Koebe distortion estimates.

In Appendix D we describe the global critical locus of the map R in $\mathbb{CP}^2$.

In Appendix E we re-prove the classical Lee-Yang Theorem and extend it to the LY Theorem with Boundary Conditions used in §8. We then describe the Lee-Yang zeros in the one-dimensional model, and explain in what sense the hierarchical lattices give an approximation to the standard lattices $\mathbb{Z}^d$.

In Appendix F we collect several open problems. Finally, in Appendix G we provide a list of notation that are frequently used throughout the paper.

1.6. **Basic notation and terminology.** $\mathcal{I} = [0,1]$, $\mathbb{C}^* = \mathbb{C} \smallsetminus \{0\}$, $\mathbb{T} = \{|z| = 1\}$, $\mathbb{D}_r = \{|z| < r\}$, $\mathbb{D} \equiv \mathbb{D}_1$, $\mathbb{D}^* = \mathbb{D} \smallsetminus \{0\}$, $\mathbb{N} = \{0, 1, 2 \dots\}$. Given two variables x and y, $x \asymp y$ means that $c \leq |x/y| \leq C$ for some constants $C > c > 0$. A *path* (in some topological space) is an embedded interval.

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2. Description of the model

2.1. **Background: Ising models on graphs.** Let Γ be a graph representing a magnetic matter in a certain scale. Let $\mathcal{V}$ and $\mathcal{E}$ stand respectively for the set of its vertices (representing atoms) and edges (representing magnetic bonds between the atoms). Two vertices, v and w, connected by an edge are called neighbors: we respectively write $(v, w) \in \mathcal{E}$ or $\{v, w\} \in \mathcal{E}$ for the corresponding oriented or unoriented edge, respectively.

A *spin configuration* on Γ is a function $\sigma : \mathcal{V} \to \{\pm 1\}$. The spin $\sigma(v)$ represents a magnetic momentum of an atom $v \in \mathcal{V}$. The total magnetic momentum of the configuration is equal to[10]

$$M(\sigma) = \frac{1}{2} \sum_{(v,w)\in\mathcal{E}} (\sigma(v) + \sigma(w)) = n_+(\sigma) - n_-(\sigma), \tag{2.1}$$

where $n_+(\sigma)$ and $n_-(\sigma)$ stand respectively for the number of $\{++\}$ and $\{--\}$ bonds.

The Ising model depends on three physical parameters:

- $J \in \mathbb{R}$ – the coupling constant (strength of the magnetic bonds between the atoms);
- $h \in \mathbb{R}$ – strength of the external magnetic field;
- $T > 0$ – temperature.

The Ising model is called *ferromagnetic* if $J > 0$, and *anti-ferromagnetic* otherwise. The Gibbs distribution of a ferromagnetic model favors neighboring spins with the same orientation. In this paper we consider the ferromagnetic model.

The total energy of the configuration σ is given by the Hamiltonian

$$H(\sigma) = -JI(\sigma) - hM(\sigma),$$

where

$$I(\sigma) := \sum_{\{v,w\}\in\mathcal{E}} \sigma(v)\sigma(w) = n_+(\sigma) + n_-(\sigma) - n_0(\sigma) \tag{2.2}$$

is called the *interaction* of the configuration, with $n_0(\sigma)$ being the number of $\{+-\}$ bonds in σ.

Let $\mathrm{Conf} = \mathrm{Conf}(\Gamma)$ be the *configuration space*, i.e., the space of all spin configurations. The *Gibbs weight* of a configuration σ is equal to[11]

$$W(\sigma) \equiv W(\sigma; J/T, h/T) = e^{\frac{-H(\sigma)}{T}} = t^{-I(\sigma)/2} z^{-M(\sigma)}, \tag{2.3}$$

where $z = e^{-h/T}$ and $t = e^{-2J/T}$ are field-like and temperature-like variables. Since we assume the model is ferromagnetic ($J > 0$), the physical values $h \in \mathbb{R}$ and $T > 0$ correspond to $0 < z < \infty$, $0 < t < 1$. However, it is insightful to extend magnetic observables beyond this region–we will do so starting in §2.3.

The *partition function* (or the *statistical sum*) is the total Gibbs weight of the space:

$$\mathsf{Z}_\Gamma = \mathsf{Z}_\Gamma(z,t) = \sum_{\sigma\in\mathrm{Conf}} W(\sigma).$$

It is a Laurent polynomial in z and t.

The *Gibbs distribution* is the probability measure on Conf with probabilities of the configurations proportional to the Gibbs weights:

$$P(\sigma) = \frac{W(\sigma)}{\mathsf{Z}_\Gamma}.$$

Note that it gives a bigger weight to less energetic configurations.

Let $\mathcal{P} = \{(z,t)\} \subset \mathbb{R}^2$ stand for a relevant parameter space. The Gibbs weights are invariant under simultaneous change of sign of the external field and the spins:

$$W(-\sigma; J/T, -h/T) = W(\sigma; J/T, h/T).$$

[10] As we have mentioned in the introduction, this definition is different from (1.1) as the summation here is taken over the bonds rather than the atoms: see Appendix E.4 for a motivation for this unconventional definition.

[11] We let the Boltzmann constant $k = 1$.

This is the *basic symmetry* of the Ising model. It can be also formulated as follows. Consider the "total configuration space" $\widehat{\mathrm{Conf}} = \mathrm{Conf} \times \mathcal{P}$ fibered over $\mathcal{P}$. The Gibbs weights $W(\sigma; z, t)$ endow it with the fibered Gibbs measure. The basic symmetry translates into invariance of this measure under the involution

$$\hat{\iota} : \widehat{\mathrm{Conf}} \to \widehat{\mathrm{Conf}}; \quad \hat{\iota} : (\sigma; z, t) \mapsto (-\sigma; z^{-1}, t). \tag{2.4}$$

As a consequence of this basic symmetry, the partition function Z_Γ is invariant under the involution $\iota : (z, t) \mapsto (z^{-1}, t)$, so it has a form

$$\mathsf{Z}_\Gamma = \sum_{n=0}^{d} a_n(t)(z^n + z^{-n}), \quad \text{where } d = |\mathcal{E}| \text{ and } a_d(t) = t^{-d/2}. \tag{2.5}$$

Thus, for any given $t \neq 0$, $\mathsf{Z}_\Gamma(t, z)$ has $2|\mathcal{E}|$ roots $z_i(t) \in \mathbb{C}$. They are called *Lee-Yang zeros.*

The entropy of a configuration σ is defined as

$$S(\sigma) = -\log P(\sigma) = \log \mathsf{Z} + H(\sigma)/T.$$

The *free energy* is defined as

$$F_\Gamma = H(\sigma) - TS(\sigma) = -T \log \mathsf{Z}_\Gamma. \tag{2.6}$$

It is independent of the configuration σ (in the Gibbs state) and hence coincides with its average over Conf.

Remark 2.1. One can define in the same way the entropy and the free energy for an arbitrary probability distribution on Conf. Then the Gibbs distribution is singled out by one of two equivalent properties: (i) *it minimizes the free energy*; (ii) *the free energy is evenly distributed over configurations.*

The *magnetization* of the matter is the average of the magnetic momentum over the Gibbs distribution:

$$M_\Gamma = \sum M(\sigma)P(\sigma) = -\frac{\partial F_\Gamma}{\partial h} = -z \sum \frac{1}{z - z_i(t)} + |\mathcal{E}|. \tag{2.7}$$

(The last equality is obtained using (2.5)).

Recall that physical values of temperature $T > 0$ correspond to $t = e^{-2J/T} \in (0, 1)$, where $t = 0$ and $t = 1$ correspond respectively to zero and infinite temperature.

Lee-Yang Theorem ([YL, LY])**.** *For a ferromagnetic Ising model, for any temperature $t \in (0, 1)$, the Lee-Yang zeros $z_i(t)$ lie on the unit circle $\mathbb{T}$.*

This is a fundamental theorem of statistical mechanics. In Appendix E we will provide a proof of it in this general form (in fact, even in a slightly more general one). In §5 we will prove it for DHL using dynamics of the Migdal-Kadanoff renormalization.

Given a subsystem of atoms, $\mathcal{U} \subset \mathcal{V}$, and a partial configuration $\sigma_\mathcal{U} : \mathcal{U} \to \{\pm 1\}$, we can define *conditional configurations* as all configurations $\sigma : \mathcal{V} \to \{\pm 1\}$ that agree with $\sigma_\mathcal{U}$ on $\mathcal{U}$. Let $\mathrm{Conf}(\Gamma | \, \sigma_\mathcal{U})$ stand for the space of all such configurations. The *conditional partition function* is defined as the total weight of this space:

$$\mathsf{Z}_{\Gamma | \, \sigma_\mathcal{U}} = \sum_{\sigma \in \mathrm{Conf}(\Gamma | \sigma_\mathcal{U})} W(\sigma).$$

Lee-Yang Theorem with Boundary Conditions. *Consider a ferromagnetic Ising model on a connected graph Γ and let $\sigma_{\mathcal{U}} \equiv +1$ on a nonempty $\mathcal{U} \subsetneq \mathcal{V}$. Then, for any temperature $t \in (0,1)$ the Lee-Yang zeros $z_i^+(t)$ of the conditional partition function $\mathsf{Z}_{\Gamma|\,\sigma_{\mathcal{U}}}$ lie outside the closed disc $\overline{\mathbb{D}}$.*

This interpretation follows directly from the proof of the classical Lee-Yang Theorem; see Appendix E. From the Basic Symmetry of the Ising model we get that for $\sigma_{\mathcal{U}} \equiv -1$ and $t \in (0,1)$ the Lee-Yang zeros $z_i(t)$ lie in the open disc $\mathbb{D}$.

2.2. **Multiplicativity of the partition function.** For a subgraph $\Gamma' \subset \Gamma$, let $\bar{\Gamma}'$ stand for its *closure* obtained by adding to Γ' all of the vertices adjacent to Γ' and all of the edges connecting them to Γ'. Let $\partial\Gamma' = \bar{\Gamma}' \smallsetminus \Gamma'$.

Lemma 2.1. *Let $\sigma_{\mathcal{U}}$ be a conditional configuration and let Γ_i be the connected components obtained after removing the vertices in $\mathcal{U}$ and all of the edges ending at them from Γ. Then*

$$\mathsf{Z}_{\Gamma|\,\sigma_{\mathcal{U}}} = \prod_i \mathsf{Z}_{\bar{\Gamma}_i|\,\sigma_{\partial\Gamma_i}}.$$

Proof. Clearly,

$$\operatorname{Conf}(\Gamma|\,\sigma_{\mathcal{U}}) \cong \prod \operatorname{Conf}(\bar{\Gamma}_i|\,\sigma_{\partial\Gamma_i}). \tag{2.8}$$

Since there are no interactions between the partial configurations $\sigma|\,\Gamma_i$, we have the additivity property for the energy:

$$H(\sigma|\,\sigma_{\mathcal{U}}) = \sum H(\sigma_{\bar{\Gamma}_i}|\sigma_{\partial\Gamma_i}).$$

This implies multiplicativity for the corresponding Gibbs weights and (together with (2.8)) for the conditional partition functions. □

2.3. **Complexification.** Nearly all of the quantities defined in §2.1 complexify in the obvious way. Moreover, the basic symmetry (2.4) holds for any relevant parameter space $\mathcal{P} \subset \mathbb{C}^2$ and (2.5) implies for any $t \in \mathbb{C}^*$ the partition $\mathsf{Z}_\Gamma(z,t)$ has $2|\mathcal{E}|$ roots $z_i(t) \in \mathbb{C}$.

The only caveat is that if one complexifies expression (2.6) for the free energy F_Γ by means of analytic continuation, singularities of the logarithm lead to monodromy. This could be addressed by extending to a suitable Riemann surface. Instead, we introduce a modulus under the logarithm:

$$F_\Gamma(z,t) := -T\log|\mathsf{Z}_\Gamma(z,t)| = -T\sum \log|z - z_i(t)| + |\mathcal{E}|\,T\,(\,\log|z| + \frac{1}{2}\log|t|\,), \tag{2.9}$$

where the summation is taken over the $2|\mathcal{E}|$ Lee-Yang zeros $z_i(t)$ of $\mathsf{Z}(\cdot,t)$ (here $\log|z|$- and $\log|t|$-terms account respectively for the denominator and the leading coefficient of $\mathsf{Z}(\cdot,t)$). In this way, for fixed t, the $F_\Gamma - |\mathcal{E}|\,T\,\log|z|$ is superharmonic on $\mathbb{C}$. We will still refer to this extension as the "free energy".

We complexify the magnetization M_Γ as a meromorphic function on $\mathbb{C}^2$ by applying (2.7) to complex values of z and t. By (2.9), we have

$$F_\Gamma(z,t) = -\frac{T}{2}\log\mathsf{Z}_\Gamma(z,t) - \frac{T}{2}\log\overline{\mathsf{Z}_\Gamma(z,t)} = -\frac{T}{2}\log\mathsf{Z}_\Gamma(z,t) - \frac{T}{2}\log\mathsf{Z}_\Gamma(\bar{z},\bar{t}).$$

If $\frac{\partial}{\partial h}$ denotes the complex (Wirtinger) partial derivative (as opposed to the real partial derivative used in (2.7)), we have $\frac{\partial}{\partial h}\mathsf{Z}_\Gamma(\bar{z},\bar{t}) = 0$. Therefore,

$$-2\frac{\partial}{\partial h}F_\Gamma(z,t) = -T\frac{\partial \log \mathsf{Z}_\Gamma(z,t)}{\partial h} = M_\Gamma(z,t)$$

for all $(z,t) \in \mathbb{C}^2$.

2.4. **Thermodynamic limit.** For finite graphs, the partition function Z is a Laurent polynomial with non-negative coefficients, so the free energy $F = -T\log\mathsf{Z}$ is real analytic in the physical region – there are no phase transitions. To observe phase transitions, one should pass to a thermodynamic limit. Already in the original paper by Lee & Yang [LY], the phase transitions were explicitly related to the asymptotic distribution of the zeros of the partition functions. In this section we will give a more rigorous account of these classical results.

Assume that we have a "lattice" given by a "hierarchy" of graphs Γ_n of increasing size (corresponding to finer and finer scales of the matter)[12] with partition functions Z_n, free energies F_n and magnetizations M_n. To pass to the thermodynamic limit we normalize these quantities *per bond*.[13] Let us say that our hierarchy of graphs has a *thermodynamic limit* if

$$\frac{1}{|\mathcal{E}_n|}F_n(z,t) \to F(z,t) \quad \text{for any } z \in \mathbb{R}_+,\ t \in (0,1). \tag{2.10}$$

In this case, the function F is called the *free energy* of the lattice. For many[14] lattices (e.g. $\mathbb{Z}^d$), existence of the thermodynamic limit can be justified by van Hove's Theorem [vH, R3]. For the DHL, existence of the thermodynamic limit was proved in [BZ3] using the Migdal-Kadanoff RG Equations (1.7). (We include their proof in §2.6.)

When combined with the Lee-Yang Theorem, existence of the thermodynamic limit (2.10) will allow us to extend the limiting free energy $F(z,t)$ and also the limiting magnetization $M(z,t)$ to all $z \in \mathbb{C}$ for any $t \in (0,1)$:

Proposition 2.2. *Assume that a hierarchy of graphs has a thermodynamic limit. Then for any $t \in (0,1)$, the limit (2.10) in z exists in $L^1_{\mathrm{loc}}(\mathbb{C})$ and the zeros of the partition functions Z_n are asymptotically equidistributed with respect to some measure μ_t on the unit circle $\mathbb{T}$. Moreover, the limiting free energy $F(z,t)$ admits the following electrostatic representation:*

$$F(z,t) = -2T\int_{\mathbb{T}} \log|z-\zeta|\, d\mu_t(\zeta) + T\big(\log|z| + \frac{1}{2}\log|t|\big) \quad \textit{for a.e. } z \in \mathbb{C}, \tag{2.11}$$

so $F(z,t) - T\log|z|$ is superharmonic in z on the whole plane $\mathbb{C}$, and is harmonic in $\mathbb{C} \smallsetminus \operatorname{supp}\mu_t$.

[12]At this moment, the terms "lattice" and "hierarchy" are used in a purely heuristic sense. Formally speaking, we just have a sequence of graphs with $|\Gamma_n| \to \infty$.

[13]Viewing $|\mathcal{E}|$ as the "volume" of the system, the normalized quantities get interpreted as "specific" free energy and magnetization.

[14]Note that the DHL is not in this class—instead, dynamical techniques are used to justify its classical thermodynamic limit.

Furthermore, the magnetizations $\frac{1}{|\mathcal{E}_n|}M_n$ *converge locally uniformly on* $\mathbb{C}\setminus\mathbb{T}$, *and the limiting magnetization* M *admits the following Cauchy integral representation:*

$$M(z,t) = -2z\int_{\mathbb{T}}\frac{d\mu_t(\zeta)}{z-\zeta}+1 \quad \textit{for } z\in\mathbb{C}\setminus\operatorname{supp}\mu_t, \tag{2.12}$$

so $M(z,t)$ *is holomorphic in* z *on* $\mathbb{C}\setminus\operatorname{supp}\mu_t$.

Proof. We will fix some $t\in(0,1)$ and will consider all the functions in the z-variable only. Let z_i^n stand for the zeros of the Z_n. Let us clear up the denominators of the Laurent polynomials Z_n to obtain ordinary polynomials $\tilde{\mathsf{Z}}_n = z^{d_n}\mathsf{Z}_n$ (where $d_n = |\mathcal{E}_n|$). They have the same zeros as the Z_n, so by the Lee-Yang Theorem, they do not vanish on $\mathbb{D}$. Hence they admit well defined roots $\phi_n := \tilde{\mathsf{Z}}_n^{1/d_n}$ on $\mathbb{D}$ that are positive on the real line.

Since the polynomials $\tilde{\mathsf{Z}}_n$ have positive coefficients, we have:

$$|\phi_n(z)|\le\phi_n(1)\le\exp\left\{-\frac{1}{Td_n}F_n(1)\right\}\le C \quad \text{for any } z\in\bar{\mathbb{D}}, \tag{2.13}$$

where the last bound follows from existence of the thermodynamic limit.

By Montel's Theorem, the sequence of functions ϕ_n is normal on $\mathbb{D}$. Since it converges on $(0,1)$, it converges locally uniformly on $\mathbb{D}$ to a holomorphic function ϕ. Hence the free energies $d_n^{-1}F_n = -T(\log|\phi_n| - \log|z|)$ converge locally uniformly on $\mathbb{D}^*$ to the harmonic function $F := -T(\log|\phi|-\log|z|)$.

By the basic symmetry $z\mapsto 1/z$, we have $d_n^{-1}F_n\to F$ locally uniformly on $\mathbb{C}\setminus\bar{\mathbb{D}}$. Moreover, by the same symmetry and (2.13), the functions $d_n^{-1}F_n$ are uniformly bounded from above globally on $\mathbb{C}$. By the Compactness Theorem for superharmonic functions (see [Ho, Thm 4.1.9]), the function F admits a superharmonic extension to the punctured plane $\mathbb{C}^*$ and

$$\frac{1}{d_n}F_n\to F \text{ in } L^1_{\mathrm{loc}}(\mathbb{C}). \tag{2.14}$$

Let δ_z stand for the unit mass located at z, and let $\mu^n\equiv\mu^n_t = \frac{1}{2|\mathcal{E}|}\sum\delta_{z_i^n}$. Since $\frac{1}{2\pi}\log|\zeta|$ is the fundamental solution of the Laplace equation, (2.9) implies that

$$-\frac{1}{4\pi T}\frac{\Delta F_n}{d_n} = \frac{1}{2d_n}\sum\delta_{z_i^n}-\frac{1}{2}\delta_0, \tag{2.15}$$

where the Laplacian Δ is understood in the sense of distributions.

Since the distributional Laplacian is a continuous operator, (2.14) implies that $d_n^{-1}\Delta F_n(\cdot,t)\to\Delta F(\cdot,t)$ in the weak topology on the space of measures. Together with (2.15), this implies that the Lee-Yang zeros are equidistributed with respect to measure

$$\mu\equiv\mu_t := -\frac{1}{4\pi T}\Delta F(\cdot,t)+\frac{1}{2}\delta_0.$$

Let $u^n(z)\equiv u^n_t(z)$ and $u(z)\equiv u_t(z)$ stand for the electrostatic potentials of μ^n and μ respectively. For $z\in\mathbb{C}\setminus\mathbb{T}$, the kernel $\zeta\mapsto\log|z-\zeta|$ is a continuous function on $\mathbb{T}$ depending continuously (in the uniform topology) on z. It follows that $u^n(z) \to u(z)$ locally uniformly on $\mathbb{C} \smallsetminus \mathbb{T}$. But by (2.9),

$$\frac{1}{d_n} F_n = -2T\, u^n(z) + T(\log|z| + \frac{1}{2}\log|t|).$$

Since (2.14) implies that for some subsequence n_k, $\dfrac{1}{d_{n_k}} F_{n_k}(z) \to F(z)$ a.e., representation (2.11) follows.

Taking (-2) times the $\partial/\partial h$-derivative (where $h = -T\log z$), we obtain representation (2.12).

□

The basic symmetry of the Ising model implies that the free energy is an even function of the field h, while the magnetization is odd. In terms of the (z,t)-variables, F & M are respectively even & odd under the involution ι (which is also clear from explicit representations (2.11) and (2.12)). If the magnetization has different limits at $z = 1$ from above and below, $M^+(1) > 0$ and $M^-(1) = -M^+(1) < 0$, then one says that the *first order phase transition* occurs (at $h = 0$), and call $M^+(1)$ the *spontaneous magnetization* of the model. The following statement makes physical relevance of the Lee-Yang zeros particularly clear:

Corollary 2.3. *Assume the distribution μ_t is absolutely continuous on the unit circle and its density $\rho_t(\phi) = 2\pi d\mu_t(\phi)/d\phi$ is Hölder continuous at $\phi = 0$. Then the first order phase transition at $h = 0$ occurs if and only if $\rho_t(0) \neq 0$, and the corresponding spontaneous magnetization $M^+(1)$ is equal to $\rho_t(0)$.*

Proof. Formula (2.12) with $d\mu_t = \rho_t d\phi/2\pi$ can be also written as follows:

$$M(z,t) = -2\left(\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\rho_t(\zeta)d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\rho_t(\zeta)d\zeta}{\zeta}\right) - 1.$$

By the Sokhotsky Theorem (see [Kre, Theorem 7.6]), the jump at $z = 1$ (from inside to outside of $\mathbb{T}$) of the Cauchy integral in parentheses is equal to $-\rho(0)$. Hence the jump of M (from inside to outside) is equal to $2\rho_t(0)$. On the other hand, it is equal to $2M^+(1)$. □

Remark 2.2. If the limiting distribution μ_t has a density $\rho_t(\phi)$ that is real-analytic in a neighborhood of $\phi = 0$, then (2.11) allows for a analytic continuation of $F(z,t)$ in a neighborhood of $z = 1$ by a deformation of contours.

For the $\mathbb{Z}^d$ Ising model, $d > 1$, Isakov [Isa] has shown that no such analytic continuation exists at sufficiently low temperatures. Thus, for these models $\rho_t(\phi)$ cannot be real-analytic $\phi = 0$ for low temperatures.

2.5. **Diamond Hierarchical Lattice (DHL) and Migdal-Kadanoff renormalization.** Let us start with the simplest possible graph Γ_0: just two vertices, a and b, connected with one edge. The space Conf_{Γ_0} consists of four configurations with the following energies:

$$H\begin{pmatrix}\oplus\\ |\\ \oplus\end{pmatrix} = -J - h, \quad H\begin{pmatrix}\oplus\\ |\\ \ominus\end{pmatrix} = H\begin{pmatrix}\ominus\\ |\\ \oplus\end{pmatrix} = J, \quad H\begin{pmatrix}\ominus\\ |\\ \ominus\end{pmatrix} = -J + h,$$

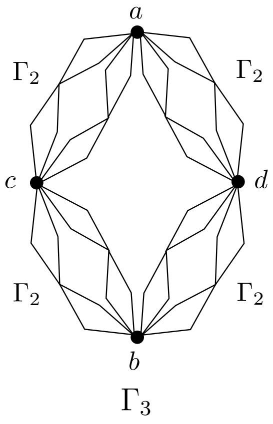


FIGURE 2.1. Graph Γ_3 built from four copies of Γ_2.

and the following Gibbs weights:

$$\begin{aligned} U &= W\begin{pmatrix}\oplus\\ |\\ \oplus\end{pmatrix} = z^{-1}t^{-1/2}, \\ V &= W\begin{pmatrix}\oplus\\ |\\ \ominus\end{pmatrix} = W\begin{pmatrix}\ominus\\ |\\ \oplus\end{pmatrix} = t^{1/2}, \qquad (2.16)\\ W &= W\begin{pmatrix}\ominus\\ |\\ \ominus\end{pmatrix} = zt^{-1/2}. \end{aligned}$$

They sum up to the following partition function:

$$\mathsf{Z} \equiv \mathsf{Z}_{\Gamma_0} = U + 2V + W = \frac{z^2+2tz+1}{z\sqrt{t}}. \qquad (2.17)$$

Let us now replace the interval Γ_0 with a diamond Γ_1 with vertices a, b, c, d (so that it shares with Γ_0 the vertices a and b), see Figure 1.1. Restricting that the spins at a and b are both $+$ and summing over the four spin configurations $(+,+)$ $(+,-)$ & $(-,+)$, and $(-,-)$ at the vertices c and d yields a sum of four conditional partition functions (two of which are equal):

$$U_1 := \mathsf{Z}_{\Gamma_1|++} = U^4 + 2U^2V^2 + V^4 = (U^2+V^2)^2$$

Similarly

$$V_1 := \mathsf{Z}_{\Gamma_1|+-} = \mathsf{Z}_{\Gamma_1|-+} = U^2V^2 + 2UV^2W + V^2W^2 = V^2(U+W)^2,$$

$$W_1 := \mathsf{Z}_{\Gamma_1|--} = V^4 + 2V^2W^2 + W^4 = (W^2+V^2)^2.$$

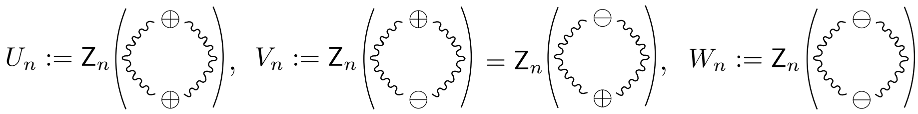
The full partition function of Γ_1 is equal $\mathsf{Z}_{\Gamma_1} = U_1 + 2V_1 + W_1$.

Replacing each edge of the diamond with Γ_1, we obtain a lattice Γ_2 with 16 edges. Inductively, replacing each edge of the diamond with the lattice Γ_{n-1}, we obtain the lattice Γ_n with 4^n edges,[15] see Figure 2.1.

All lattices Γ_n share four vertices, a, b, c and d, with the original diamond. Restricting the spins at $\{a, b\}$ we obtain three conditional partition functions, U_n, V_n

[15]This description of the DHL is "dual" to the one given in the Introduction, §1.2.

and W_n as follows:



The total partition function is equal to

$$\mathsf{Z}_n = \mathsf{Z}_{\Gamma_n} = U_n + 2V_n + W_n.$$

Similarly to the above formulas for U_1, V_1 and W_1, we have:

Migdal-Kadanoff RG Equations:

$$U_{n+1} = (U_n^2 + V_n^2)^2, \quad V_{n+1} = V_n^2(U_n + W_n)^2, \quad W_{n+1} = (V_n^2 + W_n^2)^2.$$

Proof. Let us check the first equation (the others are similar). There are four spin configurations at the vertices (c, d): $(+,+)$, $(+,-)$ & $(-,+)$ and $(-,-)$, as shown in Figure 2.2. By the multiplicativity of the partition function (Lemma 2.1), the corresponding conditional partition functions are equal respectively to U_n^4, $U_n^2 V_n^2$ (twice) and V_n^4. Summing these up, we obtain the desired equations.

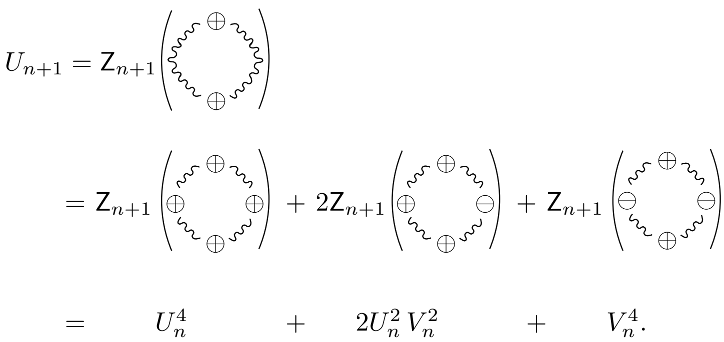


Figure 2.2. Derivation of the Migdal-Kadanoff Equations.

□

Let us consider the following homogeneous degree 4 polynomial map $\hat{R} : \mathbb{C}^3 \to \mathbb{C}^3$

$$\hat{R} : (U, V, W) \mapsto ((U^2 + V^2)^2,\ V^2(U + W)^2,\ (V^2 + W^2)^2) \tag{2.18}$$

called the *Migdal-Kadanoff Renormalization*. By the Migdal-Kadanoff RG Equations, the conditional partition functions of Γ_n are given by the orbits of this map, $(U_n, V_n, W_n) = \hat{R}^n(U, V, W)$. The full partition function is obtained by the $\hat{R}^n$-pullback of the linear form $\mathsf{Z} = U + 2V + W$:

$$\mathsf{Z}_n = \mathsf{Z} \circ \hat{R}^n. \tag{2.19}$$

The renormalization operator $\hat{R}$ descends to a rational transformation $R : \mathbb{CP}^2 \to \mathbb{CP}^2$. In the affine coordinates $u = U/V$, $w = W/V$, it assumes the form:

$$R : (u, w) \mapsto \left(\frac{u^2 + 1}{u + w},\ \frac{w^2 + 1}{u + w} \right)^2, \tag{2.20}$$

where the external squaring stands for the squaring of both coordinates, $(u,w)^2 = (u^2, w^2)$. According to (2.16), these coordinates are related to the "physical" (z,t)-coordinates as follows:

$$(u,w) = \Psi(z,t) = \left(\frac{1}{zt}, \frac{z}{t}\right). \tag{2.21}$$

In (z,t)-coordinates, the renormalization transformation assumes the form:

$$\mathcal{R} : (z,t) \mapsto \left(\frac{z^2+t^2}{z^{-2}+t^2}, \frac{z^2+z^{-2}+2}{z^2+z^{-2}+t^2+t^{-2}}\right). \tag{2.22}$$

The iterates $(z_n, t_n) = \mathcal{R}^n(z,t)$ are related to (U_n, V_n, W_n) by means of (2.16): $U_n = z_n^{-1} t_n^{-1/2}$, etc. Physically, they are interpreted as the *renormalized* field-like and temperature-like variables.

2.6. **Existence of the thermodynamic limit for the DHL.**

Proposition 2.4. *The thermodynamic limit exists for the DHL.*

The following is adapted from [BZ3, Sec. 3]:

Proof. The basic symmetry of the Ising model implies that $F_n(z,t) = F_n(1/z,t)$ for every $n \geq 0$. Thus, it suffices to prove that (2.10) holds for every $(z,t) \in (0,1] \times (0,1)$.

One can check that $(0,1] \times (0,1)$ is invariant under $\mathcal{R}$. Let $(z_0, t_0) \in (0,1] \times (0,1)$ and let $(z_n, t_n) := \mathcal{R}^n(z_0, t_0)$. For any $n \geq 0$

$$U_n = (U_{n-1}^2 + V_{n-1}^2)^2 = U_{n-1}^4 (1 + z_{n-1}^2 t_{n-1}^2)^2.$$

By induction, we have

$$U_n = U_0^{4^n} \prod_{j=0}^{n-1} \left(1 + z_j^2 t_j^2\right)^{2\ 4^{n-1-j}}.$$

Meanwhile,

$$Z_n = U_n + 2V_n + W_n = U_n \left(1 + 2 z_n t_n + z_n^2\right).$$

Since $U_0 = z_0^{-1} t_0^{-1/2}$, we have

$$Z_n = (z_0^{-1} t_0^{-1/2})^{4^n} \prod_{j=0}^{n-1} \left(1 + z_j^2 t_j^2\right)^{2\ 4^{n-1-j}} \left(1 + 2 z_n t_n + z_n^2\right).$$

For the DHL we have $|\mathcal{E}_n| = 4^n$. Using that $z_0 = e^{-h/T}$ and $t_0 = e^{-2J/T}$, we find

$$\begin{aligned} \frac{1}{|\mathcal{E}_n|} F_n(z,t) &= -\frac{T}{4^n} \log Z_n \\ &= -h - J - \frac{T}{2} \sum_{j=0}^{n-1} \frac{1}{4^j} \log(1 + z_j^2 t_j^2) - \frac{1}{4^n} \log\left(1 + 2 z_n t_n + z_n^2\right). \end{aligned} \tag{2.23}$$

Therefore,

$$F(z,t) = \lim_{n\to\infty} \frac{1}{|\mathcal{E}_n|} F_n(z,t) = -h - J - \frac{T}{2} \sum_{j=0}^{\infty} \frac{1}{4^j} \log(1 + z_j^2 t_j^2), \tag{2.24}$$

where we have used that $(z_n, t_n) \in (0,1] \times (0,1)$ for each $n \geq 1$ to conclude that the last term from (2.23) converges to 0 and that the sum in (2.24) converges. □

Together, Propositions 2.2 and 2.4 imply that the limiting distribution μ_t of Lee-Yang zeros exists for the DHL at each $t \in (0, 1)$. Our proof of the main theorem will not need this fact and will thus give an alternate proof that the limiting distribution (and hence the thermodynamic limit) exists.

2.7. **Basic Symmetries.** By the basic symmetry of the Ising model, the change of sign of h interchanges the conditional partition functions U_n and W_n keeping V_n and the total sum Z_n invariant. Consequently, the RG transformation $\hat{R}$ commutes with the involution $(U, V, W) \mapsto (W, V, U)$, which is also obvious from the explicit formula (2.18). Accordingly, the transformation R commutes with the permutation $(u, w) \mapsto (w, u)$, while $\mathcal{R}$ commutes with $(z, t) \mapsto (z^{-1}, t)$.

All these transformations have real coefficients, so all of them commute with the corresponding complex conjugacies: $(U, V, W) \mapsto (\bar{U}, \bar{V}, \bar{W})$, etc.

Finally, there is an extra "accidental" symmetry of the DHL: the generating diamond Γ_1 is symmetric under reflection across the vertical axis. It results in the squared terms in the Migdal-Kadanoff RG Equations that makes the LY distributions μ_t symmetric under the half-period translation $\phi \mapsto \phi + \pi$. It will play an important role in §8.2.

3. Structure of the RG transformation I: Invariant cylinder

We will now begin to explore systematically the RG dynamics. Its generator was represented above in several coordinate systems, in particular, as a transformation R (2.20) in the affine coordinates (u, w) and as a transformation $\mathcal{R}$ (2.22) in the physical coordinates (z, t). From a physical point of view we are primarily interested in the latter. However, R possess better global dynamical properties. For these reasons we will treat both mappings in parallel. In order to help keep track of the various corresponding objects in these two different coordinate systems, we have included a table of notation in Appendix G.

Since R and $\mathcal{R}$ represent the same map in different coordinates, the corresponding change of variables (2.21) should be equivariant, i.e., the following diagram must commute:

$$
\begin{array}{ccc}
\mathbb{CP}^2 & \xrightarrow{\mathcal{R}} & \mathbb{CP}^2 \\
\downarrow \Psi & & \downarrow \Psi \\
\mathbb{CP}^2 & \xrightarrow{R} & \mathbb{CP}^2
\end{array}
\tag{3.1}
$$

wherever the maps are well defined. And this is indeed true and can be verified directly using the explicit formulas for the maps.

In this section we will describe basic features of $\mathcal{R}$ (viewed statically) on the invariant cylinder (that supports the Lee-Yang zeros) and of R on the corresponding invariant Möbius band.

3.1. **Invariant cylinder and Möbius band.** Let us consider the round cylinder $\mathcal{C} = \mathbb{T} \times \mathcal{I}$ naturally sitting in $\mathbb{C}^2$. It is obvious from (2.22) that $\mathcal{C}$ is invariant under $\mathcal{R}$.

Remark 3.1. Note that the whole bi-infinite cylinder $\hat{\mathcal{C}} = \mathbb{T} \times \mathbb{R}$ is also invariant under $\mathcal{R}$, and in fact, $\mathcal{R}(\hat{\mathcal{C}}) \subset \mathcal{C}$. (Here the upper cylinder ($t > 1$) corresponds to the *anti-ferromagnetic* region.)

We keep identifying the unit circle $\mathbb{T}$ with $\mathbb{R}/2\pi\mathbb{Z}$, so that points (z,t) $(z \in \mathbb{T})$ of the cylinder $\mathcal{C}$ will often be written in the angular coordinate as (ϕ, t), where $z = e^{i\phi}$, $\phi \in \mathbb{R}/2\pi\mathbb{Z}$. Let $\mathbb{T}_t = \mathbb{T} \times \{t\}$ be the horizontal sections of the cylinder. We will use special notation for the bottom and the top sections:

$$\mathcal{B} = \mathbb{T}_0 \qquad \mathcal{T} = \mathbb{T}_1.$$

We will also use special notation $\mathcal{C}_1 := \mathcal{C} \smallsetminus \mathcal{T}$ and $\mathcal{C}_0 := \mathcal{C} \smallsetminus \mathcal{B}$ respectively for the topless and the bottomless cylinder.

The cylinder is foliated by the vertical intervals

$$\mathcal{I}_\phi = \{(\phi, t) : \ 0 \le t \le 1\}.$$

The interval $\mathcal{I}_0 = \{\phi = 0\}$ plays a distinguished role, both physically and dynamically. Physically, it corresponds to the *vanishing* magnetic field. Dynamically, it is singled out by the property of being *invariant* under $\mathcal{R}$. Its endpoints $\beta_0 = (0,0) \in \mathcal{B}$ and $\beta_1 = (0,1) \in \mathcal{T}$ are *superattracting fixed points* for $\mathcal{R}|\,\mathcal{I}_0$, and there is a unique *repelling fixed point* $\beta_c = (0, t_c) \in \mathcal{I}_0$. This is exactly the *critical point of the Ising model*[16] mentioned in the introduction and marked on Figure 1.2. Orbits of all points $\beta \in \mathcal{I}_0$ below β_c converge to β_0, while orbits of all points above it converge to β_1.

Let us now switch to the affine coordinates $(u, w) = \Psi(z, t)$ from (2.21). Consider a topological annulus

$$C_0 = \{(u, w) \in \mathbb{C}^2 : \ w = \bar{u}, \ |u| \ge 1\}. \tag{3.2}$$

in $\mathbb{C}^2$, and let C stand for its closure in $\mathbb{CP}^2$. Let $\mathrm{T} = \{(u, \bar{u}) : \ |u| = 1\}$ be the "top" circle of C, while B be the slice of C at infinity. Let $C_1 = C \smallsetminus \mathrm{T}$.

Though the change of variables Ψ is not globally invertible, it is nearly such on the cylinder $\mathcal{C}$:

Proposition 3.1. (i) Ψ *restricts to a diffeomorphism* $\mathcal{C}_0 \to C_0$*;*

(ii) $C = \Psi(\mathcal{C})$ *is an* R*-invariant Möbius band, and* B *is an* R*-invariant circle, a "median" of the band (given as the unit circle in the coordinate* $\zeta = w/u$*).*

(iii) R *acts on* C_0 *as* $u \to \left(\dfrac{u^2+1}{2\,\mathrm{Re}\,u}\right)^2$*, and it acts on* B *as* $\zeta \mapsto \zeta^4$*;*

(iv) *The map* $\Psi : \mathcal{B} \to \mathrm{B}$ *is 2-to-1, and* $\Psi : \mathcal{C}_1 \to C_1$ *continuously semiconjugates* $\mathcal{R}$ *to* R*.*

Proof. It is obvious from (2.21) that Ψ maps $\mathcal{C}_0$ smoothly to C_0. Moreover, (ϕ, t) can be recovered from $(u, v) = \Psi(\phi, t)$ as the polar coordinates of $u^{-1} = te^{i\phi}$. This yields (i).

Let us use coordinates $(\xi = 1/u, \zeta = w/u)$ near the line at infinity $\{\xi = 0\}$. In these coordinates, the map Ψ assumes the form $\xi = tz$, $\zeta = z^2$, which makes obvious its continuity. Hence it maps $\mathcal{C}$ onto C.

Moreover, the circle $\mathcal{B}$ is mapped to the circle $\mathrm{B} = \{\xi = 0, |\zeta| = 1\}$ by $\zeta = z^2$. So, topologically C is obtained from the cylinder $\mathcal{C}$ by identifying the antipodal points z and $-z$ on $\mathcal{B}$. This makes the Möbius band.

Invariance of C and B follow from the semi-conjugacy or directly from (2.20). Expression (iii) is straightforward and property (iv) follows from the semi-conjugacy Ψ.

[16] We hope it will be clear from the context whether the term "critical" is used in the physical or dynamical sense.

(We will soon see that indeterminate points for iterates of $\mathcal{R}$ form a dense set in $\mathcal{T}$, this is why we omit it in (iv).) $\square$

In the coordinate u on C, the interval $\mathcal{I}_0$ becomes the real ray $I_0 = \{u \in \mathbb{R}, u \geq 1\}$ and the fixed points $\beta_1, \beta_c, \beta_0$ on $\mathcal{I}_0$ become fixed points $b_1 = \{u = 1\}, b_c$, $b_0 = \{\zeta = 1\}$ for R.

3.2. **Decomposition $\mathcal{R} = f \circ Q$ and structure of f on the cylinder.** To understand further the geometric structure (of the first iterate) of the renormalization $\mathcal{R}$, it is convenient to decompose it as $f \circ Q$, where $Q(z,t) = (z^2, t^2)$ and

$$f(z,t) = \left(\frac{z+t}{z^{-1}+t}, \frac{\cos\phi + 1}{\cos\phi + s} \right), \quad s = \frac{1}{2}(t + t^{-1}). \tag{3.3}$$

As a reflection of the basic symmetry of the Ising model, f commutes with the involution $\sigma : (\phi, t) \mapsto (-\phi, t)$.

The cylinder $\mathcal{C}$ is invariant under both Q and f. However, we should be careful: f is not well defined on the whole cylinder; it has a *point of indeterminacy* $\alpha = (\pi, 1) \in \mathcal{T} \cap \mathcal{I}_\pi$ which decisively influences the dynamics. When we approach α from inside the cylinder at angle $\omega \in [-\pi/2, \pi/2]$ with the leaf $\mathcal{I}_\pi$, the map f converges to the point

$$(\phi, t) = (2\omega, \sin^2\omega) := \mathcal{G}(\omega) \in \mathcal{C} \tag{3.4}$$

(see [BZ3, p. 419] and also calculations in Appendix B). Thus, the point α *blows up* to the *blow-up locus*

$$\mathcal{G} = \left\{ t = \sin^2\phi/2 \equiv \frac{1-\cos\phi}{2} \right\}. \tag{3.5}$$

Note that $\mathcal{G}$ touches the top $\mathcal{T}$ at α itself, and touches the bottom $\mathcal{B}$ at β_0 (see Figures 3.1 and 3.2). It divides the cylinder into two pieces: $\mathcal{C}_-$ (below $\mathcal{G}$) and $\mathcal{C}_+$ (above it).

The interval $\mathcal{I}_\pi$ (that ends at the point of indeterminacy α) is *critical*: it collapses under f to the fixed point β_0.

In the angular coordinate, we have:

$$Df = \frac{2}{\eta^2} \begin{pmatrix} \eta & 0 \\ 0 & 1-t \end{pmatrix} \cdot \begin{pmatrix} 1 + t\cos\phi & -\sin\phi \\ -t(1-t)\sin\phi & (1+t)(1+\cos\phi) \end{pmatrix}, \tag{3.6}$$

where $\eta = 1 + 2t\cos\phi + t^2 \in [0,4]$. It follows that

$$\operatorname{Jac} f = \frac{4(1-t)(1+\cos\phi)}{\eta^2} \geq 0, \tag{3.7}$$

and $\operatorname{Jac} f = 0$ only on the critical interval $\mathcal{I}_\pi$ and the top of the cylinder $\mathcal{T}$. Thus, f is an *orientation preserving local diffeomorphism* on $\mathcal{C}_1 \setminus \mathcal{I}_\pi$. (In fact, the same is true on $\mathcal{C} \setminus \mathcal{I}_\pi$.)

Note that $f|\mathcal{B} : z \mapsto z^2$ while $f|\mathcal{T} = \text{id}$. This drop in the degree is caused by the point of indeterminacy in the following way. Let us consider the zero level Lee-Yang locus $\mathcal{S} = \{\mathsf{Z} = 0\} \cap \mathcal{C}$, where Z is the partition function (2.17):

$$\mathcal{S} \equiv \mathcal{S}_0 = \{z^2 + 2tz + 1 = 0\} \cap \mathcal{C} = \{t = -\cos\phi : \ \phi \in [\pi/2, 3\pi/2]\}. \tag{3.8}$$

It has two branches over I (symmetric with respect to $\mathcal{I}_\pi$) each of which is mapped diffeomorphically onto $\mathcal{I}_\pi$ (see Figure 3.1).

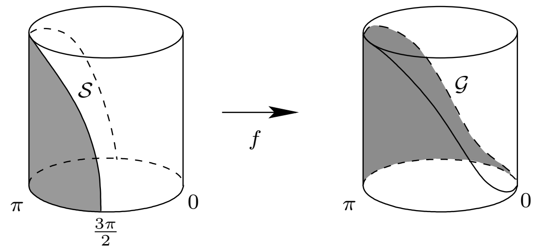


FIGURE 3.1. The map $f : \mathcal{C} \to \mathcal{C}$.

The curve $\mathcal{S}$ divides $\mathcal{C}$ into two domains: Λ^s (containing $\mathcal{I}_\pi \smallsetminus \{\alpha\}$) and Λ^r (containing $\mathcal{I}_0$). The domain $\Lambda^s \smallsetminus \mathcal{I}_\pi$ is composed of two topological triangles mapped diffeomorphically[17] onto the corresponding triangles of $\mathcal{C}_- \smallsetminus \mathcal{I}_\pi$, namely, the right-hand side triangle of $\Lambda_s \smallsetminus \mathcal{I}_\pi$ is mapped onto the left-hand side triangle of $\mathcal{C}_- \smallsetminus \mathcal{I}_\pi$,[18] see Figure 3.2. On the other hand, Λ^r is mapped diffeomorphically onto the whole $\mathcal{C} \smallsetminus \mathcal{I}_\pi$. Accordingly, we have two diffeomorphic branches of the inverse map, the "singular" branch $f_s^{-1} : \mathcal{C}_- \smallsetminus \mathcal{I}_\pi \to \Lambda^s$ and the "regular" one, $f_r^{-1} : \mathcal{C} \smallsetminus \mathcal{I}_\pi \to \Lambda^r$.

In particular, we conclude that *the map f has degree 2 over* $\operatorname{int}\mathcal{C}_-$ *and degree 1 over* $\operatorname{int}\mathcal{C}_+$. So, *$f : \mathcal{C}_1 \to \mathcal{C}_1$ is not a proper map.*

A path $\gamma : [0,1] \to \mathcal{C}$ is called *proper* if it connects the bottom of the cylinder to its top without passing through $\partial\mathcal{C}$ in between.[19] A crucial property of the cylinder dynamics is that f^{-1} acts properly on proper paths:

Lemma 3.2. *If $\gamma : [0,1] \to \mathcal{C}$ is a proper path then the full preimage $f^{-1}\gamma$ contains two proper paths, δ_1 and δ_2. These two paths can meet only at α. Moreover, if γ crosses $\mathcal{G}$ only once, then $f^{-1}\gamma = \delta_1 \cup \delta_2 := \delta_r \cup \delta_s$, where $\delta_r = f_r^{-1}\gamma$ is the "regular" lift of γ while $\delta_s = f_s^{-1}\gamma$ is the "singular" lift ending at α.*[20]

Proof. Since the endpoints of γ belong to different components of $\partial\mathcal{C}$, we can orient it so that $\gamma(0) \in \mathcal{B}$. This initial point has two preimages on $\mathcal{B}$; let p be either of them. We will show that there is a proper path $\delta \subset f^{-1}\gamma$ that begins at p.

Let $\alpha_0 = (\pi, 0) \in \mathcal{B}$ stand for the bottom point of the critical interval $\mathcal{I}_\pi$ (which collapses to β_0). If $p = \alpha_0$ then $\gamma(0) = \beta_0$ and the interval $\mathcal{I}_\pi$ is a desired path δ.[21]

So, assume $p \neq \alpha_0$. Then $f : \mathcal{C} \to \mathcal{C}$ is a local diffeomorphism near p, so there is a local lift δ of γ that begins at p. Continuing lifting it as far as possible, we obtain a lift $\delta(\lambda)$, $0 \le \lambda < \lambda_* \in (0,1]$, that cannot be extended further.

What can go wrong at λ_*? If $x := \gamma(\lambda_*) \notin \mathcal{T} \cup \mathcal{G}$ then x would have a disk neighborhood $U \subset \operatorname{int}\mathcal{C}$ such that $f : f^{-1}(U) \to U$ were a covering map (of degree 1 or 2), and the lift would admit a further extension. So, $x \in \mathcal{T} \cup \mathcal{G}$.

[17]Each of these assertions is proved by verifying that the the mapping is a proper local diffeomorphism between simply connected spaces.

[18]The map is quite peculiar on the boundary of $\Lambda^s \smallsetminus \mathcal{I}_\pi$ as it blows up α to the $\mathcal{G}$-boundary of $(\mathcal{C}_- \smallsetminus \mathcal{I}_\pi)$ while collapses $\mathcal{I}_\pi$ to β_0.

[19]An open path $\gamma_1 : (0,1) :\to \operatorname{int}\mathcal{C}_1$ or a half-open path $\gamma_1 : [0,1) :\to \mathcal{C}_1$ that extends to a proper path $\gamma : [0,1] \to \mathcal{C}$ will also be called "proper".

[20]In case $\gamma(1) = \alpha$, both lifts end at α.

[21]If an initial piece of γ lies in $\mathcal{C}_-$ then there is a lift δ' of γ that begins at α_0. This possible extra lift is disregarded in our discussion.

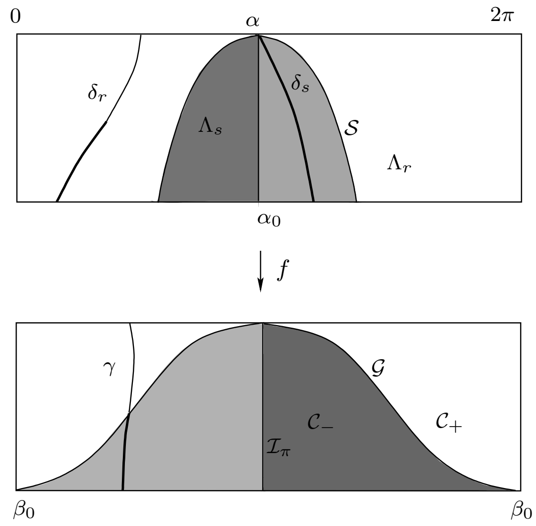


FIGURE 3.2. Cylinder $\mathcal{C}$ shown in (ϕ, t) coordinates. The left-hand side triangle of $\Lambda_s \smallsetminus \mathcal{I}_\pi$ is mapped onto the right-hand side triangle of $\mathcal{C}_- \smallsetminus \mathcal{I}_\pi$. The proper path γ is lifted by f to two paths, the regular lift δ_r and the singular lift δ_s. The singular lift δ_s reaches $\mathcal{T}$ at α.

If $x \in \mathcal{T} \smallsetminus \{\alpha\}$ then $\lambda_* = 1$ and x has a relative half-disk neighborhood $U \subset \mathcal{C}$ such that $f^{-1}(U)$ is a half-disk neighborhood of $f^{-1}x = x$ mapped homeomorphically onto U. In this case we let $\delta(1) = x$ and obtain the desired lift.

If $x \in \mathcal{G} \smallsetminus \{\alpha\}$ then x has a neighborhood $U \subset \operatorname{int} \mathcal{C}$ such that $f^{-1}U = U^r \sqcup U^s$ where $U^r = f_r^{-1}(U)$ is a disk homeomorphically mapped onto U, while $U^s = f_s^{-1}(U \cap \mathcal{C}_-)$ is a "wedge centered at α" homeomorphically mapped onto $U \cap \mathcal{C}_-$. Then for all λ near λ_*,

$$\text{either } \delta(\lambda) = f_r^{-1}(\gamma(\lambda)) \subset U^r \text{ or } \delta(\lambda) = f_s^{-1}(\gamma(\lambda)) \subset U^s. \tag{3.9}$$[22]

But in the former case, $\delta(\lambda)$ can be extended beyond λ_*, contrary to our assumption. In the latter case, $\delta(\lambda)$ is forced to converge (as $\lambda \to \lambda_*$) to the center α of the wedge U^s. This gives us the desired proper path terminating at α.

Finally, if $x = \alpha$ then $\lambda_* = 1$ and α can be the only accumulation point for $\delta(\lambda)$ as $\lambda \to 1$ (for, if $y \in \mathcal{C} \smallsetminus \{\alpha\}$ is another accumulation point then $\gamma(\lambda)$ would accumulate on $f(y) \neq \alpha$ as $\lambda \to 1$). Thus, $\delta(\lambda) \to \alpha$ as $\lambda \to 1$, and we obtain a proper path again.

Remark that if $\delta \neq \mathcal{I}_\pi$ then the path $\delta(\lambda)$ cannot meet $\mathcal{I}_\pi \smallsetminus \{\alpha\}$. Indeed, under this assumption, $\beta_0 \notin \gamma$, while $\mathcal{I}_\pi$ collapses to β_0.

So, we have constructed two proper paths, δ_1 and δ_2. They cannot meet at any point of $\mathcal{C} \smallsetminus \mathcal{I}_\pi$ since f is a local homeomorphism over there. By the above remark, they cannot meet at any point of $\mathcal{I}_\pi \smallsetminus \{\alpha\}$ either. Hence α is their only possible meeting point.

[22] Note that the curve $\gamma(\lambda)$ can cross $\mathcal{G}$ infinitely many times at $\lambda \to \lambda_*$. If this happens then $\delta(\lambda) = f_r^{-1}(\gamma(\lambda))$ for λ near λ_*.

Assume now that γ crosses $\mathcal{G}$ only once, and let $\gamma(\lambda_*) \in \mathcal{G}$ be this intersection point. Assume first that $\gamma(0) \neq \beta_0$. Then by the previous argument, the arc $\gamma(\lambda)$, $0 \leq \lambda < \lambda_*$, has two lifts $\delta_r(\lambda)$ and $\delta_s(\lambda)$ as in (3.9). Then $\delta_s(\lambda)$, $0 \leq \lambda \leq \lambda_*$, is a proper path terminating at α (the singular lift of γ), while $\delta_r(\lambda)$ extends further to a proper path parameterized by the full interval $[0, 1]$ (the regular lift of γ).

Thus, for $\lambda < \lambda_*$, both preimages of $\gamma(\lambda)$ are captured by the above lifts $\delta_r(\lambda)$ and $\delta_s(\lambda)$, while for $\lambda > \lambda_*$, the only preimage of $\gamma(\lambda)$ is $\delta_r(\lambda)$. We conclude that $f^{-1}(\gamma) = \delta_r \cup \delta_s$.

Finally, if $\gamma(0) = \beta_0$ then $\gamma(\lambda) \in \mathcal{C}^+$ for $\lambda > 0$, so such $\gamma(\lambda)$ has only one preimage. This preimage is captured by the lift δ_r that begins at β_0. It follows that

$$f^{-1}(\gamma) = \delta_r \cup f^{-1}(\beta_0) = \delta_r \cup \mathcal{I}_\pi := \delta_r \cup \delta_s.$$

□

Figure 3.2 shows $\mathcal{C}$ in (ϕ, t) coordinates, a proper path γ that crosses $\mathcal{G}$ in only one point, and the regular and singular lifts δ_r and δ_s of γ under f.

If η is not a full proper path, but merely a proper path in $\mathcal{C}_-$ (connecting $\mathcal{B}$ to $\mathcal{G}$ without passing through $\partial\mathcal{C}_-$ in between) we will also call the lift of η under f_s^{-1} the "singular lift" of η.

3.3. **Structure of $\mathcal{R}$ on the cylinder.** The above properties of f immediately translate into the following properties of the renormalization operator $\mathcal{R}$:

(P1) *Symmetries*: As we have already mentioned in §2.7, the Basic Symmetry of the Ising model implies that R commutes with the involution $\iota : (z, t) \mapsto (z^{-1}, t)$. On the other hand, since $R = f \circ Q$, we have: $R \circ \rho = R$, where $\rho : (z, t) \mapsto (-z, t)$. It follows that the basins of the top and the bottom of $\mathcal{C}$ are invariant under the Klein group $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ comprising id and three involutions, ι, ρ and $\iota \circ \rho$. These symmetries are clearly visible on Figure 1.2 as it is[23] π-periodic and is invariant under reflections in the axes $\phi = 0, \pi$, $\phi = \pm\pi/2$.

(P2) $\mathcal{R}$ has *two points of indeterminacy*, $\alpha_\pm = (\pm\pi/2, 1) \in \mathcal{T}$. Each of them blows up onto the singular curve $\mathcal{G}$. (See Appendix B for detailed formulae.)

(P3) Formula (3.7) implies that the *critical locus* of $\mathcal{R}|\mathcal{C}$ comprises the bottom $\mathcal{B}$, the top $\mathcal{T}$, and two vertical intervals, $\mathcal{I}_{\pm\pi/2}$, terminating at the points of indeterminacy. These intervals collapse under $\mathcal{R}$ to the fixed point $\beta_0 \in \mathcal{B}$. $\mathcal{R}$ is an orientation preserving local diffeomorphism on the complement of the critical set, $\mathcal{C} \setminus \left(\mathcal{I}_{\pm\pi/2} \cup \mathcal{T} \cup \mathcal{B}\right)$.

(P4) $\mathcal{R}$ is *postcritically finite* in the following sense: $\mathcal{C} \setminus \left(\mathcal{I}_{\pm\pi/2} \cup \mathcal{T} \cup \mathcal{B}\right)$ is backward invariant, and $\mathcal{R}$ is a local diffeomorphism on this set. (Note: while postcritically finite maps typically have rather simple dynamics, $\mathcal{R}$ is not *postsingularly finite*, since images of the curve $\mathcal{G}$ are not eventually periodic, leading to dynamical complexity of $\mathcal{R}$.)

(P5) $\mathcal{R}|\,\mathcal{B} : z \mapsto z^4$, while $\mathcal{R}|\,\mathcal{T} : z \mapsto z^2$. Moreover, the bottom circle is *uniformly superattracting*, namely, letting $\mathcal{R}(z, t) = (z', t')$, we have: $t' = O(t^2)$ for t near 0. The top circle is *non-uniformly superattracting*, namely, near the top we have

$$\tau' = O\left(\frac{\tau^2}{\cos^2 \phi}\right) = O\left(\frac{\tau^2}{\epsilon^2}\right), \quad \tau = 1 - t,\ \epsilon = \pi/2 - \phi \mod \pi\mathbb{Z},$$

[23] or rather, its 2π-periodic unfolding

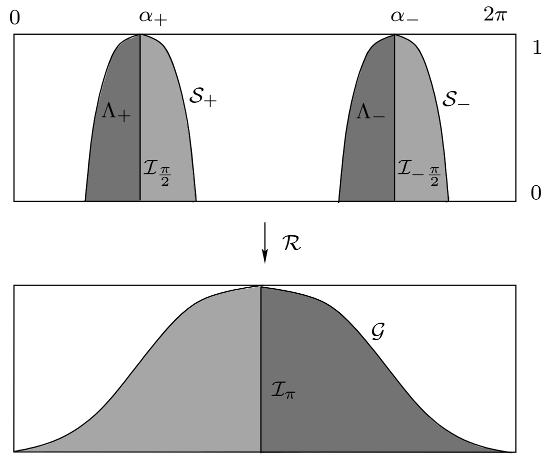


FIGURE 3.3. The mapping $\mathcal{R} : \mathcal{C} \to \mathcal{C}$, the regions $\Lambda_\pm$ in grey, and the region $\Pi = \mathcal{C} \smallsetminus \Lambda_\pm$ in white (above).

so that, the superattraction rate explodes near the points of indeterminacy. See Figure 1.2 for a computer image of the basins of attraction for $\mathcal{B}$ and $\mathcal{T}$.

(P6) The preimage $Q^{-1}(\mathcal{S})$ comprises two curves $\mathcal{S}_\pm$ that are tangent to $\mathcal{T}$ at the indeterminacy points $\alpha_\pm$ and are symmetric with respect to $\mathcal{I}_\pm$ respectively. The domains below them are called the *(primary) central tongues* $\Lambda_\pm$, see Figure 3.3. The vertical intervals $\mathcal{I}_{\pm\pi/2}$ cut the corresponding tongues $\Lambda_\pm$ into two topological triangles. Each of these (open) triangles is mapped diffeomorphically by $\mathcal{R}$ onto the appropriate triangle of $\mathcal{C}_- \smallsetminus \mathcal{I}_\pi$. The inverse diffeomorphisms are called *singular branches* of $\mathcal{R}^{-1}$.

(P7) The complement $\Pi := \mathcal{C} \smallsetminus \Lambda_+ \cup \Lambda_-$ consists of two domains each of which is mapped diffeomorphically onto the cut rectangle $\mathcal{C} \smallsetminus \mathcal{I}_\pi$. The inverse diffeomorphisms are called *regular branches* of $\mathcal{R}^{-1}$.

(P8) $\mathcal{R}$ has degree 4 over $\mathcal{C}_-$ and it has degree 2 over $\mathcal{C}_+$.

(P9) By Lemma 3.2, every proper path γ in $\mathcal{C}$ has at least 4 proper lifts δ_i. These lifts can meet only at the indeterminacy points $\alpha_\pm$. If γ crosses $\mathcal{G}$ at a single point, then $\mathcal{R}^{-1}\gamma = \cup\,\delta_i$. Two of these lifts (contained in $\Lambda_\pm$) are "singular": they terminate at the points $\alpha_\pm$; the other two are "regular".

(P10) $\mathcal{R}$ acts with *degree 4* on closed curves: If γ is any closed curve on $\mathcal{C}$ wrapping once around $\mathcal{C}$, then $\mathcal{R}(\gamma)$ is a closed curve wrapping four times around $\mathcal{C}$.

Proposition 3.3. *For any $n \geq 0$ the Lee-Yang locus at level n is given by*

$$\mathcal{S}_n = (\mathcal{R}^n)^* \mathcal{S}_0. \tag{3.10}$$

Proof. Let $S_0 = \{U + 2V + W = 0\} \cap C$ and note that $\mathcal{S}_0 = \Psi^* S_0$. It follows from (2.19) that $\mathcal{S}_n = (R^n \circ \Psi)^* S_0$. On $\mathcal{C}_1$ we obtain (3.10) from the semi-conjugacy $R^n \circ \Psi = \Psi \circ \mathcal{R}^n$. The result extends by continuity to all of $\mathcal{C}$. □

3.4. **Structure of R on the Möbius band.** Because of the conjugacy $\Psi : \mathcal{C}_0 \to C_0$ from Proposition 3.1, the structural properties (P1-P10) for $\mathcal{R} : \mathcal{C} \to \mathcal{C}$ discussed in §3.3 have immediate analogs for $R : C \to C$. Particularly important is

(P9′) Every proper path γ in C_0 lifts under R to at least 4 proper paths in C_0. If γ crosses G at a single point, then $R^{-1}\gamma = \cup\, \delta_i$.

(Here a path in C_0 is called *proper* if it goes from $\mathbb{T}$ to ∞).

Also, we have:

- The principal LY locus $\mathcal{S}$ in $\mathcal{C}$ (see (3.8)) is turned into the the vertical line $S = \{\mathrm{Re}\, u = -1\}$ in C. (This is seen directly from the formula (2.17) for the partition function). We will refer to S as the *principal LY locus* in the affine coordinates.
- The indeterminacy points $\alpha_\pm \in \mathcal{T}$ for $\mathcal{R}$ are turned into indeterminacy point $\pm i \in \mathbb{T}$ for R.
- The blow-up locus $\mathcal{G}$ (3.5) is turned into the parabola

$$G = \{u :\ |u| = \mathrm{Re}\, u + 2\} = \{x + iy :\ x = \frac{1}{4}y^2 - 1\} \tag{3.11}$$

(use $u^{-1} = te^{i\phi}$). See Figure 6.3.

Let us rotate the LY locus S around the circle $\mathbb{T}$. We obtain a family of lines $S_\phi = e^{i(\phi-\pi)}S$ tangent to $\mathbb{T}$ at $e^{i\phi}$. Let $S^c_\phi = \{e^{-i(\phi-\pi)}U + 2V + e^{i(\phi-\pi)}W = 0\}$ be the corresponding complex line in $\mathbb{CP}^2$. By Corollary 4.5, the pullback $R^*(S^c_\phi)$ is a complex algebraic curve of degree 4. By Bezout's Theorem, it intersects the conic $L_1 = \{uw = 1\}$ (which is the complexification of the circle $\mathbb{T}$) at 8 points counted with multiplicity.

Lemma 3.4. (i) *If $\phi \neq \pi$ (i.e., $S_\phi \neq S$), then the conic L_1 intersects the pullback $R^*(S^c_\phi)$ transversally at four transverse double points ($\pm e^{\phi/2}$ and the indeterminacy points $\pm i$). So, each intersection has multiplicity 2.*

(ii) *If $\phi = \pi$ (i.e., $S_\phi = S$), then L_1 intersects $R^*(S^c_\phi)$ tangentially at two first order tangential double points (the indeterminacy points $\pm i$). So, each intersection has multiplicity 4.*

Proof. (i) By Lemma D.3, the map R is a Whitney fold at $\pm e^{i\phi/2}$. By Lemma D.4, the germ of $R^*(S^c_\phi)$ at $\pm e^{i\phi/2}$ is a transverse double point transversely intersected by the critical locus L_1.

To understand the germ of $R^*(S^c_\phi)$ at an indeterminacy point $a \in \{\pm i\}$, let us blow it up and lift R to a map $\tilde{R} : \tilde{\mathbb{CP}}^2 \to \mathbb{CP}^2$, see Appendix A.2. Since $\phi \neq \pi$, S^c_ϕ intersects the blow-up locus $G = \tilde{R}(\mathcal{E}_{\mathrm{exc}})$ transversely at two points. If $\phi \neq \pm\pi/2$, these two points are regular values for $\tilde{R}$ (Appendix D.1.2). Hence, the curve $\tilde{R}^*(S^c_\phi)$ intersects the exceptional divisor $\mathcal{E}_{\mathrm{exc}}$ transversely at two points. Projecting the corresponding germs to $\mathbb{CP}^2$, we obtain two branches of $R^*(S^c_\phi)$ at a.

If $\phi = \pm\pi/2$, then the intersection of S^c_ϕ with G at $b_0 \in L_0$ is not a regular value for $\tilde{R}$. According to Appendix B.1.1, the point $p \in \mathcal{E}_{\mathrm{exc}}$ at slope $\chi = -1$ satisfies $\tilde{R}(p) = b_0$. A simple calculation yields that $D\tilde{R}(p)$ is tangent to G at b_0 and therefore transverse to S^c_ϕ. Thus, $\tilde{R}^*(S^c_\phi)$ is smooth in a neighborhood of p, with the tangent direction given by $\mathrm{Ker} D\tilde{R}(p)$, which is transverse to $\mathcal{E}_{\mathrm{exc}}$. Thus, $\tilde{R}^*(S^c_\phi)$ still intersects the exceptional divisor $\mathcal{E}_{\mathrm{exc}}$ transversely at two points, and we proceed as before.

(ii) By Lemma D.3, the blow-up map $\tilde{R} : \tilde{L}_1 \to L_1$ is a Whitney fold at the intersection point $\tilde{a} = \tilde{L}_1 \cap \mathcal{E}_{\mathrm{exc}}$. Hence the germ of $\tilde{R}^*(S_\phi)$ has a transverse double point at $\tilde{a}$ and intersects $\mathcal{E}_{\mathrm{exc}}$ generically. Its projection to $\mathbb{CP}^2$ is a pair of regular curves tangent to L_1 at a (see Lemma D.5). $\square$

Remark 3.2. The above lemma is reflected in the geometry of the initial LY loci illustrated on Figure 1.3. The locus $\mathcal{S}_1 = \mathcal{R}^{-1}\mathcal{S}_0$ looks like two tangent parabolas near the indeterminacy points $\alpha_\pm$ (part (ii) of the lemma). The next locus, $\mathcal{S}_2 = \mathcal{R}^{-1}\mathcal{S}_1$, comprises 32 branches meeting transversely at the top, as part (i) asserts.

4. Structure of the RG transformation II: Global properties in $\mathbb{CP}^2$

The Lee-Yang Theorem places special emphasis of the dynamics of $\mathcal{R}$ on the cylinder $\mathcal{C}$. However, it is instructive to understand the global dynamics of $\mathcal{R}$ on the projective space $\mathbb{CP}^2$, which has important consequences for the dynamics of $\mathcal{R}|\mathcal{C}$. In this section we will describe basic global properties of $\mathcal{R}$, along with those of $R : \mathbb{CP}^2 \to \mathbb{CP}^2$.

4.1. **Semiconjugacy Ψ.** The mapping Ψ given by (2.21) is a degree two rational map $\mathbb{CP}^2 \to \mathbb{CP}^2$. In homogeneous coordinates $[Z : T : Y]$ in the domain (with $z = Z/Y,\ t = T/Y$) and $[U : V : W]$ in the image (with $u = U/V,\ w = W/V$), it assumes the form

$$U = Y^2, \quad V = ZT, \quad W = Z^2.$$

A generic point $[U : V : W]$ has two preimages under Ψ. The critical locus of Ψ is the union of the vertical axis $\{Z = 0\}$ and the line at infinity $\{Y = 0\}$. Under Ψ, the former collapses to an R-fixed point $e = [1 : 0 : 0]$, while the latter maps onto the vertical axis $\{U = 0\}$. Since e does not lie on this axis, the intersection $\gamma = [0 : 1 : 0]$ of the two critical lines must be an indeterminacy point for Ψ (and in fact, this is the only one).

This collapsing line $\{Z = 0\}$ and the associated indeterminacy point γ created by the change of variable Ψ is what makes the physical coordinates (z, t) less suitable for describing the global structure of the renormalization.

4.2. **Indeterminacy points for $\mathcal{R}$ and R.** In homogeneous coordinates on $\mathbb{CP}^2$, the map R has the form:

$$R : [U : V : W] \mapsto [(U^2 + V^2)^2 :\ V^2(U + W)^2 :\ (V^2 + W^2)^2)] \tag{4.1}$$

which is just (2.18) with (U, V, W) interpreted as the homogeneous coordinates. We find two points of indeterminacy: $a_+ := [i : 1 : -i]$ and $a_- := [-i : 1 : i]$. They lie on the Möbius band C and correspond under $\Psi|\mathcal{C}$ to the indeterminate points $\alpha_\pm \in \mathcal{C}$ that we discussed in §3.1.

If we now write $\mathcal{R}$ in homogeneous coordinates we obtain

$$\begin{aligned} &\mathcal{R} : [Z : T : Y] \mapsto \\ &[Z^2(Z^2 + T^2)^2 : T^2(Z^2 + Y^2)^2 : (Z^2 + T^2)(T^2 Z^2 + Y^4)]. \end{aligned} \tag{4.2}$$

We find the indeterminate points $\alpha_\pm = (\pm i, 1) \in \mathcal{C}$, two symmetric points $(\pm i, -1)$, and two additional points of indeterminacy, $\mathbf{0} = (0, 0)$ and $\gamma = [0 : 1 : 0]$ (here all the points except the last one are written in the physical coordinates $z = Z/Y$, $t = T/Y$). In this way, when we turn R into $\mathcal{R}$ by the change of variable Ψ we create two accidental points of indeterminacy, $\mathbf{0}$ and γ, which makes the global properties of the map more awkward.

One can resolve all the indeterminacies of $\mathcal{R}$ using suitable blow-ups. We will only need resolutions of $\alpha_\pm$, which are described in Appendix B.

4.3. **Superattracting fixed points and their separatrices.**

4.3.1. *Description in terms of R (4.1).* We will often refer to $L_0 := \{V = 0\} \subset \mathbb{CP}^2$ as the *line at infinity.* It contains two symmetric fixed points, $e = (1 : 0 : 0)$ and $e' = (0 : 0 : 1)$. In local coordinates ($\xi = W/U$, $\eta = V/U$) near e, the map R assumes form

$$\xi' = \left(\frac{\xi^2+\eta^2}{1+\eta^2}\right)^2 \sim (\xi^2+\eta^2)^2, \quad \eta' = \eta^2\left(\frac{1+\xi}{1+\eta^2}\right)^2 \sim \eta^2, \tag{4.3}$$

so $|Rx| \le 2|x|^2$ for small $x = (\xi, \eta)$. This shows that e is superattracting:

$$|R^n x| \le |2x|^{2^n}.$$

By symmetry, e' is superattracting as well. Let $\mathcal{W}^s(e)$ and $\mathcal{W}^s(e')$ stand for the attracting basins of these points.

Moreover, the line at infinity $L_0 = \{\eta = 0\}$ is R-invariant, and the restriction $R|L_0$ is the power map $\xi \mapsto \xi^4$. Thus, points in the disk $\{|\xi| < 1\}$ in L_0 are attracted to e, points in the disk $\{|\xi| > 1\}$ are attracted to e', and these two basins are separated by the unit circle B. We will also call L_0 the *fast separatrix* of e and e'.

Let us also consider the conic

$$L_1 = \{\xi = \eta^2\} = \{V^2 = UW\} \tag{4.4}$$

passing through points e and e'. It is an embedded $\mathbb{CP}^1$ that can be uniformized by coordinate $w = W/V = \xi/\eta$. Formulas (4.3) show that L_1 is R-invariant, and the restriction $R|\,L_1$ is the quadratic map $w \mapsto w^2$. Thus, points in the disk $\{|w| < 1\}$ in L_1 are attracted to e, points in the disk $\{|w| > 1\}$ are attracted to e', and these two basins are separated by the unit circle T. We will call L_1 the *slow separatrix* of e and e'.

If a point x near e (resp. e') does not belong to the fast separatrix L_0, then its orbit is "pulled" towards the slow separatrix L_1 at rate ρ^{4^n}, with some $\rho < 1$, and converges to e (resp. e') along L_1 at rate r^{2^n}, with some $r < 1$.

The second formula of (4.3) also shows that the strong separatrix L_0 is transversally superattracting: all nearby points are pulled towards L_0 uniformly at rate r^{2^n} (see also the proof of Lemma D.3). It follows that these points either converge to one of the fixed points, e or e', or converge to the circle B.

Given a neighborhood Ω of B, let

$$\mathcal{W}^s_{\mathbb{C},\mathrm{loc}}(\mathrm{B}) = \{x \in \mathbb{CP}^2 : \ R^n x \in \Omega \ (n \in \mathbb{N}) \text{ and } R^n x \to \mathrm{B} \text{ as } n \to \infty\} \tag{4.5}$$

(where Ω is implicit in the notation, and an assertion involving $\mathcal{W}^s_{\mathbb{C},\mathrm{loc}}$ means that it holds for arbitrary small suitable neighborhoods of B).

We conclude:

Lemma 4.1. $\mathcal{W}^s(e) \cup \mathcal{W}^s(e') \cup \mathcal{W}^s_{\mathbb{C},\mathrm{loc}}(\mathrm{B})$ *fills in some neighborhood of* L_0.

As the weak separatrix L_1 is concerned, formula (D.1) from the proof of Lemma D.3 shows that it is transversally superattracting away from the indeterminacy points $a_\pm$. On the other hand, the latter act as strong repellers. We will see in §10 that this competition makes T a non-uniformly hyperbolic attractor.

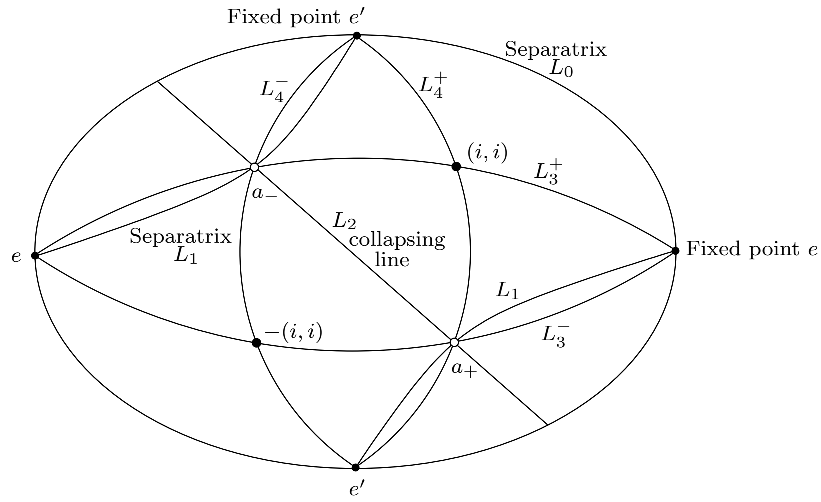


FIGURE 4.1. Critical locus for R shown with the separatrix L_0 at infinity.

4.3.2. *Description in terms of* $\mathcal{R}$. In the physical coordinates, the superattracting fixed points become $\eta = (0,1)$ and $\eta' = [1 : 0 : 0]$. The pullback of the line at infinity L_0 under the semi-conjugacy Ψ comprises two lines, $\mathcal{L}_0 = \{t = 0\}$ and $\{z = 0\}$, where the latter is the blow-up of the fixed point e under Ψ^{-1}. These two lines form the fast separatrix of the fixed points (recall that the latter collapses to $\eta = (0,1)$ under $\mathcal{R}$), which is an annoying artifact of the physical coordinates. A related nuisance is that $\mathcal{L}_0$, unlike L_0, is not transversally superattracting any more. Namely, it is superattracting away from the origin $\mathbf{0}$, but the latter blows up to the whole line $\{z = 0\}$. Still, we will sometimes refer to $\mathcal{L}_0$ itself as the "fast separatrix", as long as it does not lead to confusion.

The slow separatrix of the fixed points is the line $\{t = 1\}$.

The restrictions of $\mathcal{R}$ to the separatrices $\mathcal{L}_0$ and $\mathcal{L}_1$ become the power maps $z \mapsto z^4$ and $z \mapsto z^2$ respectively. The invariant circles on these lines (separating the basins of the fixed points) become $\mathcal{B} = \mathbb{T} \times \{0\}$ and $\mathcal{T} = \mathbb{T} \times \{1\}$, which are the bottom and the top of the physical cylinder $\mathcal{C}$ that we discussed in §3.

Lemma 4.1 implies:

Lemma 4.2. $\mathcal{W}^s(\eta) \cup \mathcal{W}^s(\eta') \cup \mathcal{W}^s_{\mathbb{C},\mathrm{loc}}(\mathcal{B})$ *fills in some neighborhood of* $\mathcal{L}_0 \smallsetminus \{\mathbf{0}\}$.

4.4. **Critical locus.** The critical locus of R is described in Appendix, §D. Besides the separatrices L_0 and L_1, it comprises the line L_2 that collapses to the low temperature fixed point b_0, and two symmetric pairs of lines, L_3 and L_4. The latter wander under the dynamics.

The critical locus is schematically depicted on Figure 4.1, while its image, the critical value locus, is depicted on Figure 4.2.

In terms of the physical coordinates, the critical locus comprises:

- $\Psi^{-1}L_0$: the fast separatrix $\mathcal{L}_0 \cup \{z = 0\}$;
- $\Psi^{-1}L_1$: the slow separatrix $\mathcal{L}_1$ and its companion $\{t = -1\}$ (mapped to $\mathcal{L}_1$ under $\mathcal{R}$);
- $\Psi^{-1}L_2$: two collapsing lines, $z = \pm i$;

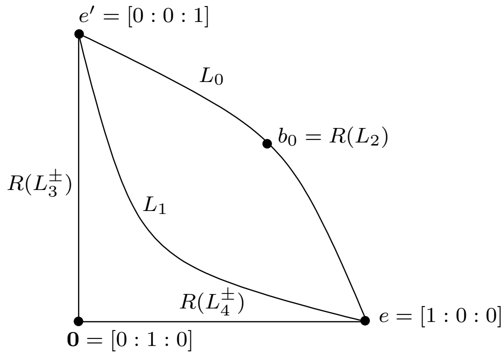


FIGURE 4.2. Critical values locus of R.

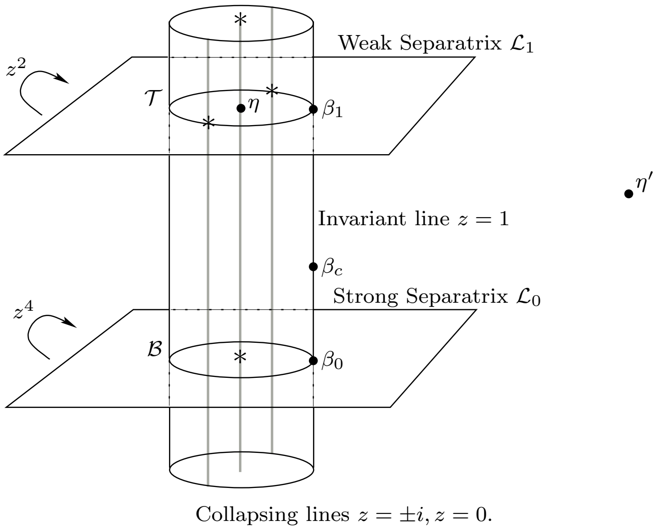


FIGURE 4.3. The LY cylinder $\mathcal{C}$ situated between the strong separatrix $\mathcal{L}_0$ and the weak separatrix $\mathcal{L}_1$. The collapsing lines at $z = 0, \pm i$ are shown in grey and the indeterminate points $\alpha_\pm, \mathbf{0}, \gamma$ are depicted by stars. The superattracting fixed point $\eta' = (0, \infty)$ is "symbolically" shown at a finite location.

- $\Psi^{-1}L_3$: two conics $zt = \pm i$;
- $\Psi^{-1}L_4$: two lines $z = \pm it$ (symmetric to the above conics).

4.5. **Topological degree.** The *topological degree* $\deg_{top}(f)$ of a rational mapping $f : \mathbb{P}^k \to \mathbb{P}^k$ is the number of preimages under f of a generic point $\zeta \in \mathbb{P}^k$.

Proposition 4.3. *We have* $\deg_{\mathrm{top}}(R) = \deg_{\mathrm{top}}(\mathcal{R}) = 8$.

Proof. This can be seen for R by taking, e.g., a point $\zeta = (u, v) \in \mathbb{C}^2$ far away, and hence close to the separatrix L_0. Such a point has 8 preimages under R, since the transverse degree of R at L_0 is equal to 2, while $\deg(R|\, L_0) = 4$.

Similarly for $\mathcal{R}$, consider a generic point ζ sufficiently close to, but not on, the separatrix $\mathcal{L}_0$. $\square$

4.6. **Algebraic degrees and pullbacks of curves.** The reader can consult Appendix A.3 for needed background in elementary algebraic geometry.

4.6.1. *Case of* R. Since R is given in homogeneous coordinates by relatively prime equations of degree 4, we have $\deg R = 4$.

The notion of algebraic stability is essential to understanding pullbacks of curves (considered as divisors) under iterates of rational mappings, see Appendix A.3.

Proposition 4.4. *The mapping* $R : \mathbb{CP}^2 \to \mathbb{CP}^2$ *is algebraically stable.*

Proof. The only collapsing curve is L_2, whose orbit lands on the low-temperature fixed point b_0. $\square$

It follows that $\deg R^n = (\deg R)^n = 4^n$, and hence we have:

Corollary 4.5. *If* D *is an algebraic curve of degree* d*, then the pullback* $(R^n)^*D$ *is a divisor of degree* $d \cdot 4^n$*.*

4.6.2. *Case of* $\mathcal{R}$. Since $\mathcal{R}$ is given in homogeneous coordinates by relatively prime equations of degree 6, we have $\deg \mathcal{R} = 6$. In particular, for any algebraic curve X we have $\mathcal{R}^*X$ is a divisor of degree $6 \cdot \deg X$. The degrees of pullbacks under iterates of $\mathcal{R}$ are less organized:

Observation 4.6. *The mapping* $\mathcal{R} : \mathbb{CP}^2 \to \mathbb{CP}^2$ *is not algebraically stable.*

Indeed, $\mathcal{R}$ maps the lines $Z = \pm iT$ to the point of indeterminacy $\gamma = (0 : 1 : 0)$ since the first and third coordinates of (4.2) contain the factor $(Z^2 + T^2)$.

Remark 4.1. In this case, algebraic instability results in a drop of degree for the second iterate of $\mathcal{R}$. We have $\deg(\mathcal{R}^2) = 28 < 36 = (\deg\mathcal{R})^2$, since the common factor of $(Z^2 + T^2)^4$ appears in the expression for $\mathcal{R}^2$, which must be canceled (compare Remark A.1).

Remark 4.2. The relationship between dynamical degrees of semi-conjugate rational maps was studied by Dinh-Nguyên [DN]. If the semi-conjugacy is generically finite (like Ψ) their work implies that the two mappings have equal dynamical degrees in each codimension. In particular,

$$\delta(\mathcal{R}) := \lim_{n\to\infty} (\deg \mathcal{R}^n)^{1/n} = \lim_{n\to\infty} (\deg R^n)^{1/n} = \deg R = 4.$$

It also gives an alternative proof that $\deg_{\mathrm{top}}(\mathcal{R}) = \deg_{\mathrm{top}}(R)$, which we saw in Prop. 4.3.

Remark 4.3. Let $S^c = \{U + 2V + W = 0\} \subset \mathbb{CP}^2$. By (2.19), the Lee-Yang-Fisher locus of level n is given by $\mathcal{S}_n^c := \{Z_n(z,t) = 0\} = \Psi^{-1}(R^{-n}S^c)$, which has degree $2 \cdot 4^n$. Meanwhile, $\mathcal{R}^{-n}(\mathcal{S}^c) = R^{-n}(\Psi^{-1}S^c)$ contains, besides $\mathcal{S}_n^c$, some "junk" components that collapse to the indeterminacy point γ under some iterate $\mathcal{R}^k$, $k = 0, 1, \ldots, n-1$. So, the commutative diagram (3.1) should be applied with caution. (This was not a danger for the Lee-Yang zeros $\mathcal{S}_n \subset \mathcal{C}$, as explained in Proposition 3.3.)

5. Proof of the Lee-Yang Theorem for the DHL

In this section we will give an easy proof of the Lee-Yang Theorem for the DHL by means of "enumerative dynamics".

Theorem 5.1. *The locus $Z_n(z,t)=0$ intersects any complex line $\Pi_t := \mathbb{C}\times\{t\}$, $t\in[0,1)$, in $2\cdot 4^m$ distinct points on the unit circle $\mathbb{T}$.*

Proof. By (2.5), the partition function Z_n is a symmetric Laurent polynomial in z of degree 4^n, so it has $2\cdot 4^n$ zeros on every complex line in question. Meanwhile, Proposition 3.3 gives that $\mathcal{S}_n = (\mathcal{R}^n)^*\mathcal{S}_0$, so Property P9 supplies us with $2\cdot 4^n$ such zeros on the unit circle of Π_t. Hence they account for all of the zeros. $\square$

Let us formulate the corresponding statement in the Migdal coordinates. In these coordinates, the horizontal complex lines Π_t turn into the conics

$$P_t := \{uw = t^{-2}\}.$$

A complex line $L=\{au+bw+c=0\}$ is called *Hermitian* if it is invariant under the antiholomorphic involution $(u,w)\mapsto(\bar w,\bar u)$.[24] The slice of such a line by the real plane $\{w=\bar u\}$ is a real line (otherwise it would be a single point).

Theorem 5.2. *Let $t\in[0,1)$ and let L be any Hermitian complex line crossing the top $\mathcal{T}=\mathbb{T}$ of the cylinder C. Then the pullback $(R^n)^*L$ intersects the horizontal complex line P_t in $2\cdot 4^n$ simple points, all on the cylinder C.*

Proof. By Corollary 4.5, $(R^n)^*(L)$ has degree 4^n, so by Bezout Theorem, it has $2\cdot 4^n$ intersection points with the conic P_t. On the other hand, $L\cap C$ comprises two vertical intervals on the cylinder C. By property (P9) from §3.3, $(R^n)^*(L)\cap C$ comprises at least 2×4^n vertical curves on C (connecting the top $\mathrm{T}=\mathbb{T}$ to the bottom B at infinity). They have at least $2\cdot 4^n$ different intersections with the circle $\{|u|=t^{-1}\}=P_t\cap C$. Hence all the intersection points of $(R^n)^*(L)$ with P_t are captured on the cylinder C, and all of them are simple. $\square$

6. Algebraic cone field

In this section we will construct a horizontal invariant cone field on the cylinder. It appears as the tangent cone field to a pair of transverse algebraic foliations obtained by translating the principal Lee-Yang locus around the cylinder or the Möbius band. In the affine coordinates on the Möbius band, these foliations assume a particular simple linear form.

6.1. **Algebraic cone fields.** Let us consider the Möbius band C introduced in §3.1,

$$\operatorname{int} C = \{(u,w)\in\mathbb{C}^2 : u=\overline{w} \text{ and } |u|>1\} \approx \mathbb{C}\smallsetminus\bar{\mathbb{D}}.$$

Recall that S_ψ stands for the line tangent to $\mathbb{T}$ at $e^{i\psi}$. We define an algebraic horizontal cone field $K^{ah}(u)$ on $\operatorname{int} C$ as follows. For any $u\in\mathbb{C}\smallsetminus\bar{\mathbb{D}}$, there are two tangent lines S_{ψ_1} and S_{ψ_2} passing through u. Then, $K^{ah}(u)$ is the open cone[25] bounded by S_{ψ_1} and S_{ψ_2} that does not contain $\mathbb{T}$ (see Figure 6.1).

This construction can be described in terms of the principal LY locus $S=\{\operatorname{Re} u=-1\}$ on C (see §3.4). The line S is a concatenation of two rays, $S^+ =$

[24]Equivalently, $b=\bar a$, $c\in\mathbb{R}\smallsetminus\{0\}$ or $|b|=|a|$, $c=0$.

[25]Here a "cone" comprises two symmetric wedges. Also, more precisely one should think of $K^{ah}(u)$ as the *tangent* cone at u.

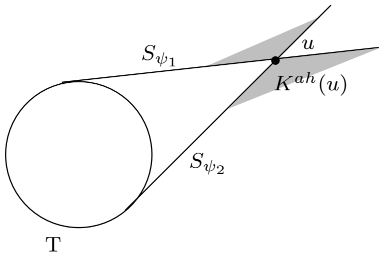


FIGURE 6.1. An algebraic horizontal cone $K^{ah}(u)$.

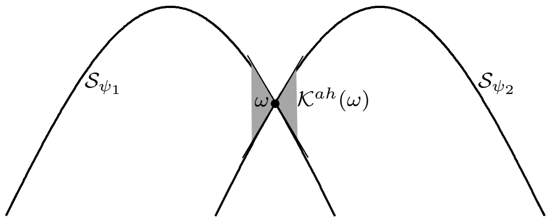


FIGURE 6.2. A horizontal cone $\mathcal{K}^{ah}(\omega)$.

$\{u \in S : \operatorname{Im} u > 0\}$ and $S^- = \{u \in S : \operatorname{Im} u < 0\}$, meeting at $-1 \in \mathbb{T}$. Rotating these rays around the origin, we obtain two linear foliations $\Phi^\pm$ of $\operatorname{int} C$ (comprised of the leaves $S^\pm_\theta = e^{i\theta} S^\pm$). The cone $K^{ah}(u)$ is bounded by (the tangent lines to) the leaves of these foliations passing through u.

Described in this way, the construction can be immediately transferred to the cylinder $\mathcal{C}$. Rotating the principal LY locus $\mathcal{S}$ around the cylinder, we obtain two transverse foliations of $\operatorname{int} \mathcal{C}$. Then the horizontal cone field is formed by the tangent cones $\mathcal{K}^{ah}(\omega)$ bounded by the tangent lines to the two leaves meeting at ω (see Figure 6.2). Clearly $\mathcal{K}^{ah}(\omega) = D\Psi^{-1}(K^{ah}(u))$, where $u = \Psi(\omega)$.

Remark 6.1. In fact, this cone field extends to the bottom $\mathcal{B}$ of $\mathcal{C}$, so it is well defined on the topless cylinder $\mathcal{C}_1$. However, it degenerates to a line field at the top.

A smooth path $\gamma(t)$ in C is called *horizontal* if it goes through the cones $K^{ah}(x)$, i.e. $\gamma'(t) \in K^{ah}(\gamma(t))$ whenever $\gamma(t) \in \operatorname{int} C$. (The same definition applies to the cylinder $\mathcal{C}$.)

Lemma 6.1. *The blow-up locus G (respectively $\mathcal{G}$) is horizontal.*

Proof. See Figure 6.3. □

Let us now define the *vertical cones* as the complements to the horizontal ones, $K^{av}(u) = T_u C \smallsetminus K^{ah}(u)$. A smooth path $\gamma(s)$ is called *vertical* if it goes through the vertical cones, i.e., $\gamma'(s) \in K^{av}(x)$. (The same definitions apply to the cylinder $\mathcal{C}$.) We call a path *strictly vertical* if it goes through the $\operatorname{int} K^{av}(u)$.

Lemma 6.2. *If γ is a smooth proper vertical path in C, then $R^{-1}\gamma$ comprises four proper paths (and similarly, for proper paths in $\mathcal{C}$).*

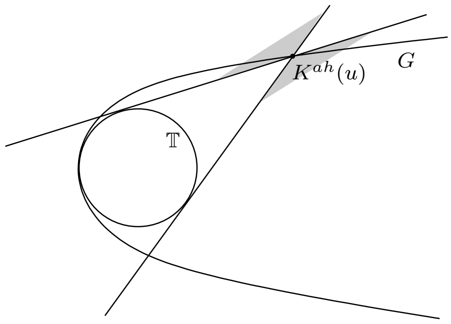


FIGURE 6.3. The blow-up locus G is horizontal.

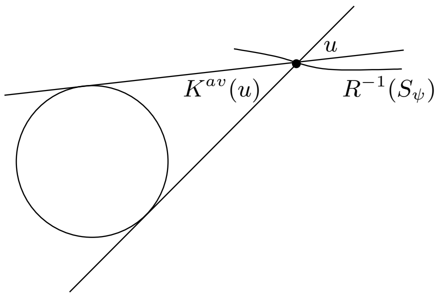


FIGURE 6.4. Illustration to the proof of invariance of the cone field.

Proof. Obviously, vertical paths are transverse to horizontal ones – so, by Lemma 6.1, they are transverse to the blow-up locus G. Combined with Property (P9′), this yields the assertion. □

Given a cone K in a linear space E, let $PK \subset PE$ stand for its projectivization. Let us say that a cone field $K(u)$ is *strictly forward invariant* if

$$DR(PK(u)) \Subset \operatorname{int} PK(Ru).$$

Proposition 6.3. *The horizontal cone fields $K^{ah}(u)$ and $\mathcal{K}^{ah}(u)$ are strictly forward invariant under the corresponding dynamics, R and $\mathcal{R}$.*

Proof. Equivalently, the vertical cone field $K^{av}(u)$ is strictly backward invariant. Since the cones are tangent to the pair of foliations $\Phi^\pm$, this is equivalent to the property that the pullbacks $R^{-1}(S_\psi^\pm)$ of the $\Phi^\pm$-leaves are strictly vertical.

By Lemma 6.2, each pullback $R^{-1}(S_\psi^\pm)$ comprises four disjoint proper paths in C. As the line S_ψ is the concatenation of two rays $S_\psi^\pm$, the pullback $R^{-1}(S_\psi)$ comprises eight disjoint proper paths $\gamma_i = \gamma_{i,\psi}$ in $\operatorname{int} C$.

To prove the desired, it suffices to show that for any angles ψ and θ, each path $\gamma_{i,\psi}$ has at most one intersection point with any line S_θ, and the intersection is transverse. In so, γ_i could not cross S_θ through the horizontal cone $K^{ah}(u)$, for it would be disjoint from the whole closed vertical cone $\operatorname{cl} K^{av}(u)$ (viewed as a subset of $\mathbb{C}$). But the latter contains $\mathbb{T}$, so γ_i would fail to land on $\mathbb{T}$ (see Figure 6.4).

Let S_ψ^c be the complexification of S_ψ. Since $R^*(S_\psi^c)$ is a complex algebraic curve of degree 4 (by Corollary 4.5), its slice by the complex line S_θ^c consists of 4 points counted with multiplicity. We will show that the intersection points that lie in $\operatorname{int} C$ are transverse and belong to distinct radial components of $R^{-1}(S_\psi)$. Let us consider several cases.

Case 1 (generic). Let $\theta \neq \psi/2, \pi/2 \mod \pi$. Then S_θ does not meet $R^{-1}(S_\psi)$ on $\mathbb{T}$. Since R is even, both pullbacks $R^{-1}(S_\psi^\pm)$ are symmetric with respect to the origin. Hence the rays comprising $R^{-1}(S_\psi^\pm)$ come in symmetric pairs γ_i, γ_i', $i = 1, \dots, 4$. We will show that one ray from each symmetric pair intersects S_θ somewhere on C_1.

Consider one such symmetric pair γ_i, γ_i'. If one (and hence both) of the rays meets S_θ at infinity (on B), then we're done. Otherwise, one ray from the pair (say γ_i) near infinity is separated by S_θ from $\mathbb{T}$. Then, since γ_i lands on $\mathbb{T}$, it must intersect S_θ.

Since the γ_i are pairwise disjoint, this gives us 4 distinct intersection points of $R^{-1}(S_\psi)$ with S_θ, Since the total number of intersection points counted with multiplicity is at most 4, we have accounted for all of them. Thus, each γ_i intersects S_θ exactly once and the intersection is transverse (while the γ_i' are disjoint from S_θ).

Case 2. Let $\theta = \psi/2$ or $\pi/2 \mod \pi$, but $\psi \neq \pi \mod 2\pi$. By Lemma 3.4 (i), the algebraic curve $R^*(S_\psi^c)$ has a double point at $e^{i\theta}$, and S_θ intersects it non-tangentially with multiplicity 2. It must intersect two other branches γ_i in C_1, and thus we have accounted for all four intersection points. The conclusion follows.

Case 3. Finally, let $\theta = \psi/2 = \pi/2 \mod \pi$ (this is the most degenerate case, but it occurs exactly when $S_\psi = S$ is the principal LY locus, see Figure 1.3). In this case, $e^{i\theta} = e^{\pm i\pi/2}$ is one of the two indeterminate points $a_\pm$ and all four branches γ_i meet at $e^{i\theta}$. Then, S_θ^c intersects $R^*(S_\psi^c)$ at this point with multiplicity 4 (as described Lemma 3.4 (ii)). It accounts for all intersection points, so no intersections occur in $\operatorname{int} C$. □

Remark 6.2. Eric Bedford has informed us that a similar algebraic method for constructing an invariant cone field (for certain birational maps) had been earlier used in [BD, §5].

Corollary 6.2 and Proposition 6.3 imply:

Corollary 6.4. *If γ is a proper vertical path in C, then $R^{-1}\gamma$ comprises four proper strictly vertical paths (and similarly, in $\mathcal{C}$).*

6.2. **Modified algebraic cone field.** The algebraic cone field we have just constructed has a disadvantage that it degenerates near the top. We will now modify it near the top so that it will become non-degenerate everywhere away from the indeterminacy points $\alpha_\pm$.

Recall the local coordinates $\tau = 1 - t$ and $\epsilon = \pm\frac{\pi}{2} - \phi$ around $\alpha_\pm$ that were introduced in §3.3 (P5). Given a small threshold $\bar\tau > 0$, let us consider the following annular neighborhood of the top:

$$\mathcal{V} \equiv \mathcal{V}_{\bar\tau} = \{x \in \mathcal{C} : \ \tau \leq \bar\tau\}. \tag{6.1}$$

Given $\eta > 0$, let us consider two parabolas $\mathcal{Y}_\eta^\pm = \{\tau = \eta\epsilon^2\}$ centered at the indeterminacy points $\alpha_\pm$.

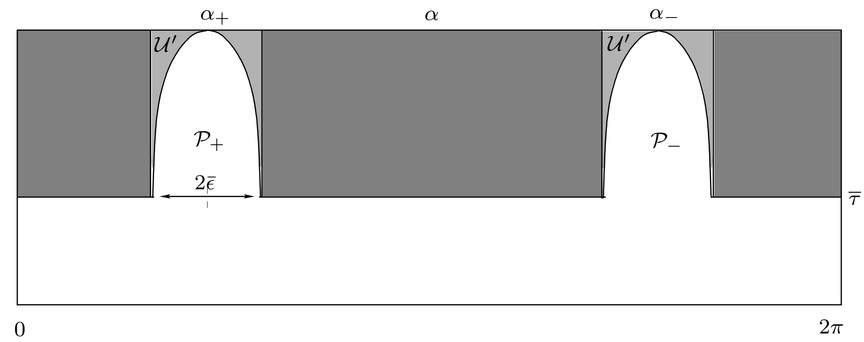


FIGURE 6.5. $\mathcal{V}'$ is the union of all shaded regions. Note that the figure is not to scale.

Consider two parabolic regions $\mathcal{P}^{\pm}_{\eta}$ below the curves $\mathcal{Y}^{\pm}_{\eta}$ (see Figure 6.5), and let

$$\mathcal{V}' \equiv \mathcal{V}'_{\eta,\bar{\tau}} = \mathcal{V}_{\bar{\tau}} \smallsetminus (\mathcal{P}^{+}_{\eta} \cup \mathcal{P}^{-}_{\eta}). \tag{6.2}$$

For a small threshold $\bar{\epsilon} > 0$, let us consider the following regions in $\mathcal{V}$:

$$\mathcal{U} \equiv \mathcal{U}_{\bar{\epsilon}} = \{x \in \mathcal{V} : \ |\epsilon(x)| < \bar{\epsilon}\} \text{ and } \mathcal{U}' = \{x \in \mathcal{V}' : \ |\epsilon(x)| < \bar{\epsilon}\}.$$

For the remainder of the construction, we let $\bar{\tau} = \eta\,\bar{\epsilon}^2$ so that the regions $\mathcal{U}$ and $\mathcal{U}'$ meet the parabolas $\mathcal{Y}^{\pm}$ at their bottom, see Figure 6.5. To be definitive, we fix $\eta = 1/18$, so the only free parameter left in our disposal is $\bar{\epsilon}$.

Note that for $x \in \mathcal{Y}^{\pm}$, the slope of the lines that bound the algebraic cone $\mathcal{K}^{ah}(x)$ is equal (in absolute value) to

$$s^a(x) = \sqrt{\tau(2-\tau)} \sim \sqrt{2\tau} = |\epsilon|/3, \quad \text{and} \quad s^a(x) \le |\epsilon|/3. \tag{6.3}$$

Lemma 6.5. *For $\bar{\epsilon} > 0$ sufficiently small, we have:*
- (a) $\mathcal{R}(\mathcal{U}) \cap \mathcal{U} = \emptyset$*;*
- (b) $\mathcal{R}^{-1}(\mathcal{U})$ *is contained in a small neighborhood of the points* $(\phi = \pi k/4,\ \tau = 0) \in \mathcal{T}$ *with* $k = \pm 1, \pm 3$.

Proof. a) By blow-up formula (B.4), the image $\mathcal{R}(\mathcal{U})$ is contained in a small neighborhood of the singular curve $\mathcal{G}$, which is disjoint from $\mathcal{U}$.

b) By (a), there are no points of $\mathcal{U}$ in $\mathcal{R}^{-1}\mathcal{U}$. But in $\mathcal{C} \smallsetminus \mathcal{U}$, the map $\mathcal{R}$ is continuous, so $\mathcal{R}^{-1}(\mathcal{U})$ is localized near $\mathcal{R}^{-1}(\alpha_{\pm})$.

□

For $x \in \mathcal{C}$, let us define a continuous horizontal cone field $\mathcal{K}^h(x) \equiv \mathcal{K}^h_{\bar{\epsilon}}(x)$ whose boundary lines have slopes with the absolute value $s(x)$ such that:
- (o) $\mathcal{K}^h(x) \supset \mathcal{K}^{ah}(x)$ *everywhere*;
- (i) $\mathcal{K}^h(x) = \mathcal{K}^{ah}(x)$ in the *white* region $\mathcal{C} \smallsetminus \mathcal{V}'$ (see Figure 6.5);
- (ii) $s(x) \sim |\epsilon(x)|/3$ in the *grey* region $\mathcal{U}'$;
- (iii) $s(x) = s_0 := \sqrt{\bar{\tau}(2-\bar{\tau})} \sim \bar{\epsilon}/3$ in the *black* region $\mathcal{V} \smallsetminus \mathcal{U} = \mathcal{V}' \smallsetminus \mathcal{U}'$.

Conditions (o)–(ii) are compatible due to (6.3). For a cone field satisfying these three conditions, we have $s(x) = s_0$ on the horizontal boundary of the black region and $s(x) \sim \bar\epsilon/3$ on the vertical one. Hence it can be extended to the black region satisfying (iii).

Note that in $\mathcal{C} \smallsetminus \mathcal{V}$, we have $s^a(x) = \sqrt{\tau(2-\tau)} \ge s_0$. Hence

$$s(x) \ge s_0 \sim \bar\epsilon/3 \quad \text{in} \quad \mathcal{C} \smallsetminus \mathcal{U}. \tag{6.4}$$

Lemma 6.6. *For sufficiently small $\bar\epsilon > 0$, the cone field $\mathcal{K}^h$ is strictly forward invariant:*

$$D\mathcal{R}(\mathcal{K}^h(x)) \Subset \mathcal{K}^h(\mathcal{R}x). \tag{6.5}$$

Proof. For $x \in \mathcal{C} \smallsetminus \mathcal{V}'$, we make use of the invariance of the algebraic cone field and conditions (o) and (i):

$$D\mathcal{R}(\mathcal{K}^h(x)) = D\mathcal{R}(\mathcal{K}^{ah}(x)) \Subset \mathcal{K}^{ah}(\mathcal{R}x) \subset \mathcal{K}^h(\mathcal{R}x).$$

So, assume $x \in \mathcal{V}'$. The cones $\mathcal{K}^h(x)$ are bounded by two lines spanned by the tangent vectors $v_\pm = (1, \pm s(x))$. Let us estimate the absolute value s' of the slope of $D\mathcal{R}(v_\pm)$.

Select $\bar\epsilon$ so small that (B.8) applies in $\mathcal{U}$. For $x \in \mathcal{U}'$, it yields:

$$s' \le \frac{|\epsilon|\tau(\tau+\epsilon^2/3)}{\sigma^2(\tau+2\epsilon^2/3)} \le \frac{|\epsilon|\tau}{\sigma^2} < \left|\frac{\tau}{\epsilon}\right| < \bar\epsilon/18 < s_0.$$

By Lemma 6.5, $\mathcal{R}x \notin \mathcal{U}$, so property (6.4) ensures (6.5).

Let $x = (\phi, \tau) \in \mathcal{V}' \smallsetminus \mathcal{U}'$. Then for $\bar\epsilon$ sufficiently small, we have:

$$|\cos\phi| \ge \bar\epsilon/2, \quad |\operatorname{tg}\phi| < 4/(3\bar\epsilon), \quad \text{and} \quad \tau \le \bar\tau \le \bar\epsilon^2/18.$$

Letting $(a, b) = D\mathcal{R}(v_\pm)$, we obtain from (B.7):

$$a \ge 2 - 2\operatorname{tg}\phi \cdot \bar\epsilon/3 \ge 10/9 > 1 \tag{6.6}$$

and

$$|b| \le \frac{2\tau^2|\operatorname{tg}\phi|}{\cos^2\phi} + \frac{\tau\bar\epsilon}{3\cos^2\phi} \le 16\,\frac{\bar\tau^2}{\bar\epsilon^3} + 2\,\frac{\bar\tau}{\bar\epsilon} \le \left(\frac{16}{18^2} + \frac{2}{18}\right)\bar\epsilon < \frac{\bar\epsilon}{6}. \tag{6.7}$$

So, the slope $s' = |b/a| < s_0$ as well. Due to property (6.4), this implies (6.5) in case $\mathcal{R}x \notin \mathcal{U}$.

Finally, let $x \in \mathcal{V}' \smallsetminus \mathcal{U}'$ and $\mathcal{R}x = (\epsilon', \tau') \in \mathcal{U}$. Then Lemma 6.5 b) gives $|\cos\phi| > C^{-1} > 0$ and $|\operatorname{tg}\phi| < C$, independent of $\bar\epsilon$. Estimate (6.7) simplifies to

$$|s'| \le |b| = O(\tau^2 + \tau\bar\epsilon) = o(\tau) = o(\sqrt{\tau'}) \text{ as } \bar\epsilon \to 0.$$

But $s(\mathcal{R}x) \ge \sqrt{\tau'}$ as follows from condition (o). This concludes the proof. $\square$

The horizontal cone field $\mathcal{K}^h$ extends continuously to the indeterminacy points as degenerate cones $\mathcal{K}^h(\alpha_\pm) = \{d\tau = 0\}$. Obviously, this continuous extension is invariant.

We will also call an smooth path $\gamma(t)$ in $\mathcal{C}$ *horizontal* if at each point $\gamma(t) \in \operatorname{int}\mathcal{C}$ we have $\gamma'(t) \in \mathcal{K}^h(\gamma(t))$. In the remainder of the paper, all horizontal paths will considered with respect to respect to $\mathcal{K}^h$ (and not $\mathcal{K}^{ah}$), unless otherwise specified.

Remark 6.3. One obtains a pushed forward cone field $K^h := D\Psi\,\mathcal{K}^h$ on C that is invariant under R and non-degenerate away from $a_\pm$.

6.3. **Vertical cone fields and laminations.** In this section we develop a language adopted, for definiteness, to the modified cone field $\mathcal{K}^h$ on the cylinder $\mathcal{C}$, but a similar language, with obvious adjustments, can be applied to the algebraic cone field $\mathcal{K}^{ah}$, as well as the corresponding cone fields on the Möbius band C.

We let $\mathcal{K}^v(x) = T_x\mathcal{C} \smallsetminus \mathcal{K}^h(x)$ be the complementary vertical cone field. (In particular, $\mathcal{K}^v(\alpha_\pm) = \{d\phi \neq 0\}$ is the complement to the horizontal line.)

Let $\mathcal{K}(x) \subset \mathcal{K}^v(x)$ be a continuous cone field on $\mathcal{C}$. Let us define the pullback $D\mathcal{R}^*(\mathcal{K})$ as follows. Let $y = \mathcal{R}x$. When $D\mathcal{R}_x$ is well defined and invertible (i.e., $x \notin \mathcal{B} \cup \mathcal{T} \cup \mathcal{I}_{\pm\pi/2}$, we let $D\mathcal{R}^*(\mathcal{K})(x) = D\mathcal{R}_x^{-1}(\mathcal{K}(y))$. When $D\mathcal{R}_x$ is well defined but is not invertible, we let $D\mathcal{R}^*(\mathcal{K})(x) = \operatorname{Ker} D\mathcal{R}_x$. Finally, for $x \in \{\alpha_\pm\}$ we let $D\mathcal{R}^*(\mathcal{K})(x) = \mathcal{K}^v(\alpha_\pm)$.

It is easy to see that the pullback is continuous (and is contained in $\mathcal{K}^v$).

Corollary 6.7. *The vertical cone field $\mathcal{K}^v$ is* backward invariant*: $D\mathcal{R}^*(\mathcal{K}^v) \subset \mathcal{K}^v$.*

In this setting, "vertical paths" are understood in the sense of the vertical cone field $\mathcal{K}^v$ rather than $\mathcal{K}^{av}$. So, by definition, a *vertical path*[26] is a a smooth path $\gamma(s)$ in $\mathcal{C}$ such that $\gamma'(s) \in \mathcal{K}^h(\gamma(s))$. Being a graph over a temperature interval, it can be parameterized accordingly: $\phi = \gamma(t)$. Moreover, $\gamma'(s)$ is finite except possibly at the indeterminacy points $\alpha_\pm = \gamma(1)$ (if γ terminates at one of them).

A vertical lamination $\mathcal{F}$ in $\mathcal{C}$ is a family of vertical paths (the *leaves* of the lamination) which are disjoint in the topless cylinder $\mathcal{C}_1$ (but are are allowed to meet on $\mathcal{T}$) that has a *local product structure*. The latter means that for any path $\gamma_0 \in \mathcal{F}$ and any point $x_0 = (\gamma_0(t_0), t_0) \in \mathcal{C}_1$ there exists an $\epsilon > 0$ such that any leaf $\gamma \in \mathcal{F}$ passing near x_0 can be locally represented as a graph over $(t_0 - \epsilon, t_0 + \epsilon)$, and this graph depends C^1-continuously on the *transverse parameter* $\phi = \gamma(t_0)$. The $\operatorname{supp}(\mathcal{F})$ is the union of the leaves of the lamination.

A vertical lamination is called *proper* if all its leaves are proper.

For instance, a finite family of disjoint proper vertical paths form a proper vertical lamination.

In case when $\operatorname{supp}\mathcal{F}$ is open in $\mathcal{C}_1$, the lamination $\mathcal{F}$ is called *a strictly vertical foliation* (of its support). For instance, the "genuinely vertical" foliation on the cylinder $\mathcal{C}_1$ is formed by the intervals $\mathcal{I}_\phi$, $\phi \in \mathbb{R}/2\pi\mathbb{Z}$.

Lemma 6.6 implies that the pullbacks $(\mathcal{R}^n)^*(\mathcal{F})$ of a (proper) vertical lamination are (proper) vertical.

We will mostly be dealing with laminations whose leaves begin on the bottom of the cylinder (in fact, mostly with proper laminations), and will use the bottom angle $\phi \in \mathbb{Z}/2\pi\mathbb{Z}$ as the transverse parameter, $\gamma = \gamma_\phi$.

7. Central line field and dominated splitting

In this section we will use the cone field constructed in §6 to prove that $\mathcal{R} : \mathcal{C} \to \mathcal{C}$ is *projectively hyperbolic*, or admits a *dominated splitting*. We will start with constructing an invariant vertical line field.

[26]In the remainder of the paper, all vertical paths will be considered with respect to $\mathcal{K}^v$ (and not $\mathcal{K}^{av}$) unless otherwise specified.

7.1. **Central line field.** A *central line field* $\mathcal{L}$ on $\mathcal{C}$ is an $\mathcal{R}$-invariant continuous tangent line field $\mathcal{L}(x) \subset \mathcal{K}^v(x)$, $x \in \mathcal{C} \setminus \{\alpha_\pm\}$. Here "invariance" means that $D\mathcal{R}_x(\mathcal{L}(x)) \subset \mathcal{L}(\mathcal{R}x)$ whenever $x \notin \{\alpha_\pm\} \cup \mathcal{R}^{-1}\{\alpha_\pm\}$.

Proposition 7.1. *There exists a unique central line field on* $\mathcal{C}$*. Moreover, if* $\mathcal{L}(x) \subset \mathcal{K}^v(x)$ *is any vertical line field then*

$$(D\mathcal{R}^n)^*\mathcal{L} \to \mathcal{L}^c$$

uniformly and exponentially on compact subsets of $\mathcal{C} \setminus \{\alpha_\pm\}$*.*

7.1.1. *Hyperbolic metric.* To prove this proposition, we will make use of the "projective hyperbolic metric" on the vertical cones. Any interval $I = (a, b) \subset \mathbb{R}$ can be viewed as the hyperbolic line endowed with the hyperbolic metric

$$\operatorname{dist}_I(x, y) = \log \frac{y-a}{x-a} + \log \frac{b-x}{b-y}, \quad a < x < y < b.$$

Since cross-ratios are projective invariants, the hyperbolic metric is invariant under Möbius isomorphisms $\phi : I \to J$. Moreover, it gets contracted under inclusions: if $I \Subset J$ then

$$\operatorname{dist}_J(x, y) \le \lambda \operatorname{dist}_I(x, y), \quad x, y \in I,$$

where $\lambda < 1$ depends only on the hyperbolic diameter of I in J. Putting these two properties together, we obtain the following "Schwarz Lemma" for projective maps:

Lemma 7.2. *Let* $\phi : I \to J$ *be a Möbius transformation with* $\phi(I) \Subset J$*. Then*

$$\operatorname{dist}_J(\phi(x), \phi(y)) \le \lambda \operatorname{dist}_I(x, y), \quad x, y \in I,$$

where $\lambda < 1$ *depends only on the hyperbolic diameter of* $\phi(I)$ *in* J*.*

Due to projective invariance, the above discussion can be carried to any intervals I, J in the projective line $P\mathbb{R}^1$.

7.1.2. *Contraction of the projective cone field.* As the cones $\mathcal{K}^v(x)$ represent intervals in the projective tangent lines, they can be endowed with the hyperbolic metrics d_x.

Lemma 7.3. *For any neighborhood* $\mathcal{U}$ *of the indeterminacy points* $\{\alpha_\pm\}$*, there exist* $C > 0$ *and* $\lambda > 1$ *such that for any* $x \in \mathcal{C} \setminus \mathcal{U}$

$$\operatorname{diam}(D\mathcal{R}^n)^*(P\mathcal{K}^v)(x) \le C\lambda^{-n},$$

where the diam *stands for the angular size of the cones.*

Proof. By Lemma 7.2, the differential $D\mathcal{R}^{-1} : P\mathcal{K}^v(Rx) \to P\mathcal{K}^v(x)$ contracts the hyperbolic metric by a factor $\mu(x) < 1$ depending only on the hyperbolic diameter of $D\mathcal{R}^*(P\mathcal{K}^v)(x)$ in $P\mathcal{K}^v(x)$. (For a critical point x, the projective cone $D\mathcal{R}^*(P\mathcal{K}^v)(x)$ collapses to a point, and we let $\mu(x) = 0$). By continuity of the cone fields, this factor is uniform away from $\mathcal{U}$. Since the $\alpha_\pm$ blow up to the curve $\mathcal{G}$ that does not contain $\alpha_\pm$, the orbit of x can visit $\mathcal{U}$ with frequency bounded by $1/2$ (provided $\mathcal{U}$ is sufficiently small). Hence the hyperbolic diameter of the cones $(D\mathcal{R}^n)^*(P\mathcal{K}^v)(x)$ decay exponentially with rate $O(\mu^{n/2})$. Since the projective intervals $P\mathcal{K}^v(x)$ have angular size bounded away from π, the $\operatorname{diam}(D\mathcal{R}^n)^*(P\mathcal{K}^v)(x)$ are O of their hyperbolic size, and the conclusion follows. □

7.1.3. *Proof of Proposition 7.1.* Let us take a vertical line field $\mathcal{L}$ on $\mathcal{C}$ and pull it back by the dynamics: $\mathcal{L}_n = (D\mathcal{R}^n)^*\mathcal{L}$. By Lemma 7.3, for any $m \le n$ we have:

$$\operatorname{dist}(\mathcal{L}_n(x), \mathcal{L}_m(x)) \le C\lambda^{-m}, \quad x \in \mathcal{C} \smallsetminus \mathcal{U}.$$

Hence the $\mathcal{L}_n$ uniformly and exponentially converge to a limit, which is the desired central line field $\mathcal{L}^c$. □

7.2. **Dominated splitting.** We say that the horizontal cone field $\mathcal{K}^h$ and central line field $\mathcal{L}^c$ give a *dominated splitting* of the map $\mathcal{R} : \mathcal{C} \to \mathcal{C}$ if for any neighborhood $\mathcal{U}$ of the indeterminacy points $\alpha_\pm$, there exist constants $c > 0$ and $\lambda > 1$ such that for any two tangent vectors $v^h \in \mathcal{K}^h(x)$ and $v^c \in \mathcal{L}^c(x)$ of unit length we have:

$$\|D\mathcal{R}^n_x v^h\| \ge c\lambda^n \|D\mathcal{R}^n_x v^c\|, \quad x \in \mathcal{C} \smallsetminus \mathcal{U}. \tag{7.1}$$

(In other words, horizontal vectors grow exponentially faster that the central ones.)

Remark 7.1. For diffeomorphism, the splitting is usually given by two sub-bundles of the tangent bundle (see [Pu]). However, such a definition is not suitable for the non-invertible case when the unstable sub-bundle may not exist. That is why we give a definition in terms of cone fields. Of course, in the invertible case, both definitions are equivalent.

Lemma 7.4. *For any $x \in \mathcal{C} \smallsetminus \{\alpha_\pm\}$ and $i \ge 3$, if $v_1, v_2 \in D\mathcal{R}^i(\mathcal{K}^h(x))$ satisfy $v_1 - v_2 \in \mathcal{L}^c(\mathcal{R}^i x)$, then $\|v_1\| \asymp \|v_2\|$.*

Proof. Let $\mathcal{U}$ be a neighborhood of $\{\alpha_\pm\}$ chosen sufficiently small so that if $x \in \mathcal{U}$ then $\mathcal{R}^n x \notin \mathcal{U}$ for $n = 1, 2$. Lemma 7.2 implies that $D\mathcal{R}^i(P\mathcal{K}^h(x))$ has uniformly bounded hyperbolic diameter for $i \ge 3$. Let L_1 and L_2 be any two lines through $D\mathcal{R}^i(P\mathcal{K}^h(x))$ and let $\theta(L_1, L_2)$, $\theta(L_1, \mathcal{L}^c)$, and $\theta(L_2, \mathcal{L}^c)$ be the angles between them and between each line and $\mathcal{L}^c \equiv \mathcal{L}^c(x)$. The uniform bound on the hyperbolic diameter implies that that there is some constant $C > 0$ so that

$$\theta(L_1, L_2) \le C \cdot \theta(L_j, \mathcal{L}^c), \qquad j = 1, 2.$$

The result then follows from basic trigonometry. □

Corollary 7.5. *For any neighborhood $\mathcal{U}$ of $\{\alpha_\pm\}$ and any $x \in \mathcal{C} \smallsetminus \mathcal{U}$, if $v^h_1, v^h_2 \in \mathcal{K}^h(x)$ are unit tangent vectors then $\|D\mathcal{R}^n v^h_1\| \asymp \|D\mathcal{R}^n v^h_2\|$.*

Proof. Since $x \notin \mathcal{U}$, the projection of v^h_1 onto v^h_2 along $\mathcal{L}^c$ will have length comparable to the length of v^h_2. Thus, the result follows for $n \ge 3$ from Lemma 7.4. If $n \le 2$, the result follows from the fact that $D\mathcal{R}$ can only contract a horizontal vector by a bounded amount (Lemma B.3). □

Note that Corollary 7.5 implies that condition (7.1) is in fact independent of the particular choice of v^h.

Proposition 7.6. *The horizontal cone field $\mathcal{K}^h$ and central line field $\mathcal{L}$ give a dominated splitting of the map $\mathcal{R} : \mathcal{C} \to \mathcal{C}$.*

Proof. Since a single iterate of $\mathcal{R}$ can only contract horizontal vectors by a bounded amount (Lemma B.3), it suffices to consider $n \ge 3$. Let $x_n := \mathcal{R}^n x$ be the orbit of any $x \in \mathcal{C} \smallsetminus \mathcal{U}$. Let $v^h(x_n)$ be the unit vector on the boundary of $D\mathcal{R}^3(\mathcal{K}^h(x_{n-3}))$ pointing "northeast" and let $v^c(x_n)$ be the unit central vector pointing "north". By definition, the vector

$$w_n = v^h(x_n) + v^c(x_n) \tag{7.2}$$

will satisfy $(D\mathcal{R}^3)^* w_n \in \mathcal{K}^v(\mathcal{R}^{n-3}x)$.

We pull back w_n under the dynamics and decompose it as

$$(D\mathcal{R}^n)^* w_n := w = w^h + w^c$$

with w^h parallel to the unit vector on the boundary of $\mathcal{K}^h(x)$ pointing "northeast" and $w^c \in \mathcal{L}^c(x)$. By Proposition 7.1,

$$\|w^c\| \geq c\lambda^n \|w^h\|. \tag{7.3}$$

But

$$w_n = D\mathcal{R}^n(w^h) + D\mathcal{R}^n(w^c) \text{ with } D\mathcal{R}^n(w^h) \in D\mathcal{R}^3(\mathcal{K}^h(x_{n-3})),\ D\mathcal{R}^n(w^c) \in \mathcal{L}^c.$$

Since $D\mathcal{R}^n(w^h)$ and $v^h(x_n)$ differ by an element of $\mathcal{L}^c(x_n)$, Lemma 7.4 gives

$$\|D\mathcal{R}^n(w^h)\| \asymp \|v^h(x_n)\| = 1,$$

and hence $\|D\mathcal{R}^n(w^c)\| = \|w_n - D\mathcal{R}^n(w^h)\| = O(1)$. We see that $\frac{\|D\mathcal{R}^n(w^h)\|}{\|D\mathcal{R}^n(w^c)\|}$ is bounded from below, and hence (7.3) can be written as

$$\frac{\|D\mathcal{R}^n(w^h)\|}{\|D\mathcal{R}^n(w^c)\|} \geq c\lambda^n \frac{\|w^h\|}{\|w^c\|}$$

But this is just the homogeneous form of the dominated splitting condition (7.1). Since this condition is independent of the particular choice of vectors w^h and w^c, we are done. ☐

7.2.1. *Central curves.* Let us say that a smooth curve is *central* if it is tangent (on $\mathcal{C} \smallsetminus \{\alpha_\pm\}$) to the central line field $\mathcal{L}^c$.

Proposition 7.7. *Through any point $x \in \mathcal{C} \smallsetminus \{\alpha_\pm\}$ passes a vertical central curve.*

Proof. It follows from the Peano Existence Theorem (see [W]) that continuous line fields are integrable, so we can find a central curve through any point $x \in \mathcal{C} \smallsetminus \{\alpha_\pm\}$. Since the central line field is transverse to the genuinely horizontal foliation, this curve is a graph over the t-axes, and for standard reasons can be extended in both ways to the boundary of the cylinder. This is the desired vertical central curve.

If $x \in \{\alpha_\pm\}$, then one can take $\mathcal{I}_\pm$ as the desired central curve. (In fact, there are whole central tongues $\Lambda_\pm$ filled with vertical central curves landing at $\alpha_\pm$ – see §12.1). ☐

Remark 7.2. At this stage we do not know yet that there exists a unique central curve through a given x. In fact, as we have just mentioned, this is not the case for the indeterminacy points $\alpha_\pm$ (and hence for their preimages). However, we will prove in §12.1 that the uniqueness holds on $\mathcal{C}_1$.

8. Horizontal expansion

In this section we will prove that the map $\mathcal{R} : \mathcal{C}_1 \to \mathcal{C}_1$ is *horizontally expanding*, in the following sense:

- $\mathcal{R}$ has an invariant horizontal cone field $\mathcal{K}^h(x)$ on $\mathcal{C}_1$;
- There exist constants $c > 0$ and $\lambda > 1$ such that

$$\|D\mathcal{R}^n(x)(v)\| \geq c\lambda^n \|v\|, \quad n = 0, 1, \ldots \tag{8.1}$$

for any $x \in \mathcal{C}_1$ and $v \in \mathcal{K}^h(x)$. Moreover, λ is called a *rate of expansion.*

Theorem 8.1. *The map $\mathcal{R} : \mathcal{C}_1 \to \mathcal{C}_1$ is horizontally expanding on $\mathcal{C}_1$ with the rate $\lambda = 2$ with respect to the horizontal cone field $\mathcal{K}^h$ from §6.2.*

We will give two proofs of Theorem 8.1 coming from two different perspectives: (1) Global Complex Dynamics on $\mathbb{CP}^2$ and (2) Combinatorics and Mathematical Physics. These two proofs both rely on the same principle and their connection will be explained at the end of the section.

8.1. **Global Approach.** Let us consider the solid cylinder

$$\mathcal{SC} := \{(z,t) : \ |z| \le 1, \ t \in [0,1]\}.$$

It is foliated by the horizontal leaves

$$\Pi_t^* := \{(z,t) : \ |z| \le 1, \quad t \in [0,1]\},$$

each of which contained in a horizontal complex projective line Π_t obtained as the closure in $\mathbb{CP}^2$ of $\{(z,t) : z \in \mathbb{C}\}$.

Because $\mathcal{R} : \mathbb{CP}^2 \to \mathbb{CP}^2$ is not algebraically stable (Observation 4.6), it will be better to switch to the mapping $R : \mathbb{CP}^2 \to \mathbb{CP}^2$, which is algebraically stable (Proposition 4.4). This won't affect the proof of Theorem 8.1 because $\mathcal{R} : \mathcal{C}_0 \to \mathcal{C}_0$ and $R : C_0 \to C_0$ are conjugate by means of $\psi : \mathcal{C}_0 \to C_0$. (Meanwhile, both maps are obviously expanding on $\mathcal{B}$ and B, respectively.)

In the (u,w) coordinates, the horizontal complex lines Π_t correspond to the conics $P_t = \overline{\{uw = t^{-2}\}}$ and the leaves of the solid cylinder become

$$P_t^* := \overline{\{(u,w) \in P_t : \ |w| \le t^{-1}\}}, \quad t \in [0,1],$$

where the closures are taken in $\mathbb{CP}^2$. The bottom leaf P_0^* becomes the closed unit disk $\overline{\mathbb{D}}$ in the coordinate $\zeta = w/u$ on the line at infinity L_0.

Recall that the cylinder C itself (or rather, the Möbius band) is given by

$$C = \overline{\{w = \bar{u}, \ |u| \ge 1\}},$$

where the closure is taken in $\mathbb{CP}^2$, and the “topless” Möbius band by $C_1 = C \setminus \mathrm{T}$, where $\mathrm{T} = \{w = \bar{u}, \ |u| = 1\}$; See §3.1. The leaf P_t^* intersects C by the round circle $S_t = \{|w| = t^{-1}\}$. All the leaves P_t^* meet at the attracting fixed point $e = (1:0:0)$, which is the center $\zeta = 0$ of the disc P_0^*.

Let us now consider the central projection from the origin to the line L_0 at infinity:

$$\pi : \mathbb{CP}^2 \setminus \{[0:1:0]\} \to L_0 \quad (u,w) \mapsto \zeta = w/u.$$

Lemma 8.2. *For any $t \in (0,1)$, the map $\psi_n = \pi \circ R^n$ sends P_t^* to P_0^* as a branched covering of degree $2 \cdot 4^n$.*

Proof. We fix some $t \in (0,1)$, and skip it from the notation, so $P^* \equiv P_t^*$, etc.

The fundamental homology class of B is a generator for the first homology $H_1(C_1, \mathbb{Z}) \cong \mathbb{Z}$. Since $t \ne 0$, $P \cap C_1$ is a horizontal curve of degree two in C_1, meaning that its homology class is twice that of B. Since C_1 is invariant and R acts with degree 4 on its first homology (Property (P10) from §3.3), $R^n(P \cap C_1)$ is a horizontal curve of degree $4^n \cdot 2$ in C_1. This implies that $\psi_n = \pi \circ R^n : P \cap C_1 \to \mathrm{B}$ is a covering map of degree $4^n \cdot 2$.

Because R is algebraically stable, Lemma A.6 gives that the push-forward $(R^n)_* P$ is a divisor of degree $4^n \cdot 2$. Bezout's Theorem gives $4^n \cdot 2$ intersections of $(R^n)_* P$ with the complexification of any radial line $L := \{\arg u = \phi \mod \pi\}$. Meanwhile, since $R^n(P \cap C_1)$ is a horizontal curve of degree $4^n \cdot 2$ in C_1, this forces all $4^n \cdot 2$ intersections between $(R^n)_* P$ and L to take place in C_1. They are therefore disjoint from the indeterminate points $a_\pm = \pm(i, -i)$ and the origin $(0,0)$. Hence, $R^n : P \to R^n P$ is a branched cover of some degree $m > 0$ and $\pi : R^n(P) \to L_0$ is a branched covering of degree $\deg(R^n P)$. The composition $\psi_n = \pi \circ R^n : P \to L_0$ is therefore a branched cover of degree

$$m \deg(R^n P) = \deg((R^n)_* P) = 4^n \cdot 2.$$

We find that $\psi_n : P \to L_0$ is a rational map between two Riemann spheres of degree $2 \cdot 4^n$ (commuting with the natural antiholomorphic involutions) that restricts to a covering map between the circles $P \cap C \to \mathrm{B}$ (the fixed points loci for the involutions) of the *same* degree. By the Argument Principle, any point $\zeta \in P_0^*$ has at least $2 \cdot 4^n$ preimages in the disk P^*.

$\square$

Remark 8.1. The symmetry $R(-u, -w) = R(u, w)$ implies that $R^n(P)$ is covered at least twice by P, so that the degree m in the proof of Lemma 8.2 is at least 2. We do not know if m can be higher than 2 for some values of $t \in (0,1)$ or if it can grow with n. These details will not be needed in the proof.

Let us now consider the fixed point $e = (1 : 0 : 0)$ at the center of both the disks P_t and P_0. Obviously, $\psi_n(e) = e$.

Lemma 8.3. *The map $\psi_n : P_t^* \to P_0^*$ has branching of degree 2^{n+1} at e.*

Proof. We will work in the local coordinates ($v = V/U$, $w = W/U$) near e. The curve P_t becomes the parabola $w = \frac{1}{t^2} v^2$ and the leaf P_t^* is parameterized by $s \mapsto \left(s, \frac{1}{t^2} s^2\right)$, with $|s| \le t$. The central projection becomes $\pi(v, w) = w$.

In these coordinates, the map $(v', w') = R(v, w)$ assumes the form

$$v' = v^2 \left(\frac{1+w}{1+v^2}\right)^2 \sim v^2, \quad w' = \left(\frac{w^2+v^2}{1+v^2}\right)^2 \sim v^4 (1 + (w/v)^2)^2. \tag{8.2}$$

Let $(v_n(s), u_n(s)) := R^n\left(s, \frac{1}{t^2} s^2\right)$. Equation (8.2) implies inductively that $v_n(s)$ and $w_n(s)$ vanish to order 2^n and 2^{n+1}, respectively, at $s = 0$. The result follows, since $\psi_n = \pi \circ R^n : P_t^* \to P_0^*$ is given by $w_n(s)$. $\square$

Lemma 8.4. *A Blaschke product $B : \mathbb{C} \to \mathbb{C}$ all of whose zeros lie in the unit disc and vanishing at the origin to order k expands the Euclidean metric on the circle $\mathbb{T}$ at least by k.*

Proof. Under these circumstances, $B(z) = z^k \tilde{B}$, where $\tilde{B}$ is another Blaschke product all of whose zeros lie in the unit disc. In the angular coordinate $z = e^{i\phi}$ it assumes a form

$$\phi \mapsto k\phi + h(\phi),$$

where $h'(\phi) > 0$ since $\tilde{B}$ is orientation preserving on the circle. The assertion follows. $\square$

First Proof of Theorem 8.1. If $t = 0$, then $R : \mathrm{B} \to \mathrm{B}$ is conjugate to $z \mapsto z^4$, so that R^n expands the cylinder metric along B by at least 4^n.

Suppose $t \in (0,1)$. Let us uniformize P by the Riemann sphere so that the $P \cap C$ becomes the unit circle $\mathbb{T} = \{|\xi| = 1\}$. Lemma 8.2 implies that in this coordinate, the map $\zeta = \psi_n(\xi)$ is a branched cover from the unit disc to itself. It is a classical fact that such a mapping is a Blaschke product, all of whose zeros lie in the unit disc. By Lemma 8.3, it vanishes of order 2^{n+1} at the origin. By Lemma 8.4, it expands the circle metric at least by factor 2^{n+1}. Hence R^n expands the cylinder metric along $P \cap C$ at least by $c\,2^{n+1}$ with some $c > 0$.

Clearly, it suffices to prove (8.1) for unit vectors. The discussion in the previous paragraph implies that it holds for vectors tangent to $P \cap C$. If N is any small neighborhood of the indeterminate points $a_\pm$, then Corollary 7.5 implies that (8.1) holds for any unit vector in $v \in K^h(x)$, $x \in C_1 \setminus N$.

Finally, if $x \in N$, then Lemma B.3 gives that one iterate of R can contract the horizontal length only by a bounded factor and the result follows. □

8.2. **Combinatorial Approach.** We will now present another proof of Theorem 8.1 that is based on a combinatorial interpretation of the DHL and the Lee-Yang Theorem with Boundary Conditions. The notation is from §2.1.

Second Proof of Theorem 8.1. Recall that the partition function of the Ising model on Γ_n is given as

$$\mathsf{Z}_n = \sum_\sigma e^{-H_n(\sigma)/T} = \sum_\sigma t^{-I(\sigma)/2} z^{-M(\sigma)}, \tag{8.3}$$

where $M(\sigma)$ and $I(\sigma)$ are the magnetic moment (2.1) and interaction (2.2) of the configuration σ. Let $\sigma_+ \equiv +1$ and $\sigma_- \equiv -1$. Clearing denominators we obtain the modified partition function

$$\check{\mathsf{Z}}_n(z) := t^{I(\sigma_-)} z^{4^n} \mathsf{Z}_n(z) = \sum_{j=0}^{N} a_j z^j, \tag{8.4}$$

with $N = 2 \cdot 4^n$. Recall the basic symmetry of the Ising model

$$a_{N-j} = a_j, \quad j = 0, 1, \dots N; \qquad a_0 = a_N = 1.$$

which is obtained under the involution $\sigma \mapsto -\sigma$ and from the invariance $I(\sigma) = I(-\sigma)$.

The generating graph Γ is symmetric under reflection across the vertical line through the marked vertices a and b. This allows us to factor[27] the conditional partition functions U_n and W_n as

$$U_n = \mathrm{U}_n^2 \qquad W_n = \mathrm{W}_n^2,$$

where U_n and W_n correspond to the conditional partition functions of the right (or left) half of Γ_n, having the same boundary conditions as U_n and W_n.

Both halves of Γ_n have valence 2^{n-1} at a and b. In particular, if $\sigma(a) = \sigma(b) = +1$, there are at most $4^n/2 - 2^n$ edges both of whose endpoints have spin -1. This gives that U_n has no terms in z of degree greater than $4^n/2 - 2^n$. Similarly, W_n has no terms in z of degree lower than $-4^n/2 + 2^n$.

[27] V_n also factors, but will not use it.

Clearing denominators, one obtains

$$\begin{aligned}\check{\mathrm{U}}_n(z) &:= t^{I(\sigma_+)/2} z^{4^n/2}\ \mathrm{U}_n(z) = \sum_{j=0}^{N_0} a_j^+ z^j \quad \text{and} \\ \check{\mathrm{W}}_n(z) &:= t^{I(\sigma_-)/2} z^{4^n/2-2^n}\ \mathrm{W}_n(z) = \sum_{j=0}^{N_0} a_j^- z^j,\end{aligned}$$

where $N_0 = 4^n - 2^n$. It follows from the Lee-Yang Theorem with Boundary Conditions that the zeros $b_1, \dots, b_{N_0}$ of $\check{\mathrm{W}}_n(z)$ all lie in $\mathbb{D}$.

The basic symmetry of the Ising model appears as the following symmetry between $\check{\mathrm{U}}_n$ and $\check{\mathrm{W}}_n$:

$$a^-_{N_0-j} = a^+_j, \qquad j = 0, 1, \dots, N_0.$$

Consequently,

$$\check{\mathrm{W}}_n(z) = \prod_{j=1}^{N_0}(z - b_j) \ \text{and}\ \check{\mathrm{U}}_n(z) = \prod_{j=1}^{N_0}(1 - b_j z) = \prod_{j=1}^{N_0}(1 - \bar{b}_j z),$$

using also that $\check{\mathrm{U}}_n(z)$ has real coefficients.

Since $z_n^2 = W_n/U_n = \mathrm{W}_n^2/\mathrm{U}_n^2$, we obtain the Blaschke Product

$$z_n = \frac{\mathrm{W}_n}{\mathrm{U}_n} = \frac{z^{2^n}\check{\mathrm{W}}_n}{\check{\mathrm{U}}_n} = z^{2^n}\prod_{j=1}^{N_0}\frac{z - b_j}{1 - \bar{b}_j z} \tag{8.5}$$

having all of its zeros in $\mathbb{D}$ and a zero at $z = 0$ of multiplicity 2^n.

The remainder of the proof continues as in §8.1. □

Remark 8.2. The two proofs of Theorem 8.1 are related by observing that

$$\Psi : \Pi_t \to P_t, \quad \text{given by} \quad (v, w) = \Psi(z, t) = (zt, z^2)$$

gives the parameterization of P_t in which $P_t \cap C$ becomes the unit circle, as required in the first proof. The composition $\psi_n \circ \Psi : \Pi_t \to L_0$ is then given by

$$\psi_n \circ \Psi(z, t) = \frac{W_n(z, t)}{U_n(z, w)} = z_n^2.$$

Moreover, this accounts for why the degree of (8.5) is half the degree of $\psi_n \circ \Psi(z, t)$. In particular, it explains why the multiplicity of $z = 0$ in (8.5) is only 2^n instead of 2^{n+1}.

9. Low temperature dynamics: basin of $\mathcal{B}$ and its stable foliation

The bottom circle $\mathcal{B}$ is superattracting within $\mathcal{C}$ by Property (P5) from §3.3, so there is an open set $\mathcal{W}^s(\mathcal{B}) \subset \mathcal{C}_1$ consisting of points whose orbits converge to $\mathcal{B}$, called the *basin of attraction* of $\mathcal{B}$. Obviously, $\mathcal{W}^s(\mathcal{B})$ is completely invariant under $\mathcal{R}|\mathcal{C}_1$.

9.1. **Density.**

Lemma 9.1. *Any horizontal curve* $\gamma \subset \mathcal{C}_1$ *intersects* $\mathcal{W}^s(\mathcal{B})$ *in a dense open set. In particular,* $\mathcal{W}^s(\mathcal{B})$ *is a dense open subset of* $\mathcal{C}$.

Proof. Since $\mathcal{W}^s(\mathcal{B})$ is open in $\mathcal{C}$, its intersection with γ is open in γ. To prove density, it suffices to prove that any horizontal curve $\gamma' \subset \gamma$ intersects $\mathcal{W}^s(\mathcal{B})$. According to Theorem 8.1 there is some iterate n so that $l^h(\mathcal{R}^n\gamma') > 2\pi$. This implies that $\mathcal{R}^n\gamma'$ intersects $\mathcal{I}_{\frac{\pi}{2}} \setminus \{\alpha_+\} \subset \mathcal{W}^s(\mathcal{B})$. Since $\mathcal{W}^s(\mathcal{B})$ is backward invariant, γ' also intersects $\mathcal{W}^s(\mathcal{B})$. □

9.2. **Low temperature cylinder $\mathcal{C}_*$.** Recall that t_c is the height of the real repelling fixed point β_c in the invariant line $\{\phi = 0\}$, and let $\beta_c' = (\pi, t_c)$,

$$\mathcal{C}_* = \{(\phi, t) \in \mathcal{C} : \; t \le t_c\}.$$

We will call $\mathcal{C}_* \smallsetminus \{\beta_c, \beta_c'\}$ the *low temperature cylinder*.

Let us begin with a simple observation:

Lemma 9.2. *The basin* $\mathcal{W}^s(\mathcal{B})$ *contains the low temperature cylinder* $\mathcal{C}_* \smallsetminus \{\beta_c, \beta_c'\}$.

Proof. We will check that points $x \in \mathcal{C}_* \smallsetminus \{\beta_c, \beta_c'\}$ converge to the bottom $\mathcal{B}$ at least as fast as they do on the low temperature interval $(\beta_0, \beta_c) \subset \mathcal{I}_0$.

For $x = (\phi, t) \in \mathcal{C}$, we let $t(x) = t$. Since the function

$$y \mapsto \frac{y+1}{y+s}$$

is increasing or constant on $(-1, 1]$ for $s \ge 1$, (3.3) gives that $t(\mathcal{R}x) \le t(\mathcal{R}(0, t(x)))$, with equality attained only if $x \in \mathcal{I}_0, \mathcal{I}_\pi$.

Let $t_n = t(\mathcal{R}^n x)$ and let $q_n = t(\mathcal{R}^n(0, t_0))$. By the above observation, if $x \in \mathcal{C}_* \smallsetminus \{\beta_c\}$ then $t_1 < t_c$, and furthermore, $t_{n+1} \le q_n \to 0$ as $n \to \infty$. Thus, $\mathcal{C}_* \smallsetminus \{\beta_c\} \subset \mathcal{W}^s(\mathcal{B})$. □

9.3. **Complex stable lamination of $\mathcal{B}$ in $\mathbb{C}^2$.** In $\mathbb{C}^2$, the circle $\mathcal{B}$ is *hyperbolic*, with a complex one-dimensional transverse stable direction (the t-direction) and a real one-dimensional transverse unstable one (the unstable direction within the plane $t = 0$).

Given an $\epsilon > 0$, the *ϵ-local basin* $\mathcal{W}^s_{\mathbb{C},\epsilon}(\mathcal{B})$ is the set of points $\zeta \in \mathbb{C}^2$ that are ϵ-close to $\mathcal{B}$ and whose orbits converge to $\mathcal{B}$ while remaining ϵ-close to $\mathcal{B}$. Alternatively, we can use any forward invariant open set containing $\mathcal{B}$ to define a local stable set $\mathcal{W}^s_{\mathbb{C},\mathrm{loc}}(\mathcal{B})$, with the specific open set that is used implicit in the notation.

Similarly, for $x = (\phi, 0) \in \mathcal{B}$, a local stable manifold $\mathcal{W}^s_{\mathbb{C},\mathrm{loc}}(x) \equiv \mathcal{W}^s_{\mathbb{C},\mathrm{loc}}(\phi)$ is a 1-dimensional holomorphic curve containing x consisting of all points ζ near x whose orbits are forward asymptotic to the orbit of x, while remaining close to the orbit of x.

In this subsection we will construct the local stable lamination[28] of the bottom circle $\mathcal{B}$ in $\mathbb{C}^2$ and will show that the leaves $\mathcal{W}^s_{\mathbb{C},\mathrm{loc}}(\phi)$, $\phi \in \mathbb{T}$, of this lamination are holomorphic curves filling in a topological real 3-manifold contained within $\mathcal{W}^s_{\mathbb{C},\mathrm{loc}}(\mathcal{B})$.

[28]Here, a lamination is a family of disjoint holomorphic curves that has a local product structure, in a sense similar to that given in §6.3.

In the case of diffeomorphisms, the construction of the stable laminations for hyperbolic sets is a standard background of the general theory, see [HPS, PM, Sh]. However, we have not been able to find an adequate reference in the non-invertible case (notice, however, the remark in [PM, p. 79]), so we will give a direct argument in our situation.

We will make use of the simple structure of the postcritical locus near $\mathcal{B}$ (comprising two lines $\{t = \pm\pi/2\}$ collapsing to the fixed point β_0) and of the holomorphic λ-lemma. This is similar to the method used in [HP, §2.4] and [R, §4].

Proposition 9.3. *For sufficiently small $\epsilon > 0$, local stable manifolds $\mathcal{W}^s_{\mathbb{C},\mathrm{loc}}(x)$, $x \in \mathcal{B}$, are holomorphic curves whose union $\bigcup_{x\in\mathcal{B}} \mathcal{W}^s_{\mathbb{C},\mathrm{loc}}(x)$ form a lamination supported on the local basin of $\mathcal{W}^s_{\mathbb{C},\mathrm{loc}}(\mathcal{B})$. Moreover, $\bigcup_{x\in\mathcal{B}} \mathcal{W}^s_{\mathbb{C},\mathrm{loc}}(x)$ is a topological real 3-manifold.*

Proof. Let $\mathcal{Q} \subset \mathcal{B}$ be the set of iterated preimages of 1 under $z \mapsto z^4$. We will construct a family of analytic discs $\mathcal{D}_z$ through the set $\mathcal{Q}$ in such a way that each disc intersects $\mathcal{B}$ in exactly one point $z \in \mathcal{Q}$ and so that $\mathcal{D}_z$ is obviously the stable disc of z. We will furthermore verify that each $\mathcal{D}_z$ can be expressed as the graph of a function $z = \eta(t)$ for all t in an appropriate small disc $\mathbb{D}_\rho$.

This family of discs provides a *holomorphic motion* of $\mathcal{Q}$, parameterized by $\mathbb{D}_\rho$:

$$h : \mathcal{Q} \times \mathbb{D}_\rho \to \mathbb{C}.$$

Then, the λ-Lemma [Ly, MSS] for holomorphic motions immediately gives that h extends to the closure providing a continuous mapping $\bar{h} : \mathcal{B} \times \mathbb{D}_\rho \to \mathbb{C}$.

Recall that the line $t = 0$ is superattacting away from the origin. Therefore, choosing $0 < a < 1$ and $C > 0$ we can easily further restrict $\rho > 0$ so that $|t_n| \le Ca^n$ for any choice of $(z,t) \in \cup_{z\in\mathcal{Q}}\mathcal{D}_z$. (Here $t_n = t(\mathcal{R}^n(z,t))$.) Therefore, points in the closure, and in particular, points in the image of $\mathcal{B} \times \mathbb{D}_\rho$ under $(\phi, t) \mapsto \big(\bar{h}(\phi,t), t\big)$ converge at least geometrically to $\mathcal{B}$.

Given $(\phi, 0) \in \mathcal{B}$, the stable leaf $\mathcal{W}^s_{\mathbb{C},\mathrm{loc}}(\phi)$ is parameterized by $t \to \big(\bar{h}(\phi,t), t\big)$ for $|t| < \rho$. The union of all stable leaves, which is given by the image of $\mathcal{B} \times \mathbb{D}_\rho$ under $(\phi, t) \mapsto \big(\bar{h}(\phi,t), t\big)$, is the desired lamination. It is contained within $\mathcal{W}^s_{\mathbb{C},\mathrm{loc}}(\mathcal{B})$ by the discussion in the previous paragraph.

We now construct the holomorphic motion $h : \mathcal{Q} \times \mathbb{D}_\rho \to \mathbb{C}$.

Consider a neighborhood $\Delta_{\delta,\rho}$ of $\mathcal{B}$ of the form

$$\Delta_{\delta,\rho} = \{(z,t) \in \mathbb{C}^2 \ : 1 - \delta < |z| < 1 + \delta,\ |t| < \rho\}$$

Let $\partial^v \Delta_{\delta,\rho}$ be the vertical boundary $|z| = 1 \pm \delta$ and $\partial^h \Delta_{\delta,\rho}$ the horizontal boundary $|t| = \rho$.

Lemma 9.4. *We can choose $\delta, \rho > 0$ sufficiently small so that*

(1) *The complex vertical cone-field $|dt(v)| > |dz(v)|$ is backward invariant under $\mathcal{R}$.*
(2) *$\mathcal{R}(\partial^v \Delta_{\delta,\rho})$ is entirely outside of $\Delta_{\delta,\rho}$ and $\mathcal{R}(\partial^h \Delta_{\delta,\rho})$ is entirely contained within $|t| < \rho$.*

Proof. On $\mathcal{B}$ we have

$$D\mathcal{R} = \begin{bmatrix} 4z^3 & 0 \\ 0 & 0 \end{bmatrix}$$

so that, by continuity the $(1,1)$ term of $D\mathcal{R}$ dominates the remaining terms in any sufficiently small neighborhood of $\mathcal{B}$. This is sufficient for the desired invariance of the conefield in (1).

We now further restrict $\partial^h \Delta_{\delta,\rho}$ so that (2) holds. However, this again follows easily by continuity because on $t = 0$ we have that $\mathcal{R}$ is given by $z \mapsto z^4$ which maps the boundary of any annulus of the form $1 - \delta < |z| < 1 + \delta$ well outside of the annulus and because the line $t = 0$ is superattracting. □

We will call an analytic disc $\mathcal{D}$ in $\Delta_{\delta,\rho}$ *admissible* if:

(1) $\mathcal{D}$ intersects $\mathcal{B}$ in a single point,
(2) $\partial\mathcal{D}$ intersects $\partial\Delta_{\delta,\rho}$ only in the vertical boundary $\partial^v \Delta_{\delta,\rho}$, and
(3) The tangents to $\mathcal{D}$ lie within the vertical cone-field Lemma 9.4 above.

These properties are chosen to ensure that any admissible disc can be written as the graph $z = \eta(t)$ for some analytic function $\eta : \mathbb{D}_\rho \to \mathbb{C}$.

Let $\mathcal{D}_1 := \{1\} \times \mathbb{D}_\rho$, which is clearly a stable disc and admissible. We now inductively define admissible stable discs $\mathcal{D}_z$ over any $z \in \mathcal{Q}$. Suppose that z is an n-th preimage of 1. By the induction hypothesis, there is an admissible stable disc $\mathcal{D}_{z'}$ over $z' = z^4$. Within $\Delta_{\delta,\rho}$ the critical points of $\mathcal{R}$ consist of points on the line $\{t = 0\}$ and on the collapsing lines $\{z = \pm i\}$ both of which map to $\{t = 0\}$. At any such critical point, the image of $D\mathcal{R}$ spans the tangent direction to $\{t = 0\}$ and is therefore transverse to $\mathcal{D}_{z'}$. Thus, $\mathcal{R}^{-1}(\mathcal{D}_{z'})$ is a finite union of analytic discs. Let $\mathcal{D}_z$ be the component of $\mathcal{R}^{-1}(\mathcal{D}_{z'}) \cap \Delta_{\delta,\rho}$ containing z. By properties (1) and (2) from the lemma, we see that since $\mathcal{D}_{z'}$ was an admissible disc, so is $\mathcal{D}_z$.

Continuing in this way one defines a family of admissible stable discs over every $z \in \mathcal{Q}$. The result is the desired holomorphic motion $h : \mathcal{Q} \times \mathbb{D}_\rho \to \mathbb{C}$.

The map $(\phi, t) \mapsto \big(\bar{h}(\phi, t), t\big)$ is clearly an immersion of $\mathcal{B} \times \mathbb{D}_\rho$ into $\mathbb{C}^2$ with image $\bigcup_{x\in\mathcal{B}} \mathcal{W}^s_{\mathbb{C},\mathrm{loc}}(x)$. Shrinking ρ slightly, if necessary, this immersion can be made into an embedding. □

The λ-Lemma gives the following regularity for $\bigcup_{x\in\mathcal{B}} \mathcal{W}^s_{\mathbb{C},\mathrm{loc}}(x)$. Globally, it is just a topological manifold, however each slice with $t = t_0$ for $|t_0| < \rho$ is the image of the unit circle $\mathcal{B}$ under a quasiconformal homeomorphism with dilatation

$$K \leq \frac{\rho + |t_0|}{\rho - |t_0|}.$$

9.4. **Stable foliation of $\mathcal{B}$ in the cylinder.** Let us now consider the slices of the local stable manifolds by the cylinder,

$$\mathcal{W}^s_{\mathrm{loc}}(x) \equiv \mathcal{W}^s_{\mathrm{loc}}(\phi) = \mathcal{W}^s_{\mathbb{C},\mathrm{loc}}(\phi) \cap \mathcal{C}, \quad x = (\phi, 0),\ \phi \in \mathbb{T}.$$

They are real analytic curves that form a foliation of a neighborhood of $\mathcal{B}$ in $\mathcal{C}$. Note that the $\mathcal{W}^s_{\mathrm{loc}}(\pm\pi/2)$ are arcs of the collapsing intervals $\mathcal{I}_{\pm\pi/2}$.

We will now globalize this foliation. Let $\mathcal{W}^s_n(\phi)$ stand for the lift of $\mathcal{W}^s(4^n\phi, 0)$ that begins at $(\phi, 0)$. Since $\mathcal{R}(\mathcal{W}^s_{\mathrm{loc}}(\phi)) \subset \mathcal{W}^s_{\mathrm{loc}}(4\phi)$, we have:

$$\mathcal{W}^s_{\mathrm{loc}}(\phi) \equiv \mathcal{W}^s_0(\phi) \subset \mathcal{W}^s_1(\phi) \subset \mathcal{W}^s_2(\phi) \subset \dots$$

By Property (P4), for $\phi \neq \pm\pi/2$, each $\mathcal{W}^s_n(\phi)$ is a real analytic curve, while $\mathcal{W}^s_n(\pm\pi/2) = \mathcal{I}_{\pm\pi/2}$ for all $n \geq 1$. Hence the sets

$$\mathcal{W}^s(\phi) = \bigcup_{n=0}^{\infty} \mathcal{W}^s_n(\phi).$$

are real analytic curves for all $\phi \in \mathbb{T}$. They are called the *global stable manifolds* of the points $x = (\phi, 0) \in \mathcal{B}$.[29] Note that $\mathcal{W}^s(\pm\pi/2) = I_{\pm\pi/2}$, while $\mathcal{W}^s(0) = [\beta_0, \beta_c)$.

By construction, $\mathcal{R}(\mathcal{W}^s(\phi)) \subset \mathcal{W}^s(4\phi)$, and, in fact, $\mathcal{W}^s(\phi)$ is the lift of $\mathcal{W}^s(4\phi)$ by $\mathcal{R}$ that begins at $(\phi, 0) \in \mathcal{B}$ (compare Lemma 3.2).

Lemma 9.5. *The stable manifolds $\mathcal{W}^s(\phi)$ are strictly vertical curves.*

Proof. The stable manifolds $\mathcal{W}^s(x)$ are tangent to the Ker $D\mathcal{R}(x)$. From representation $\mathcal{R} = f \circ Q$ in §3.2 we see that the Ker $D\mathcal{R}(x) =$ Ker $DQ(x)$ are orthogonal to $\mathcal{B}$. Hence the local stable manifolds $\mathcal{W}^s_\epsilon(x)$ go through the vertical algebraic cones $\mathcal{K}^v(x)$ (for $\epsilon > 0$ sufficiently small), so they are strictly vertical. Since the cone field $\mathcal{K}^v(x)$ is backward invariant (see Cor. 6.7), the global stable manifolds $\mathcal{W}^s(x)$ are strictly vertical as well. □

9.5. **Stable tongues.** We will now describe "tongues" in $\mathcal{W}^s(\mathcal{B})$ connecting from the bottom of $\mathcal{C}$ to the top of $\mathcal{C}$. The reader may wish to refer to Figure 1.2, where many of these tongues are visible in blue (dark). It may also be helpful to recall the combinatorial description of $\mathcal{R} : \mathcal{C} \to \mathcal{C}$ from §3.2 and §3.3, including the Primary Central Tongues $\Lambda_\pm$ defined in §3.3 (P6).

A *tongue* Υ attached to a "tip" $x \in \mathcal{T}$ is a domain in $\mathcal{C}$ bounded by two proper vertical paths meeting only at x. Note that Υ meets $\mathcal{B}$ in an interval $\mathcal{B}_\Upsilon$, called its *bottom*. A tongue is called *stable* if it is contained in $\mathcal{W}^s(\mathcal{B})$ and is foliated by proper stable manifolds $\mathcal{W}^s(\phi)$, $(\phi, 0) \in \mathcal{B}_\Upsilon$ (terminating at x). A stable tongue is *maximal* if it is not a proper subset of another stable tongue.

Proposition 9.6. *There are two maximal stable tongues $\Upsilon(\alpha_\pm)$ attached to the indeterminacy points $\alpha_\pm$ respectively. They are symmetric with respect to the collapsing intervals $\mathcal{I}_{\pm\pi/2}$ and have positive angles at the tip.*

Proof. By the blow-up formula (3.4) points x approaching $\alpha_\pm$ at angle ω from $\mathcal{I}_{\pm\pi/2}$ are mapped to the point $\mathcal{G}(\omega) = (2\omega, \sin^2 \omega)$. This curve intersects the critical level $\{t = t_c\}$ at two points $\mathcal{G}(\pm\omega_c)$ with $\omega_c > 0$. Moreover,

$$\mathcal{G}[-\omega_c, \omega_c] \subset \mathcal{C}_* \smallsetminus \{\beta_c, \beta'_c\} \subset \mathcal{W}^s(\mathcal{B}).$$

Hence there is a region U under the arc $\mathcal{G}[-\omega_c, \omega_c]$ (comprised of two symmetric topological triangles) foliated by stable manifolds $\mathcal{W}^s(\phi)$ (see Figure 9.1). By Lemma 3.2, this region lifts by $\mathcal{R}$ to two stable tongues $\Upsilon'(\alpha_\pm)$ attached to $\alpha_\pm$. By construction, each $\Upsilon'(\alpha_\pm)$ has angle $2\omega_c > 0$ at the tip. The desired tongues $\Upsilon(\alpha_\pm)$ are the maximal stable tongues containing $\Upsilon'(\alpha_\pm)$. (As such, they are unique and also have positive angle at the tip.) By $\pm\pi/2$-symmetry of the basin $\mathcal{W}^s(\mathcal{B})$ (see Property (P1) in §3.3), they are symmetric with respect to the corresponding axes $\mathcal{I}_{\pm\pi/2}$. □

The above tongues $\Upsilon(\alpha_\pm)$ will be called the *primary* stable tongues of $\mathcal{W}^s(\mathcal{B})$.

For $n \in \mathbb{N}$, let

$$\mathcal{A}_n = \mathcal{R}^{-n}\{\alpha_\pm\} = \{(\phi_{n,k}, 1)\}_{k=0}^{2^n-1}, \text{ where } \phi_{n,k} = \frac{\pm\pi/2 + 2\pi k}{2^n}. \tag{9.1}$$

[29]This terminology is not completely standard, as usually the global stable manifold of x is defined as the set of *all* points that are forward asymptotic to the orb x. In our situation, it would be $\cup \mathcal{R}^{-n}\mathcal{W}^s(4^n\phi)$, which is disconnected.

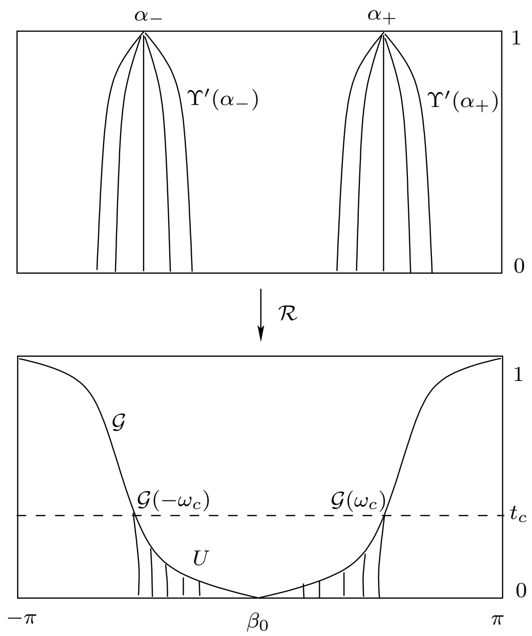


FIGURE 9.1. Topological triangles foliated by stable manifolds (below) and the corresponding stable tongues attached to the indeterminacy points $\alpha_\pm$ (above).

and let $\mathcal{A} = \cup \mathcal{A}_n$ be the *pre-indeterminacy set*. By the end of §9.6 we will see that there are infinitely many maximal stable tongues attached to each $\alpha \in \mathcal{A}$. We begin with the following:

Proposition 9.7. *There is a family of maximal stable tongues* $\{\Upsilon_k(\alpha)\}_{\alpha\in\mathcal{A},k\in K(\alpha)}$ *where* $\Upsilon_k(\alpha)$ *is attached to* α *for each* $\alpha \in \mathcal{A}$ *and either* $K(\alpha) = \{0, \dots, k(\alpha)\}$ *for some* $k(\alpha) \geq 0$ *or* $K(\alpha) = \mathbb{N}$.

The family $\{\Upsilon_k(\alpha)\}_{\alpha\in\mathcal{A},k\in K(\alpha)}$ *is pairwise disjoint and satisfies:*

(i) $\Upsilon_0(\alpha_\pm) \equiv \Upsilon(\alpha_\pm)$*;*
(ii) *If* $\alpha \neq \alpha_\pm$ *then* $\Upsilon_k(\alpha)$ *is a regular lift of some* $\Upsilon_j(\mathcal{R}(\alpha))$*;*
(iii) *If* $\alpha = \alpha_\pm$ *but* $k > 0$ *then* $\Upsilon_k(\alpha)$ *is a singular lift of some* $\Upsilon_j(\beta)$ *with* $\beta \in \mathcal{A}$*;*
(iv) *The union* $\bigcup_{\alpha\in\mathcal{A},k\in K(\alpha)} \Upsilon_k(\alpha)$ *is backward invariant;*
(v) *The union of the bottoms of all the tongues is an open set of full Lebesgue measure in* $\mathcal{B}$.

Proof. Lemmas 9.5, 6.1, 3.2, and Corollary 6.7 imply that the lifts of stable tongues by $\mathcal{R}$ are stable tongues. So, taking all possible lifts of the principal tongues $\Upsilon(\alpha_\pm)$ by the iterates of $\mathcal{R}$, we obtain an infinite family of stable tongues attached to each point of $\mathcal{A}$.

Since the stable manifolds are disjoint, the stable tongues with different tips are disjoint. If one of these tongues overlaps with a principal tongue $\Upsilon(\alpha_\pm)$ then it must be contained in it, by maximality of the latter. It follows that any two tongues in the family are either disjoint or nested. Keeping only the maximal tongues, we obtain the desired family of tongues.

All the properties of this family are straightforward except the last one. This one follows from ergodicity of the map $z \mapsto z^4$ with respect to the Lebesgue measure on $\mathbb{T}$, which implies that $\cup \mathcal{R}^{-n}(\mathcal{B}_{\Upsilon_\pm})$ has full measure. □

Proposition 9.8. *The family of stable tongues $\{\Upsilon_k(\alpha_+)\}_{k\in K(\alpha_+)}$ is contained in the central tongue Λ_+, and their bottoms form an open set of full Lebesgue measure in the bottom of Λ_+ $(= (\pi/4, 3\pi/4))$. Moreover, each of these tongues has positive angle at its tip. The same property is valid for α_-.*

Proof. Let us say that a topological rectangle $\Pi \subset \mathcal{C}_-$ is a *singular stable rectangle* if it is bounded by an interval on $\mathcal{B}$, an arc of $\mathcal{G}$, and two proper vertical paths in $\mathcal{C}_-$, and is foliated by stable leaves.[30] For instance, the intersection of any stable tongue $\Upsilon_k(\alpha_+)$ with $\mathcal{C}_-$ is singular stable rectangle (by Lemmas 9.5 and 6.1).

Let Π_k be the family of *maximal* singular stable rectangles. Proposition 9.7 (v) implies that their bottoms have full measure in $\mathcal{B}$.

Any singular stable rectangle lifts to a stable tongue attached to the indeterminacy points α_+ with positive angle at the tip (equal to $|\omega_1 - \omega_2|$ where $\mathcal{G}(\omega_i)$ are the upper vertices of Π). Lifting the rectangles Π_k, we obtain the desired family of tongues. □

The above discussion does not imply that there are infinitely many stable tongues $\Upsilon_k(\alpha)$ attached to each $\alpha \in \mathcal{A}$: this will be justified in the following piece.

9.6. **Long hairs growing from $\mathcal{T}$.** Let us recall dynamics on the invariant interval $\mathcal{I}_0$ (see §3.1). Let

$$\mathcal{I}_0^+ = \{(\phi, t) \in \mathcal{I}_0 :\ t > t_c\}, \quad \mathcal{I}_0^\delta = \{(\phi, t) \in \mathcal{I}_0 :\ t \geq t_c + \delta\}.$$

The interval $\mathcal{I}_0^+$ is the stable manifold of the high temperature fixed point $\beta_1 \in \mathcal{T}$. In this section it will be convenient to orient $\mathcal{I}_0^+$ so that it begins at β_1.

We say that a sequence of curves γ_k *stretch along* $\mathcal{I}_0^+$ if for any $\delta > 0$ there is k_0 such that the curves γ_k, $k \geq k_0$, contain arcs that can be represented as graphs $\phi = \gamma_k(t)$ over $\mathcal{I}_0^\delta$ such that $\gamma_k \to 0$ in $C^1(\mathcal{I}_0^\delta)$.

Let us consider a sequence of preimages $\beta_k \in \mathcal{T}$ of β_1 converging to β_1, say $\beta_k = (2\pi/2^{k-1}, 1)$. Let γ_k stand for the lift of $\mathcal{I}_0^+$ by $\mathcal{R}^k$ that begins at β_k.

Lemma 9.9. *The curves γ_k are pairwise disjoint, and the orbits of points $x \in \cup\gamma_k$ converge to β_1. Moreover, the curves γ_k stretch along $\mathcal{I}_0^+$.*

Proof. The first assertion is obvious. The last one follows from the Dynamical λ-Lemma (see [PM, pp. 80-85]) applied near the hyperbolic fixed point β_1. □

Proposition 9.10. *There are infinitely many maximal stable tongues $\Upsilon_k(\alpha_\pm)$. Each of them sticks at a positive angle out of the top $\mathcal{T}$.*

Proof. Let γ_k' be the curves γ_k translated horizontally by π. They stretch along the interval $\mathcal{I}_\pi^+ = \{(\phi, t) \in \mathcal{I}_\pi :\ t > t_c\}$ and hence (for k sufficiently big) intersect the singular curve $\mathcal{G}$ transversally near the top.

By the symmetry $\mathcal{R}(\phi + \pi) = \mathcal{R}(\phi)$ (see Property (P1)), the orbits of points $x \in \cup\gamma_k'$ converge to β_1. So, $\cup\gamma_k'$ is disjoint from the basin $\mathcal{W}^s(\mathcal{B})$, and hence the singular stable rectangles from the proof of Proposition 9.8 can meet $\mathcal{G}$ only in between the curves.

[30] We allow one of the vertical sides to degenerate to β_0, making Π degenerate to a triangle.

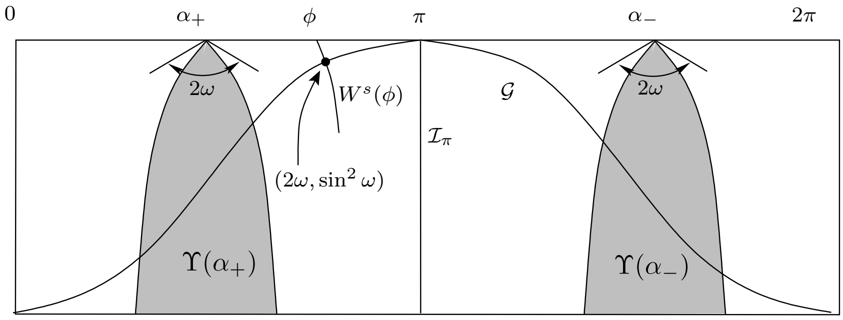


FIGURE 9.2. The primary stable tongues.

By Proposition 9.7 there is at least one maximal stable tongue attached to each α in $\mathcal{A}$, with $\mathcal{A}$ a dense subset of $\mathcal{T}$. Therefore, there is a tongue $\Upsilon(\alpha_k)$ "squeezed" in between any pair of curves γ'_k and γ'_{k+1}. Then the corresponding rectangles $\Upsilon(\alpha_k) \cap \mathcal{C}_-$ are contained in disjoint (for k big enough) maximal rectangles Π_k. The latter lift to disjoint maximal stable tongues $\Upsilon_k(\alpha_\pm)$. This proves the first assertion.

The second one follows as well since any singular stable rectangle Π is separated from $\mathcal{I}_\pi$ by some curve γ'_k, hence the corresponding tongue (the singular lift of Π) meets the top non-tangentially. □

Corollary 9.11. *For any $\alpha \in \mathcal{A}$ there are infinitely many maximal stable tongues $\Upsilon_k(\alpha)$ attached at α.*

Proof. Since there are infinitely many maximal stable tongues attached at $\alpha_\pm$, this follows from Proposition 9.7, Part (ii). □

Corollary 9.12. *The stable tongues $\Upsilon_k(\alpha_\pm)$ have angles $0 < \omega_k(\alpha_\pm) < \pi$ at their tips. All other stable tongues $\Upsilon_k(\alpha)$, $\alpha \neq \alpha_\pm$, have cusps at their tips.*[31]

Proof. The angles $\omega_k(\alpha_\pm)$ are positive by Propositions 9.6 and 9.10. Since $\sum \omega_k(\alpha_+) = \sum \omega_k(\alpha_-) \leq \pi$, each of the angles is also strictly smaller than π.

By Property (ii) of Proposition 9.7, the other stable tongues $\Upsilon_k(\alpha)$ are regular pullbacks of the $\Upsilon_k(\alpha_\pm)$. Since $\mathcal{R}$ is transversely super-attracting at $\mathcal{T}$ (away from $\alpha_\pm$) and expanding along $\mathcal{T}$, these pullbacks have cusps at the tips. □

9.7. **Regularity.**

Proposition 9.13. *$\mathcal{F}^s(\mathcal{B})$ is a C^∞ foliation of $\mathcal{W}^s(\mathcal{B})$.*

Since the leaves of $\mathcal{F}^s(\mathcal{B})$ are integral curves of the central line field $\mathcal{L}^c$, it suffices to prove the following proposition.

Proposition 9.14. *The sequence $B_n(x) := \frac{1}{4^n} D\mathcal{R}^n(x)$ converges uniformly on compact subsets of $\mathcal{W}^s(\mathcal{B})$ at super-exponential rate to a C^∞ matrix-valued function $B(x)$. Moreover, $\mathcal{L}^c(x) = \ker B(x)$.*

[31]This is a reason why the tongues do not appear to reach $\mathcal{T}$ on Figure 1.2.

Proof of Proposition 9.14. It suffices to prove the statement in any neighborhood of $\mathcal{B}$, since one can use the invariance

$$4B_n(x) = B_{n-1}(\mathcal{R}x)D\mathcal{R}(x) \text{ and } 4B(x) = B(\mathcal{R}x)D\mathcal{R}(x), \tag{9.2}$$

to extend the result to any compact subset of $\mathcal{W}^s(\mathcal{B})$. For example, $\ker B(x) = \mathcal{R}^* \ker B(\mathcal{R}x)$ follows automatically from (9.2) at the regular points of $\mathcal{R}$ and is a simple check near the critical points $\mathcal{I}_{\pm\pi/2}$.

For $x \in \mathcal{C}_*$ we have

$$t_n \le Cq^{2^n} \tag{9.3}$$

with $C > 1$ and $0 < q < 1$. It is noteworthy that $C \equiv C(\epsilon)$ and $q \equiv q(\epsilon)$ can be chosen uniformly on the region $\mathcal{C}_*^\epsilon := \{x \in \mathcal{C} \,:\, t(x) < t_c - \epsilon\}$ for any $\epsilon > 0$.

Note that $A(x) := \frac{1}{4}D\mathcal{R}(x)$ is real-analytic and, by (B.6), it satisfies

$$A(\phi, 0) = A_0 := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

It follows from (9.3) that

$$|A(\mathcal{R}^n x) - A_0| < C_0 q^{2^n} \tag{9.4}$$

for any $x \in \mathcal{C}_*^\epsilon$.

By the chain rule

$$B_n(x) = A(\mathcal{R}^{n-1}x)A(\mathcal{R}^{n-2}x) \cdots A(x).$$

Moreover, Equation (9.4) is sufficient for $B_n(x)$ to converge uniformly (and super-exponentially fast) to some continuous $B(x)$ on $\mathcal{C}_*^\epsilon$ for any ϵ. It satisfies $B(\phi, 0) = A_0$ and also $4B(x) = B(\mathcal{R}x)D\mathcal{R}(x)$.

Since $\mathcal{B}$ is superattracting, there is some forward invariant neighborhood $\mathcal{N}$ of $\mathcal{B}$ so that for any $x \in \mathcal{N}$, any $v \in \mathcal{L}^c(x)$ has its length contracted under $D\mathcal{R}$ by a definite factor, thus satisfying $v \in \ker B(x)$. Moreover, since $B(\phi, 0) = A_0$ we can trim this neighborhood, if necessary, so that $\operatorname{rank} B(x) = 1$ and thus $\mathcal{L}^c(x) = \ker B(x)$ for all $x \in \mathcal{N}$.

We now show that $B(x)$ is C^∞ in $\mathcal{C}_*$. The proof depends on the superattacting nature of $\mathcal{B}$. Let $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2)$. Then, $\mathcal{R}_2$ vanishes quadratically in t when $t = 0$, giving that for any multi-index $\boldsymbol{\beta}$ there is some $M_{\boldsymbol{\beta}}$ such that

$$|\partial_{\boldsymbol{\beta}} \mathcal{R}_2(\phi, t)| \le \begin{cases} \frac{t^2}{C^2} M_{\boldsymbol{\beta}} & \text{if } \beta_2 = 0 \\ \frac{t}{C} M_{\boldsymbol{\beta}} & \text{if } \beta_2 = 1 \\ M_{\boldsymbol{\beta}} & \text{if } \beta_2 \ge 2 \end{cases} \tag{9.5}$$

for all $(\phi, t) \in \mathcal{C}_*$. Furthermore, since $A(x) - A_0$ vanishes at $t = 0$, we have that

$$\|\partial_{\boldsymbol{\beta}} A\| < C_{\boldsymbol{\beta}}\, t \quad \text{if } \beta_2 = 0. \tag{9.6}$$

We'll first observe that $B(x)$ is C^1. It is a consequence of the following estimates

$$\|D\mathcal{R}^n(x)\| \le \lambda^n, \tag{9.7}$$

$$|\partial_x t_n| \le \mu^n q^{2^n}, \text{ and} \tag{9.8}$$

$$\|\partial_x A(\mathcal{R}^n x)\| \le C_1 \nu^n q^{2^n}. \tag{9.9}$$

for appropriate λ, μ, ν, and C_1.

The first follows a bound $\|D\mathcal{R}(x)\| \leq \lambda$ on $\mathcal{C}_*$ and the chain rule. Meanwhile (9.8) follows from induction on n, since the chain rule and (9.5) give

$$\begin{aligned} \partial_x t_n &= \partial_\phi \mathcal{R}_2(\phi_{n-1}, t_{n-1})\partial_x\phi_{n-1} + \partial_t \mathcal{R}_2(\phi_{n-1}, t_{n-1})\partial_x t_{n-1} \\ &\leq \frac{t_{n-1}^2}{C^2} M_\phi \cdot \partial_x \phi_{n-1} + \frac{t_{n-1}}{C} M_t \cdot \partial_x t_{n-1}. \end{aligned} \tag{9.10}$$

Equation (9.9) follows from similar use of the chain rule, together with (9.6), (9.8), and (9.7).

Then (9.9) is sufficient for the series

$$\partial_x B(x) = \partial_x A(x) A(\mathcal{R}x) A(\mathcal{R}^2 x)\cdots + A(x)\partial_x A(\mathcal{R}x)A(\mathcal{R}^2x)\cdots + \cdots \tag{9.11}$$

to converge uniformly on $\mathcal{C}_*^\epsilon$ for any $\epsilon > 0$.

To prove the convergence of higher derivatives of $B(x)$ we will use

Lemma 9.15. *For $x \in \mathcal{C}_*^\epsilon$ and any multi-index $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \neq 0$ we have*

$$\|\partial_{\boldsymbol{\alpha}} \mathcal{R}^n(x)\| < \lambda_{\boldsymbol{\alpha}}^n, \tag{9.12}$$

$$|\partial_{\boldsymbol{\alpha}} t_n| < \mu_{\boldsymbol{\alpha}}^n q^{2^n}, \text{ and} \tag{9.13}$$

$$\|\partial_{\boldsymbol{\alpha}} A(\mathcal{R}^n x)\| < C_{\boldsymbol{\alpha}} \nu_{\boldsymbol{\alpha}}^n q^{2^n}, \tag{9.14}$$

for suitable $\lambda_{\boldsymbol{\alpha}}, \mu_{\boldsymbol{\alpha}}, \nu_{\boldsymbol{\alpha}} > 1$ and $C_{\boldsymbol{\alpha}} > 0$.

Proof. The proof is similar to that for (9.7-9.9) except that in place of the chain rule we will use the the Faà di Bruno formula [CS] to estimate the higher partial derivatives of a composition.

If

$$h(x_1, \dots, x_d) = f(g^{(1)}(x_1, \dots, x_d), \dots, g^{(m)}(x_1, \dots, x_d))$$

and $|\boldsymbol{\alpha}| := \sum \alpha_i$, it gives:

$$\partial_{\boldsymbol{\alpha}} h = \sum_{1 \leq |\boldsymbol{\beta}| \leq |\boldsymbol{\alpha}|} \partial_{\boldsymbol{\beta}} f \sum_{s=1}^{|\boldsymbol{\alpha}|} \sum_{p_s(\boldsymbol{\alpha},\boldsymbol{\beta})} \boldsymbol{\alpha}! \prod_{j=1}^{s} \frac{(\partial_{\boldsymbol{l}_j} \boldsymbol{g})^{\boldsymbol{k}_j}}{(\boldsymbol{k}_j!)(\boldsymbol{l}_j!)^{|\boldsymbol{k}_j|}}. \tag{9.15}$$

Each $\boldsymbol{\beta}$ and $\boldsymbol{k}_j$ is a n-dimensional multi-index and each $\boldsymbol{l}_j$ is a d-dimensional multi-index. The final sum is taken over the set

$$\begin{aligned} p_s(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \{(\boldsymbol{k}_1, \dots, \boldsymbol{k}_s; \boldsymbol{l}_1, \dots, \boldsymbol{l}_s) \ : |\boldsymbol{k}_i| > 0, \\ \boldsymbol{0} \prec \boldsymbol{l}_1 \prec \cdots \prec \boldsymbol{l}_s, \sum_{i=1}^{s} \boldsymbol{k}_i = \boldsymbol{\beta} \text{ and } \sum_{i=1}^{s} |\boldsymbol{k}_i| \boldsymbol{l}_i = \boldsymbol{\alpha}.\} \end{aligned}$$

Here, $\prec$ denotes a linear order on the multi-indices (its details will not be important for us), $\boldsymbol{\eta}! := \prod \eta_i!$, and for a vector $\boldsymbol{z}$ we have $\boldsymbol{z}^{\boldsymbol{\eta}} = \prod z_i^{\eta_i}$.

We will not need the precise combinatorial details of this formula. For example, (9.12) follows directly from the existence of a polynomial expression for $\partial_{\boldsymbol{\alpha}} h$ in the partial derivatives of f and g.

Suppose that both f and g are functions of two variables. Then, all that we will need to prove (9.13) and (9.14) is that the Faà di Bruno formula gives an expression of the form

$$\partial_{\boldsymbol{\alpha}} h = \sum_i K_i \, \partial_{\boldsymbol{\beta}_i} f \prod_{j=1}^{\beta_i^1} \partial_{\boldsymbol{\gamma}_{i,j}} g^{(1)} \prod_{j=1}^{\beta_i^2} \partial_{\boldsymbol{\eta}_{i,j}} g^{(2)}, \tag{9.16}$$

having two additional properties:

(1) $1 \le \boldsymbol{\beta}_i,\ \boldsymbol{\gamma}_{i,j},\ \boldsymbol{\eta}_{i,j} \le \boldsymbol{\alpha}$ for every i, j and, if either $\boldsymbol{\gamma}_{i,j} = \boldsymbol{\alpha}$ or $\boldsymbol{\eta}_{i,j} = \boldsymbol{\alpha}$, then $|\boldsymbol{\beta}_i| = 1$; and
(2) $K_i \ge 1$ for every i.

Here, each $\boldsymbol{\beta}_i = (\beta_i^1, \beta_i^2)$.

The proof of (9.13) is done by an inductive use (9.16) similar to the usage of the chain rule for the first derivatives (9.10). It is the key step, so we'll prove it here and omit a proof of (9.14), which is simpler.

We already have (9.13) when $|\boldsymbol{\alpha}| = 1$. Therefore, can suppose that it holds for all $\boldsymbol{\beta}$ satisfying $|\boldsymbol{\beta}| < |\boldsymbol{\alpha}|$ in order to prove it for $\boldsymbol{\alpha}$. Equation (9.16) gives

$$\partial_{\boldsymbol{\alpha}} t_n = \sum_i K_i\, \partial_{\boldsymbol{\beta}_i} \mathcal{R}_2(x_{n-1}) \prod_{j=1}^{\beta_i^1} \partial_{\boldsymbol{\gamma}_{i,j}} \phi_{n-1} \prod_{j=1}^{\beta_i^2} \partial_{\boldsymbol{\eta}_{i,j}} t_{n-1}. \tag{9.17}$$

Let $\mu_{\boldsymbol{\alpha}} := \max(N_{\boldsymbol{\alpha}}, P_{\boldsymbol{\alpha}})$, where

$$N_{\boldsymbol{\alpha}} := \frac{1}{q^2} \sum_i K_i M_{\boldsymbol{\beta}_i}$$

and $P_{\boldsymbol{\alpha}}$ is the maximum of

$$\prod_{j=1}^{\beta_i^1} \lambda_{\boldsymbol{\gamma}_{i,j}} \prod_{j=1}^{\beta_i^2} \mu_{\boldsymbol{\eta}_{i,j}}$$

taken over all i in (9.17).

Equation (9.13) follows for $n = 1$, since $\mu_{\boldsymbol{\alpha}} \ge N_{\boldsymbol{\alpha}} \ge \frac{1}{q^2} M_{\boldsymbol{\alpha}} \ge \frac{1}{q^2} |\partial_{\boldsymbol{\alpha}} t_1|$.

We now suppose that (9.13) is true for the $(n-1)$-st iterate in order to prove it for the n-th iterate. By the definition of $\mu_{\boldsymbol{\alpha}}$, it suffices to show that each term in (9.17) has absolute value bounded by

$$q^{2^n} K_i M_{\boldsymbol{\beta}_i} \left(\prod_{j=1}^{\beta_i^1} \lambda_{\boldsymbol{\gamma}_{i,j}} \prod_{j=1}^{\beta_i^2} \mu_{\boldsymbol{\eta}_{i,j}} \right)^{n-1}. \tag{9.18}$$

If $\beta_i^2 \ge 2$, then the factor of q^{2^n} results from at least two factors of $|\partial_{\boldsymbol{\eta}_{i,j}} t_{n-1}|$ in the i-th term from (9.17) and the induction hypothesis. Otherwise, sufficient extra factors of $q^{2^{n-1}}$ come from from $t_{n-1} < Cq^{2^{n-1}}$ and (9.5). □

Using a generalization of the product rule, we have

$$\partial_{\boldsymbol{\alpha}} B(x) = \sum_{\boldsymbol{\alpha} = \boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1 + \cdots} \frac{\boldsymbol{\alpha}!}{\boldsymbol{\alpha}_0! \boldsymbol{\alpha}_1! \boldsymbol{\alpha}_2! \cdots} \partial_{\boldsymbol{\alpha}_0} A(x) \cdot \partial_{\boldsymbol{\alpha}_1} A(\mathcal{R}x) \cdot \partial_{\boldsymbol{\alpha}_2} A(\mathcal{R}^2 x) \cdots \tag{9.19}$$

where the sum is taken over all partitions of $\boldsymbol{\alpha}$ into a sum $\boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1 + \cdots$. We apply (9.14) to each $|\boldsymbol{\alpha}_i| \le |\boldsymbol{\alpha}|$ and take appropriate maxima to find

$$|\partial_{\boldsymbol{\alpha}_n} A(\mathcal{R}^n x)| \quad < \quad D_1 \nu^n q^{2^n} \le D_2$$

for each n and suitable $\nu > 1, D_1 > 0$, and $D_2 > 0$.

There are no more than $n^{|\boldsymbol{\alpha}|}$ partitions $\boldsymbol{\alpha} = \boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1 + \cdots$ for which n is the maximal index with $|\boldsymbol{\alpha}_i| > 0$. Each such term in the sum (9.19) can be bounded by $K\nu^n q^{2^n}$, since there are at most $|\boldsymbol{\alpha}|$ terms in the product for which $\boldsymbol{\alpha}_i \ne \mathbf{0}$, each of which is bounded by D_2, and the last one is bounded by $D_1\nu^n q^{2^n}$. Thus, we bound the sum (9.19) by

$$\sum_n n^{|\boldsymbol{\alpha}|} K\nu^n q^{2^n},$$

which is convergent.

Thus the series (9.19) for $\partial_{\boldsymbol{\alpha}} B(x)$ converges uniformly on $\mathcal{C}_*^\epsilon$ for any $\epsilon > 0$. Since the multi-index $\boldsymbol{\alpha}$ was arbitrary, we conclude that $B(x)$ is C^∞ on $\mathcal{C}_*$. □

9.8. **"Böttcher coordinate" on $\mathcal{W}^s(\mathcal{B})$ and convergence of foliations.** Since the map $\mathcal{R} : \mathcal{C} \to \mathcal{C}$ has degree 4 in the first homology of $\mathcal{C}_1$, the normalized pullback

$$\phi_1 : \mathcal{C}_1 \to \mathbb{T}, \quad \phi_1 = \frac{1}{4}\phi \circ \mathcal{R} : \mathcal{C}_1 \to \mathbb{T}$$

is a well defined map of degree 1. Iterating the pullback, we obtain a sequence of degree 1 maps

$$\phi_n : \mathcal{C}_1 \to \mathbb{T}, \quad \phi_n = \frac{1}{4^n}\phi \circ \mathcal{R}^n : \mathcal{C}_1 \to \mathbb{T}.$$

By Proposition 9.7,

$$d\phi_n(x) = d\phi(\mathcal{R}^n x) \cdot B_n(x) \to (1,0) \cdot B(x) := \omega(x), \quad x \in \mathcal{B},$$

where ω is a closed C^∞-smooth 1-form on $\mathcal{W}^s(\mathcal{B})$ with period 1. Hence $\omega = d\Phi$ where $\Phi : \mathcal{W}(\mathcal{B}) \to \mathbb{T}$ is a C^∞ map of degree 1. Moreover,

$$\Phi|\,\mathcal{B} \equiv \phi, \quad \text{and} \quad \Phi(\mathcal{R}x) = 4\Phi(x). \tag{9.20}$$

Since $d\Phi$ vanishes on the stable leaves $\mathcal{W}^s(x)$ ($x \in \mathcal{B}$) and does not vanish transversally, it is a defining function for the stable foliation.

The function Φ plays the role of the *Böttcher coordinate* on the basin of the bottom.

Instead of the angular coordinate ϕ, we can do the same construction with a more general function:

Lemma 9.16. *Let $\psi : \mathcal{C}_1 \to \mathbb{T}$ be a C^∞-map of degree l tangent to $l\phi$ at the bottom (i.e., $\psi(\phi, t) = l\phi + o(t)$ as $t \to 0$). Then*

$$\psi_n := \frac{1}{l \cdot 4^n}\psi(\mathcal{R}^n x) \to \Phi(x) \quad \textit{super-exponentially fast as } n \to \infty,$$

in the C^1-topology on compact subsets of $\mathcal{W}^s(\mathcal{B})$.

Proof. The same reason as above shows that $d\psi_n \to d\Phi$ super-exponentially fast in the C^1-topology on compact subsets of $\mathcal{W}^s(\mathcal{B})$. The assertion follows, since the ψ_n agree with ϕ on $\mathcal{B}$. □

If ψ is a defining function for some foliation $\mathcal{F}$ on $\mathcal{W}^s(\mathcal{B})$, then the ψ_n are defining functions for the pullbacks $(\mathcal{R}^n)^*(\mathcal{F})$, and Lemma 9.16 gives a strong sense in which these pullbacks converge to the stable foliation of the bottom $\mathcal{F}^s(\mathcal{B})$.

Let us finally mention that the Böttcher coordinate Φ can be extended continuously to a degree 1 map $\tilde{\Phi} : \mathcal{C}_1 \to \mathbb{T}$ satisfying Böttcher functional equation (9.20), see Remark 12.1 below.

10. High temperature dynamics: basin of the top of the cylinder

Property (P5) from §3.3 states that the top $\mathcal{T}$ of the cylinder $\mathcal{C}$ is non-uniformly superattracting. In this section we will prove that there is set of positive measure attracted to $\mathcal{T}$,

$$W^s(\mathcal{T}) = \{x \in \mathcal{C} : \ \mathcal{R}^n x \to \mathcal{T}\},$$

that supports a "stable bouquet" $\mathcal{F}^s(\mathcal{T})$ consisting of curves emanating from almost all points of $\mathcal{T}$.

Near the top, we will make use of the local coordinate τ, and near the indeterminacy points – of the local coordinates (τ, ϵ), see (P5). We say that "x is τ-*below* y" if $\tau(x) < \tau(y)$ (so, x is, in fact, above y on the cylinder $\mathcal{C}$).

Recall the neighborhoods $\mathcal{V}' \equiv \mathcal{V}'_{\bar\tau,\eta}$ of $\mathcal{T} \smallsetminus \{\alpha_\pm\}$ obtained by removing the parabolic regions $\mathcal{P}^\pm_\eta$ from the $\mathcal{V}_{\bar\tau}$ neighborhood of the top (see §6.2.) Let $q \in (0,1)$. By property (P5), if η and $\bar\tau$ are sufficiently small then

$$\tau(\mathcal{R}x) < q\,\tau(x) \quad \forall\ x \in \mathcal{V}'. \tag{10.1}$$

Let $\mathcal{W}^s_{\eta,\bar\tau}(\mathcal{T})$ be the set of points whose orbits converge to $\mathcal{T}$ while remaining in $\mathcal{V}'_{\bar\tau,\eta}$ and let $\mathcal{W}^s_\eta(\mathcal{T})$ be the set of points whose orbits eventually land in $\mathcal{V}'_{\bar\tau,\eta}$ for some $\bar\tau$ and stay there (note that this property is independent of $\bar\tau$). Then, points of $\mathcal{W}^s_\eta(\mathcal{T})$ are attracted to $\mathcal{T}$ with exponential rate $O(q^n)$. We will show below that this set supports a "stable bouquet" $\mathcal{F}^s_\eta(\mathcal{T})$ consisting of curves emanating from almost all points of $\mathcal{T}$ and that it has positive two-dimensional Lebesgue measure.

10.1. Vertical bouquet $\mathcal{F}^s(\mathcal{T})$.

A *bouquet* of curves in $\mathcal{C}$ is a family of curves that are disjoint on $\mathcal{C}_1$ (which may or may not be a lamination). The curves comprising the bouquet are called its *leaves*.

In the following proposition, horizontal and vertical curves γ are understood in the sense of the cone fields $\mathcal{K}^h$ and $\mathcal{K}^v$. Vertical curves are oriented by the local coordinate τ. Note that for sufficiently small η, the boundary of $\mathcal{V}'$ is horizontal because the tangent lines to the parabolas $\mathcal{Y}^\pm_\eta$ have slope $2\eta\epsilon$ which can be made less than $\epsilon/3$.

Proposition 10.1. *For any sufficiently small $\eta > 0$, the basin $W^s_\eta(\mathcal{T})$ supports an invariant bouquet $\mathcal{F}^s(\mathcal{T}) \equiv \mathcal{F}^s_\eta(\mathcal{T})$ by smooth vertical paths landing transversely at almost all points of $\mathcal{T}$.*

Proof. A vertical curve γ is called *semi-proper* if it begins on $\mathcal{T}$. If additionally, γ lands on the τ-upper boundary of $\mathcal{V}'$, it is called *proper*.

Let

$$\mathcal{V}'_n = \{x : \ \mathcal{R}^k x \in \mathcal{V}',\ k = 0, \dots, n-1\}.$$

Let $\gamma^0 \equiv \gamma^0_x \subset \mathcal{V}'$ be the proper genuinely vertical interval containing $x \in \mathcal{V}' \equiv \mathcal{V}'_{\bar\tau,\eta}$. For each $n > 0$ and $x \in \mathcal{V}'_n$, we will inductively construct a semi-proper vertical curve $\gamma^n = \gamma^n_x \subset \mathcal{V}'_n$ containing $x \in \mathcal{V}'_n$. Assume we have already constructed curves γ^{n-1}_y for all $y \in \mathcal{V}'_{n-1}$. Then for $x \in \mathcal{V}'_n$, we define γ^n_x as the regular lift of $\gamma^{n-1}_{\mathcal{R}x}$ truncated (if needed) by the τ-upper boundary of $\mathcal{V}'$. Since the cone field $\mathcal{K}^v$ is backward invariant, we obtain a semi-proper vertical curve. Since the boundary of $\mathcal{V}'$ is horizontal and $x \in \mathcal{V}'$, the whole curve γ^n_x is contained in $\mathcal{V}'$. Since $\gamma^{n-1}_{\mathcal{R}x} \subset \mathcal{V}'_{n-1}$, we conclude that $\gamma^n_x \subset \mathcal{V}'_n$.

Let now $x \in \mathcal{W}^s_{\eta,\bar\tau} = \bigcap \mathcal{V}'_n$, so that the curves γ^n_x exist for all $n \in \mathbb{N}$. Since each of them contains x, they all have a definite height $\tau_0 \geq \tau(x)$. Since $\mathcal{R}$ is horizontally expanding, the curves γ_x^n exponentially converge in the uniform topology to a curve γ_x containing x and of height $\geq \tau_0$. Moreover, being vertical, the curves γ_x^n are uniformly Lipschitz, so γ_x is Lipschitz as well.

The results of §7 gives that the intersections $\cap D\mathcal{R}^{-n}(\mathcal{K}^v(\mathcal{R}^n x))$ converge geometrically to the central line field $\mathcal{L}^c(x)$. Thus, γ_x is tangent to $\mathcal{L}^c(x)$ at x. Similarly, at any $y \in \gamma_x$ we have that γ_x is tangent to $\mathcal{L}^c(y)$. Since the $\mathcal{L}^c$ is a continuous line field, the entire curve γ_x is C^1.

It lands at $\mathcal{T}$ transversely since the vertical cone field $\mathcal{K}^v$ is non-degenerate on $\mathcal{T} \smallsetminus \{\alpha_\pm\}$. (Note that the curves γ_x do not land at $\alpha_\pm$ since they are vertical while $\partial\mathcal{V}'$ is horizontal.)

So, the basin $\mathcal{W}^s_{\eta,\bar{\tau}}$ supports an invariant bouquet by smooth vertical paths landing transversely at $\mathcal{T}$. Pulling it back by the dynamics, we obtain a similar bouquet supported on the whole basin $\mathcal{W}^s_\eta$. To complete the proof we need to show that the leaves of this bouquet land at almost all points of $\mathcal{T}$.

Let $\epsilon_n(\phi) = \epsilon(\mathcal{R}^n(\phi))$ (where ϕ is considered to be a point on $\mathcal{T}$). Since the doubling map $\mathcal{R} : \phi \mapsto 2\phi$ preserves the Lebesgue measure on $\mathcal{T}$, the Borel-Cantelli Lemma implies that for a.e. $\phi \in \mathcal{T}$, eventually we have: $\epsilon_n(\phi) > q^{n/4}$. Hence for a.e. $\phi \in \mathcal{T}$, there exists $c = c(\phi) > 0$ such that

$$\epsilon_n(\phi) > c\, q^{n/4}, \quad n \in \mathbb{N}. \tag{10.2}$$

Let $h(\phi) = \min\{\bar{\tau}, \eta\epsilon^2/2\}$, so that any vertical curve γ based at $\phi \in \mathcal{T}$ of τ-height $\leq h(\phi)$ is necessarily contained in $\mathcal{V}'$. (To check it, note that a vertical curve that begins at ϵ_0-distance from α_+ goes τ-below the parabola $\tau = (\epsilon_0^2 - \epsilon^2)/4$, and hence reaches the boundary parabola $\mathcal{Y}_+ = \{\tau = \eta\epsilon^2\}$ at least at τ-height $\eta\epsilon_0^2/(1+4\eta)$.)

Let us now slightly modify the above construction of semi-proper vertical curves. Let $\gamma^0_\phi \subset \mathcal{V}'$ be the proper vertical (straight) interval based at $\phi \in \mathcal{T}$. For each $n > 0$ we will inductively construct a family of semi-proper vertical curves $\gamma^n_\phi \subset \mathcal{V}'$ based at $\phi \in \mathcal{T}$. Assume we have already constructed curves γ^{n-1}_ϕ. Then we define γ^n_ϕ as the regular lift of $\gamma^{n-1}_{2\phi}$ truncated (if needed) by the τ-upper boundary of $\mathcal{V}'$. Since the cone field $\mathcal{K}^v$ is backward invariant, we obtain a family of semi-proper vertical curves.

We will now show that if ϕ satisfies (10.2) then the curves $\gamma^n = \gamma^n_\phi$ have a definite height (depending on ϕ but independent of n). Indeed, by construction, one of the curves $\mathcal{R}^k(\gamma^n)$, $k = 0, 1, \dots, n$, is proper. But the height of $\mathcal{R}^n(\gamma^n)$ is bounded by $q^n\bar{\tau}$ which is eventually smaller than

$$\frac{\eta}{2} c^2 q^{n/2} \leq \min\{\,\frac{\eta}{2}\epsilon_n^2,\ \bar{\tau}\} = h(\phi_n).$$

Hence there is $k_0 = k_0(\phi)$ (independent of n) such that all the curves $\mathcal{R}^k\gamma^n$ are not proper for $k > k_0$. It follows that one of the curves $\mathcal{R}^k\gamma^n$, $k = 0, 1, \dots, k_0$, is proper, and hence, it has a definite height. Then the same is true for the curve γ^n. This completes the proof. □

10.2. **Positive measure of $\mathcal{W}^s(\mathcal{T})$.** Let

$$\mathcal{W}^{s,o}_\eta(\mathcal{T}) := \{x \in \mathcal{W}^s_\eta(\mathcal{T}) \,:\, \text{the curve } \gamma_x \in \mathcal{F}^s_\eta \text{ containing } x \text{ extends beyond } x\}.$$

It is a completely invariant subset of $\mathcal{W}^s_\eta(\mathcal{T})$ consisting of points $x \in \mathcal{W}^s_\eta$ whose orbits $\mathcal{R}^n x$ converge to $\mathcal{T}$ within the interiors of the corresponding leaves $\gamma_{\mathcal{R}^n x}$.

Let

$$\pi : \mathcal{C} \to \mathcal{B} = \mathbb{T}, \quad (\phi, t) \mapsto \phi$$

be the natural projection onto the bottom of the cylinder.

Given a horizontal curve ξ, let $dl^h \equiv dl^h_\xi = \pi^*(d\phi)$ stand for the *horizontal length* on it, i.e., the *pullback* of the standard Lebesgue measure on $\mathbb{T}$ to ξ under π. So, if ξ projects injectively onto the horizontal axis (or equivalently, if $l^h(\xi) \le 2\pi$) then $l^h(X) = |\pi(X)|$ for any measurable set $X \subset \xi$ (where $|Y|$ stands for the Lebesgue measure of $Y \subset \mathbb{T}$). In general,

$$l^h(X) = \int |(\pi|X)^{-1}(\phi)| d\phi \le \deg(\pi|\, X)\, |\pi(X)|, \tag{10.3}$$

where $\deg(\pi|X) = \max_{\phi\in\mathbb{T}} |(\pi|X)^{-1}(\phi)|$.

By Theorem 8.1, $\mathcal{R}$ expands the horizontal length: there exists $\lambda > 1$ and $c > 0$ such that for any horizontal curve $\xi \subset \mathcal{V}'$ and any measurable set $X \subset \xi$, we have:

$$l^h(\mathcal{R}^n X) \ge c\lambda^n\, l^h(X). \tag{10.4}$$

Remark 10.1. In fact, $\lambda = 2$ but we will keep notation "λ" to distinguish it from the combinatorial appearance of "2". Note also that in region $\mathcal{V}'$ (which we are concerned with in this section) expanding property (10.4) can be easily derived from Lemma B.4.

Lemma 10.2. *For any $\eta \in (0, 1/2)$ and $\delta > 0$, there exists a threshold $\bar\tau > 0$ with the following property. Let ξ be a horizontal curve in the strip $\mathcal{V}_{\bar\tau}$ with $l^h(\xi) < 2\pi$. Then all points of ξ, except for a set of horizontal length $< \delta$, belong to $\mathcal{W}^{s,o}_\eta(\mathcal{T})$.*

Proof of Lemma 10.2: Making η and $\bar\tau$ sufficiently small, we can assume if $\gamma \subset \mathcal{V}'_{2\bar\tau,\eta}$ is any vertical curve then the vertical length[32] of $\mathcal{R}\gamma$ is at least a factor of $q < 1/4$ smaller than the vertical length γ. This slightly stronger condition than (10.1) follows from (B.7) and (B.8).

Let

(10.5)
$$\xi_n = \left\{ x \in \xi :\ |\epsilon(\mathcal{R}^k x)| \ge 2\sqrt{\frac{\bar\tau}{\eta}} \cdot 2^{-k} \text{ for } 0 \le k \le n-1 \text{ and } \ |\epsilon(\mathcal{R}^n x)| < 2\sqrt{\frac{\bar\tau}{\eta}} \cdot 2^{-n} \right\},$$

and let $X_n = \mathcal{R}^n(\xi_n)$. Note that the sets ξ_n are pairwise disjoint.

Using (10.1), one can inductively show that $\xi \smallsetminus \cup\xi_n \subset \mathcal{W}^s_\eta(\mathcal{T})$. Let us now estimate $l^h(\cup\xi_n)$. By the definition of X_n

$$|\pi(X_n)| \le 8\sqrt{\bar\tau/\eta}\; 2^{-n}.$$

Making use of (10.3), we obtain:

$$l^h(X_n) \le 8\sqrt{\bar\tau/\eta}\, \deg(\pi|\, X_n)\, 2^{-n}.$$

Together with (10.4), this implies

$$l^h(\xi_n) \le 8\sqrt{\bar\tau/\eta}\, \deg(\pi|\, X_n)\, 2^{-n}\, c^{-1}\lambda^{-n}.$$

We will show that

$$\deg(\pi|\, X_n) \le 2^n. \tag{10.6}$$

[32]The vertical length of $\mathcal{R}\gamma$ is defined as the total length of its projection onto the vertical interval $\mathcal{I}_0$.

Indeed, in this case $l^h(\xi_n(x)) \le 8\sqrt{\bar\tau/\eta}\, c^{-1}\lambda^{-n}$, hence

$$l^h(\cup\xi_n) = \sum_{n=0}^{\infty} l^h(\xi_n(x)) \le 8\sqrt{\frac{\bar\tau}{\eta}}\,\frac{\lambda}{c(\lambda-1)}, \tag{10.7}$$

which can be made arbitrarily small if $\bar\tau$ is selected small enough.

Let us prove (10.6). For $x \in \xi_n$, let $x_k = \mathcal{R}^k x$, and let $\phi = \pi(x_n)$. Let $\gamma_n \equiv \gamma_n(x) \subset \mathcal{I}_\phi$ be the genuine vertical interval connecting x_n to $\mathcal{T}$, and let $\gamma_k \equiv \gamma_k(x)$ be its lifts that connect x_k to $\mathcal{T}$. Since $x_k \in \mathcal{V}'$ for $k < n$, and the region $\mathcal{V}'$ lies above the tongues $\Lambda_\pm$ (defined by Property P6 and Figure 3.3), the lifts γ_k are regular, i.e., they lie in the regular lifts I_k^j of the interval $\mathcal{I}_\phi$ (where the lift I_k^j terminates at the point $(\phi + 2\pi j)/2^k$ of $\mathcal{T}$, $j = 0, 1, \dots, 2^k - 1$). But each lift I_k^j is vertical since the vertical cone field $\mathcal{K}^v$ is backward invariant. Hence each I_k^j crosses the horizontal curve ξ at most once. Hence, given $\phi = \pi(x_n)$, there are at most 2^n points $x \in \xi_n$ such that $\pi(x_n) = \phi$, and (10.6) follows.

It remains to prove that $\xi \smallsetminus \cup\xi_n$ is actually a subset of $\mathcal{W}^{s,o}_\eta(\mathcal{T})$. We will show that the leaf $\gamma_x \in \mathcal{F}^s_\eta(\mathcal{T})$ through any $x \in \xi \smallsetminus \cup\xi_n$ extends beyond x by a definite amount. It suffices to verify that this holds for each of the curves γ_x^n used in the proof of Proposition 10.1 to construct γ_x.

There is a constant $K > 0$ so that if $\tau(x) \le \bar\tau$ and

$$|\epsilon(x)| \ge 2\sqrt{\frac{\tau(x)}{\eta}},$$

then any proper vertical curve γ in $\mathcal{V}'_{2\bar\tau,\eta}$ containing x extends beyond x by at least $K|\epsilon(x)|^2$.

To see it, let $\tilde x$ be the point where γ reaches the τ-upper boundary of $\mathcal{V}'_{2\bar\tau,\eta}$. If $\tau(\tilde x) = 2\bar\tau$, then we are done. So, we can suppose that $x = (\epsilon, \tau)$ and $\tilde x = (\tilde\epsilon, \tilde\tau)$ are near α_+ and $\epsilon, \tilde\epsilon > 0$. If $\tilde\epsilon \ge 3/4\epsilon$, then $\tilde\tau \ge (9/16)\eta\epsilon^2$ so that $\tilde\tau - \tau \ge (5/16)\eta\epsilon^2$. Otherwise, $\tilde\epsilon \le 3/4\epsilon$. Since the vertical cones $\mathcal{K}^v(x)$ have slope $d\tau/d\epsilon \ge \epsilon(x)/3$ this forces that $\tilde\tau - \tau \ge (1/16)\epsilon^2$.

For any $x \in \xi \smallsetminus \cup\xi_n$, consider the orbit $x_i = \mathcal{R}^i x$. By (10.5), one finds that

$$|\epsilon(x_i)| \ge 2\sqrt{\frac{\bar\tau}{\eta}} \cdot 2^{-i} \ge 2\sqrt{\frac{\tau(x_i)}{\eta}}, \tag{10.8}$$

so that any proper vertical curve in $\mathcal{V}'_{2\bar\tau,\eta}$ through x_i extends beyond x_i by at least $K|\epsilon(x_i)|^2 \ge 4^{-i}K\bar\tau/\eta$.

We now inductively prove that for every n the curve $\gamma^n_{x_i}$ extends beyond x_i by at least $4^{-i}K\bar\tau/\eta$. This holds when $n = 0$, since $\gamma^0_{x_i}$ is the proper genuinely vertical interval in $\mathcal{V}'_{2\bar\tau,\eta}$ through x_i. Now suppose that it is true at step n in order to check for step $n+1$. Suppose that some $\gamma^{n+1}_{x_i}$ extends beyond x_i by less than $4^{-i}K\bar\tau/\eta$. It implies that $\gamma^{n+1}_{x_i}$ is not proper, hence $\mathcal{R} : \gamma^{n+1}_{x_i} \to \gamma^n_{x_{i+1}}$ is a homeomorphism. Since $\mathcal{R}$ contracts the vertical lengths of vertical curves in $\mathcal{V}'_{2\bar\tau,\eta}$ by at least $q < 1/4$, this would then imply that $\gamma^n_{x_{i+1}}$ extends beyond x_{i+1} by less than $4^{-(i+1)}K\bar\tau/\eta$, contradicting the induction hypothesis.

In particular, each of the curves $\gamma^n_x \equiv \gamma^n_{x_0}$ extends beyond x by at least $K\bar\tau/\eta$, implying that γ_x does as well.

□

Remark 10.2. In the above proof we could replace the 2^{-n} in (10.5) with σ^{-n} for any $\sigma > 1$, so long as we choose η sufficiently small that $q < \sigma^{-2}$, allowing (10.8) (with 2^{-i} replaced by σ^{-i}) to hold. In particular, choosing $\sigma = 4$ would let us use the more obvious degree bound by 4^n instead of the more delicate (10.6).

Corollary 10.3. *For any $\eta \in (0, 1/2)$ and $\delta > 0$, there exist a threshold $\bar\tau > 0$ such that*

$$\operatorname{area} \mathcal{W}^s_{\eta,\bar\tau}(\mathcal{T}) \geq (1-\delta) \operatorname{area} \mathcal{V}_{\bar\tau}.$$

Moreover, all the points in $\mathcal{W}^s_{\eta,\bar\tau}$ get attracted to the top of the cylinder exponentially at rate $O(q^n)$, where $q = q(\eta) \to 0$ as $\eta \to 0$.

Proof. For each $\tau \in (0, \bar\tau)$ let $\xi_\tau = \{x : \ \tau(x) = \tau\}$. Applying Lemma 10.2, we find a subset of ξ_τ of horizontal measure $> (2\pi - \delta)$ that converges to $\mathcal{T}$ exponentially at rate $O(q^n)$. An application of the Fubini Theorem completes the proof. □

Corollary 10.4. *For any $t > t_c$, the Lebesgue measure of $\mathcal{W}^s(\mathcal{T}) \cap \mathbb{T}_t$ is positive. Moreover, as $t \to 1$, the Lebesgue measure of $\mathcal{W}^s(\mathcal{T}) \cap \mathbb{T}_t$ tends to 2π.*

Proof. The point $(0, t) \in \mathbb{T}_t$ is in $\mathcal{W}^s(\beta_1)$, so, after some finite number of iterates, the horizontal curve $\mathcal{R}^n(\mathbb{T}_t)$ contains a point arbitrarily close to $\beta_1 \in \mathcal{V}'_{\bar\tau,\eta}$. Lemma 10.2 gives that $\mathcal{R}^n(\mathbb{T}_t) \cap \mathcal{W}^s(\mathcal{T})$ has positive horizontal measure of $\mathcal{R}^n(\mathbb{T}_t)$. Since $\mathcal{W}^s(\mathcal{T})$ is backward invariant, $\mathbb{T}_t \cap \mathcal{W}^s(\mathcal{T})$ also has positive horizontal measure in $\mathbb{T}_t$.

The statement about the horizontal measure of $\mathcal{W}^s(\mathcal{T}) \cap \mathbb{T}_t$ tending to 2π as $t \to 1$ follows immediately from Lemma 10.2. □

The proof of Lemma 10.2 actually gives a slightly better statement, which we record here for later use.

For a curve ξ, let $\tau(\xi) = \sup_{x \in \xi} \tau(x)$.

Lemma 10.5. *For any $\eta \in (0, 1/2)$, there exist $\bar\tau > 0$ and $C > 1$ such that $\mathcal{W}^{s,o}_\eta(\mathcal{T})$ forms at least $3/4$ of the horizontal length of any horizontal curve ξ with $\tau(\xi) \leq \bar\tau$ and $l^h(\xi) \geq C\sqrt{\tau(\xi)}$.*

Proof. By (10.7),

$$l^h(\xi \smallsetminus \mathcal{W}^s_\eta(\mathcal{T})) \leq \frac{8}{\sqrt{\eta}} \frac{\lambda}{c(\lambda - 1)} \sqrt{\tau(\xi)} \leq \frac{1}{4} C \sqrt{\tau(\xi)},$$

where we let $C = \dfrac{32}{\sqrt{\eta}} \dfrac{\lambda}{c(\lambda-1)}$. □

11. Intertwined basins of attraction

Recall the sets $\mathcal{W}^{s,o}_\eta(\mathcal{T})$ from §10.2. We let

$$\mathcal{W}^{s,o}_0(\mathcal{T}) := \bigcap_{\eta > 0} \mathcal{W}^{s,o}_\eta(\mathcal{T}). \tag{11.1}$$

Note that $\mathcal{W}^{s,o}_0(\mathcal{T})$ is a completely invariant set whose orbits get attracted to $\mathcal{T}$ superexponentially fast.

In this section we will prove the following result:

Theorem 11.1. *The union of the basins $\mathcal{W}^s(\mathcal{B})$ and $\mathcal{W}^{s,o}_0(\mathcal{T})$ is a set of full area in the cylinder $\mathcal{C}$.*

Together with Lemma B.4 we find

Corollary 11.2. *Almost every point in* $\mathcal{W}^s(\mathcal{T})$ *has characteristic exponent* $\log 2$.

11.1. **Distortion control.** If I and J are intervals, the *distortion* of a C^1 diffeomorphism $f : I \to J$ is given by

$$\mathrm{Dist}(f, I) := \sup_{x,y \in I} \log \left| \frac{f'(x)}{f'(y)} \right| . \tag{11.2}$$

The key fact we will use is that a bound on the distortion of $f : I \to J$ gives a bound relating $m(f(S))/m(J)$ to $m(S)/I$ for any measurable set $S \subset I$. (Here, m denotes the Lebesgue measure.) We refer the reader to [PM] for more background on distortion.

We will apply distortion control to horizontal curves. If ξ is a horizontal curve, we interpret (11.2) according to the parameterizations of ξ and $\mathcal{R}^n(\xi)$ by the horizontal coordinate ϕ.

To prove Theorem 11.1, we will construct a family of horizontal curves on which $\mathcal{R}$ is expanding with bounded distortion. Without the indeterminacy points, this would be straightforward from partial hyperbolicity.

We will remove the union of two wedges extending downward from $\alpha_\pm$:

$$\Delta_{\bar\kappa} = \{x \in \mathcal{C} \,:\, \tau(x) \geq \bar\kappa |\epsilon(x)|\}.$$

Lemma 11.3. *Given any* $\bar\kappa > 0$, *there is a family* $\mathcal{H}_x$ *of "admissible" horizontal curves centered each* $x \in \mathcal{C}$ *with the following property:*

If the orbit of $x \in \mathcal{C} \setminus \{\alpha_\pm\}$ *avoids the wedge regions* $\Delta_{\bar\kappa}$, *then there is a neighborhood* $\mathcal{U}$ *of* $\alpha_\pm$ *and a sequence of times* $n_i \equiv n_i(x) \in \mathbb{Z}_+$ *such that* $\mathcal{R}^{n_i} x$ *remains outside of* $\mathcal{U}$ *and for any curve* $\xi \in \mathcal{H}_x$ *we have:*

(i) $\xi \in \mathcal{H}_x$ *projects onto the horizontal interval of radius* $r(x) \asymp \mathrm{dist}^h(x, \{\alpha_\pm\})$ *centered at* x*;*

(ii) *The image* $R^{n_i}\xi$ *overflows some curve* $\eta_i \in \mathcal{H}_{R^{n_i}x}$*;*

(iii) *If* $\widetilde{\eta}_i$ *is the restriction of* η_i *to one half of its radius, then, the inverse branch* $R^{-n_i} : \widetilde{\eta}_i \to \xi$ *is uniformly exponentially contracting with bounded distortion (with the contracting factor going to* 0 *as* $n_i \to \infty$*).*

Moreover, the genuinely horizontal intervals

$$\{J_x = (\phi, t) : \ t = t(x), \ |\phi - \phi(x)| < r(x)\}$$

are admissible.

11.2. **Proof of Theorem 11.1.** Let us first derive Theorem 11.1 from Lemma 11.3.

Take a small $\eta \in (0, 1/2)$ and let X_η be the complement of $\mathcal{W}^s(\mathcal{B}) \cup \mathcal{W}^{s,o}_\eta(\mathcal{T})$. Assume $\mathrm{area}(X_\eta) > 0$. By the Lebesgue and Fubini Theorems, there is a point $x \in X_\eta$ which is a density point for the slice of X_η by any genuinely horizontal interval $J = J_x$ centered at x.

By Proposition 9.6, we can choose $\bar\kappa$ sufficiently large so that the wedge regions $\Delta_{\bar\kappa}$ are entirely contained in $\mathcal{W}^s(\mathcal{B})$. Since $x \notin \mathcal{W}^s(\mathcal{B})$, the orbit $x_n := \mathcal{R}^n x$ avoids $\Delta_{\bar\kappa}$, allowing us to apply Lemma 11.3.

Let $S \subset \mathbb{N}$ be the subsequence of times $n_i \equiv n_i(x)$ given by Lemma 11.3. We can choose a further subsequence $S' \subset S$ along which x_n converges to some $y \in \mathcal{C} \setminus \{\alpha_\pm\}$. We will keep the same notation n_i for this subsequence.

Now, let J be the genuinely horizontal interval radius $r(x)$. Since J is admissible, for any $n_i \in S'$ there is an admissible curve $\eta_i \in \mathcal{H}_{x_{n_i}}$ centered at x_{n_i} such that the inverse branch $R^{-n_i} : \tilde{\eta}_i \to J$ is contracting (exponentially in n_i) with bounded distortion.

Suppose $y \notin \mathcal{T}$. Then, the curves $\tilde{\eta}_i$ are part of a compact family of curves, each of which intersects $\mathcal{W}^s(\mathcal{B})$ in a dense open set, by Lemma 9.1. This implies that $\mathcal{W}^s(\mathcal{B})$ occupies a definite portion of each $\tilde{\eta}_i$. Since $R^{-n_i} : \tilde{\eta}_i \to J$ has a bounded distortion, the basin $\mathcal{W}^s(\mathcal{B})$ occupies a definite portion of $R^{-n_i}(\tilde{\eta}_i)$, which can be made an arbitrarily small neighborhood of $x \in J$ by taking n_i sufficiently large. This contradicts the choice of x as a density point of X_η on J.

If $y \in \mathcal{T}$, then let us consider $\bar{\tau} > 0$ and $C > 1$ from Lemma 10.5. Since $\tilde{\eta}_i$ is a horizontal curve (with respect to the algebraic cone field $\mathcal{K}^{ah}$), the horizontal length of $\gamma := \tilde{\eta}_i \cap \mathcal{V}_{\bar{\tau}}$ is at least $\sqrt{\bar{\tau}}$ for $n_i \in S'$ sufficiently big, since it lies above one of the parabolas $\mathcal{S}_\psi$. Since γ is near $\mathcal{T}$ and bounded away from $\alpha_\pm$, Lemma B.4 gives that the horizontal length of $\gamma' := \mathcal{R}\gamma$ is at least that big. But $\tau(\gamma') = O(\bar{\tau}^2)$, so $l^h(\gamma') \geq C\sqrt{\tau(\gamma')}$. By Lemma 10.5, $\mathcal{W}^{s,o}_\eta(\mathcal{T})$ occupies at least 3/4 of γ'.

Since γ is bounded away from $\alpha_\pm$, the single iterate $\mathcal{R} : \gamma \to \gamma'$ has bounded distortion (as does its inverse). Therefore, $\mathcal{R}^{-(n_i+1)} : \gamma' \to J$ is exponentially contracting with bounded distortion. Hence, by taking sufficiently large i, the basin $\mathcal{W}^{s,o}_\eta(\mathcal{T})$ occupies a definite portion of the arbitrarily small neighborhoods $R^{-(n+1)}(\gamma') \subset J$ of x, contradicting again the choice of x.

The contradictions show that $\text{area}(X_\eta) = 0$ for any $\eta > 0$, and the conclusion follows. □

11.3. **Proof of Lemma 11.3.** Let us formulate a stronger, complex version of Lemma 11.3. Here "horizontal holomorphic curves" are understood in the sense of the complex extension of the horizontal cone field $\mathcal{K}^{ah}$ constructed in Appendix C.

Let $\pi(z, w) = z$.

Lemma 11.4. *Given any $\bar{\kappa} > 0$, there is a family $\mathcal{H}_x$ of "admissible" horizontal holomorphic curves centered at each $x \in \mathcal{C}$ with the following property:*

If the orbit of $x \in \mathcal{C} \smallsetminus \{\alpha_\pm\}$ avoids the wedge regions $\Delta_{\bar{\kappa}}$, then there is a neighborhood $\mathcal{U}$ of $\alpha_\pm$ and a sequence of times $n_i \equiv n_i(x) \in \mathbb{Z}_+$ such that $\mathcal{R}^{n_i}x$ remains outside of $\mathcal{U}$ and for any curve $\xi \in \mathcal{H}_x$ we have:

(i) *$\xi \in \mathcal{H}_x$ projects under π on a complex disc of radius $r(x) \asymp \text{dist}^h(x, \{\alpha_\pm\})$ centered at $\pi(x)$.*

(ii) *The image $R^{n_i}\xi$ overflows some curve $\eta_i \in \mathcal{H}_{R^{n_i}x}$;*

(iii) *If $\widetilde{\eta}_i$ is the restriction of η_i to one half of its radius, then, the inverse branch $R^{-n_i} : \widetilde{\eta}_i \to \xi$ is uniformly exponentially contracting with bounded distortion (with the contracting factor going to 0 as $n_i \to \infty$).*

Moreover, the genuinely horizontal discs

$$\{D_x = (\phi, t) : \; t = t(x), \; |\phi - \phi(x)| < r(x)\}.$$

are admissible.

Proof. If $x \in \mathcal{C} \smallsetminus \mathcal{U}$, then $\mathcal{H}_x$ will consist of all horizontal holomorphic curves ξ that project under π onto the round disc of constant radius $r(x) = r_1$ (to be specified below) centered at $\pi(x)$. If $x \in \mathcal{U}$, then $\mathcal{H}_x$ will consist of the restrictions to half their radius of all horizontal holomorphic curves that project under π onto a round disc of radius $a|\epsilon(x)|$, where $a > 0$ will be specified below.

Since $\mathcal{R}$ is horizontally expanding (Theorem 8.1), there is an N such that $D\mathcal{R}^N$ expands horizontal vectors $v \in \mathcal{K}^{ah}(x)$, $x \in \mathcal{C} \setminus \{\alpha_\pm\}$, by a definite factor.

We can choose a neighborhood $\mathcal{U} \subset \mathcal{C}$ of $\{\alpha_\pm\}$ sufficiently small so that each of the preimages $\mathcal{R}^{-i}\mathcal{U}$ for $1 \leq i \leq N$ is in the neighborhood $\mathcal{V}'$ of $\mathcal{T}$ in which Lemma B.4 gives that each iterate of $\mathcal{R}$ expands horizontal vectors. Therefore, if $x \in \mathcal{C} \setminus \mathcal{U}$ there exists $n(x) \leq N$ so that $D\mathcal{R}^{n(x)}$ expands any $v \in \mathcal{K}^{ah}(x)$ and $\mathcal{R}^i x \notin \mathcal{U}$ for $0 \leq i < n(x)$.

Proposition C.4 gives $C > 0$ so for any sufficiently small $a > 0$ and any $x \in \mathcal{U}$ with $|\kappa(x)| < \bar{\kappa}$ there is an iterate $n(x)$ so that if ξ is an horizontal holomorphic curve centered at x of radius $a|\epsilon(x)|$ and ξ_0 is the subdisc of radius $a|\epsilon(x)|/2$, then $\mathcal{R}^{n(x)}\xi_0$ is horizontal and projects under π onto a disc has a definite radius $\geq Ca$. We further restrict $\mathcal{U}$ so that $|\epsilon(x)| < C$ for any $x \in \mathcal{U}$. It ensures that $\mathcal{R}^{n(x)}\xi_0$ will be larger than any curve ξ of radius $a|\epsilon(x)|$ that is based at any $x \in \mathcal{U}$.

Since the cone-field $\mathcal{K}^{ah}$ is defined in a definite complex neighborhood of $\mathcal{C} \setminus \mathcal{U}$, on which $D\mathcal{R}$ has bounded expansion, we can choose r_0 sufficiently small for any horizontal holomorphic curve ξ centered at $x \in \mathcal{C} \setminus \mathcal{U}$ of radius $\leq r_0$ we have that $\mathcal{R}^i\xi$ is in the domain of definition of $\mathcal{K}^{ah}$ for $1 \leq i < n(x)$. In particular, $\mathcal{R}^{n(x)}\xi$ will be horizontal. By continuity, we can also require that r_0 be sufficiently small so that $\mathcal{R}^{n(x)}$ is uniformly expanding on any such curve ξ.

If we choose a sufficiently small so that $C \cdot a < r_0$ and choose $r_1 = C \cdot a$, it will guarantee the overflowing property (ii).

The above gives a sequence of further times $n_i(x)$ and curves $\eta_i \in \mathcal{H}_{\mathcal{R}^{n_i(x)}x}$ so that $\mathcal{R}^{n_{i+1}-n_i}\eta_i$ is horizontal and overflows η_{i+1}. Consequently, the inverse $\mathcal{R}^{-n_i(x)}\eta_i \to \xi$ is well-defined (and hence univalent) for each i.

The Koebe Distortion Theorem gives that the restriction $\mathcal{R}^{-n_i(x)}\ \widetilde{\eta}_i \to \xi$ of each inverse branch to the disc of half the radius will have bounded distortion. By construction, $D\mathcal{R}^{n_i(x)}$ is exponentially expanding at the center of ξ, therefore the inverse branch is exponentially contracting.

It follows from the proof of Proposition C.4 that if $\mathcal{R}^{n_i(x)}x \in \mathcal{U}$, then $\mathcal{R}^{n_{i+1}(x)}x \notin \mathcal{U}$. Therefore, passing to a subsequence, we can suppose that $\mathcal{R}^{n_i(x)}x \notin \mathcal{U}$ for each i. □

Remark 11.1. The whole proof of Lemma 11.4 goes through in the purely real way except one problematic point: the distortion control in (iii). In fact, with some extra work it should be possible to do it as well, using the property that the horizontal non-linearity of $\mathcal{R}$ behaves like $1/\epsilon$ near the indeterminacy points $\alpha_\pm$.

12. Central foliation, its holonomy and transverse measure

In what follows all laminations in question will be assumed strictly vertical. Given a lamination $\mathcal{F}$ and $\tau \in (0, 1)$, we let $\mathcal{F}_\tau$ be the slice of $\mathcal{F}$ by the truncated cylinder $\mathcal{C}_\tau = \mathbb{T} \times [0, 1 - \tau]$.

12.1. Central foliation.

Recall that *central foliation* is a strictly vertical foliation invariant under $\mathcal{R}^*$.

Theorem 12.1. *The map $\mathcal{R}$ has a unique central foliation.*

Proof. According to Proposition 7.7, through each $x \in \mathcal{C} \smallsetminus \{\alpha_\pm\}$ is a central curve extending in both directions to the boundary of the cylinder. Taking the union of all such curves through every point on $\mathcal{C}_1$, we obtain an invariant family $\mathcal{F}^c$ of strictly vertical C^1-curves filling in the whole cylinder $\mathcal{C}_1$.

However, for a continuous vector field, there may exist many integral curves through a given point, so $\mathcal{F}^c$ may fail to be a foliation. What saves the day is that $\mathcal{R}$ is horizontally expanding (Theorem 8.1). Namely, assume that there exist two integral curves, γ_1 and γ_2, through a point $x \in \mathcal{C}_1$. Let us take two points $y_i \in \gamma_i$ on the same height, and connect them with a (genuinely) horizontal interval $\delta = [y_1, y_2]$. By Theorem 8.1, the curves $\delta_n := \mathcal{R}^n(\delta)$ are almost horizontal, and $l^h(\delta_n) \to \infty$. Hence the horizontal projections of the δ_n eventually cover the whole circle $\mathbb{T}$, and in particular, they intersect the critical interval $\mathcal{I}_{\pi/2}$. Hence δ intersects $(\mathcal{R}^n)^*(\mathcal{I}_{\pi/2})$ at some point y.

But the interval $\mathcal{I}_{\pi/2} \smallsetminus \{\alpha_+\}$ is contained in the basin of $\mathcal{B}$, where $\mathcal{F}^c$ coincides with the stable foliation $\mathcal{F}^s(\mathcal{B})$, which is a C^∞ foliation (Proposition 9.13). Pulling it back, we conclude that there is a unique integral curve $\mathcal{W}^c(y)$ through y, and $\mathcal{F}^c$ is a C^∞ foliation near it. But $\mathcal{W}^c(y)$ is squeezed in between the curves γ_1 and γ_2, and hence must merge with them at x – contradiction.

This proves that the line field $\mathcal{L}^c$ is uniquely integrable, so the family $\mathcal{F}^c$ of all integral curves forms the central foliation. Since any such foliation must be formed by integral curves to $\mathcal{L}^c$, it is unique. □

Recall from the Introduction that the *holonomy* transformations $g_t : \mathcal{B} \to \mathbb{T}_t \equiv \mathbb{T} \times \{t\},\ t \in [0,1)$, are defined by the property that x and $g_t(x)$ belong to the leaf of $\mathcal{F}^c$.

Remark 12.1. By means of the holonomy along the central foliation to the bottom of $\mathcal{C}$, we can now obtain a continuous extension $\tilde{\Phi} : \mathcal{C}_1 \to \mathbb{T}$ of the Böttcher coordinate that was constructed in §9.8 (namely, let $\tilde{\Phi}(\phi, t) = g_t^{-1}(\phi)$). Since $\mathcal{F}^c$ is $\mathcal{R}$-invariant, $\tilde{\Phi}$ also satisfies the Böttcher functional equation $\tilde{\Phi}(\mathcal{R}(\phi, t)) = 4\tilde{\Phi}(\phi, t)$. However, $\tilde{\Phi}$ has weak regularity outside of $\mathcal{W}^s(\mathcal{B})$: for example, it is not even absolutely continuous (see Corollary 12.10 below).

12.2. **Central tongues.** Recall a notion of a tongue from §9.5. A tongue is called *central* if it is bounded by two central leaves (and hence it is saturated by intermediate leaves of the central foliation $\mathcal{F}^c$). For instance, the primary central tongues $\Lambda_\pm$ from Property (P6) in §3.3 are central tongues in this sense (as they are bounded by $\mathcal{R}$-lifts of $\mathcal{I}_\pi$, which are central leaves).

Recall also the pre-indeterminacy set $\mathcal{A} = \cup \mathcal{A}^n$ (9.1).

Proposition 12.2. *There is one maximal central tongue $\Lambda(\alpha)$ attached to each pre-indeterminacy point $\alpha \in \mathcal{A}^n$, which is the regular pullback of one of the tongues $\Lambda_\pm \equiv \Lambda(\alpha_\pm)$ by $\mathcal{R}^n$. This family of tongues is dense in $\mathcal{C}$, and any central tongue is contained in one of these.*

Proof. Obviously, regular lifts of central tongues are central tongues. Lifting the $\Lambda_\pm$, we obtain a family of central tongues $\Lambda(\alpha)$ attached to points $\alpha \in \mathcal{A}$. These tongues are pairwise disjoint as they are attached to different points of $\mathcal{T}$. Moreover, a central tongue $\Lambda(\alpha)$ with $\alpha \in \mathcal{A}^n$ has the bottom of length $\pi/2^{2n+1}$. Since

$|\mathcal{A}^n| = 2^{n+1}$, we have:

$$\sum |\mathcal{B}_{\Lambda(\alpha)}| = \sum_{n=0}^{\infty} \frac{\pi}{2^n} = 2\pi,$$

so the bottoms of these tongues have full measure in $\mathcal{B}$. Hence this family of tongues is dense in $\mathcal{C}$. So, there is no room for enlarging these tongues, nor for fitting any extra tongue in between. $\square$

12.3. **Orbits of typical leaves.** Given $x \in \mathcal{C}_1$, for each $n \geq 0$ let $\gamma_n(x) \in \mathcal{F}^c$ be the leaf containing $\mathcal{R}^n x$. Invariance of $\mathcal{F}^c$ gives $\mathcal{R}\gamma_n(x) \subset \gamma_{n+1}(x)$. Note that we can have $\mathcal{R}\gamma_n(x) \subsetneq \gamma_{n+1}(x)$ if $\gamma_n(x)$ meets $\mathcal{T}$ at $\alpha_\pm$.

Proposition 12.3. *For almost every $x \in \mathcal{C}$ has either:*

(i) *there exists $N \geq 0$ so that for all $n \geq N$: $\gamma_n(x)$ meets $\mathcal{T}$ away from $\alpha_\pm$, $\mathcal{R}\gamma_n(x) = \gamma_{n+1}(x)$, there are non-trivial intervals of points on γ_n converging superexponentially to $\mathcal{B}$ and to $\mathcal{T}$, or*

(ii) *$\gamma_n(x) \subset \mathcal{W}^s(\mathcal{B})$ for all $n \geq 0$.*

Proof. According to Theorem 11.1, almost every $x \in \mathcal{C}$ is in $\mathcal{W}^s(\mathcal{B}) \cup \mathcal{W}_0^{s,o}(\mathcal{T})$. It will be helpful later if we also assume that $x \notin \mathcal{T}$.

If $x \in \mathcal{W}_0^{s,o}(\mathcal{T})$, then for any $\bar{\tau}, \eta > 0$ there exists $N \geq 0$ so that $\mathcal{R}^n x \in \mathcal{W}^s_{\bar{\tau},\eta}(\mathcal{T})$ for all $n \geq N$. (See §10.2.) For such n, some portion of γ_n agrees with a curve from the stable lamination of $\mathcal{W}^s_{\bar{\tau},\eta}(\mathcal{T})$ constructed in the proof of Proposition 10.1. In particular, γ_n meets $\mathcal{T}$ away from $\alpha_\pm$, and hence $\mathcal{R}(\gamma_n) = \gamma_{n+1}$.

Any leaf of $\mathcal{F}^c$ intersects $\mathcal{W}^s(\mathcal{B})$ in a non-trivial interval. Meanwhile, $\mathcal{R}^n x$ has orbit superattracted to $\mathcal{T}$, so all points on γ_n above $\mathcal{R}^n x \notin \mathcal{T}$ have orbits that converge superexponentially to $\mathcal{T}$.

Now suppose $x \in \mathcal{W}^s(\mathcal{B})$. Let $\mathcal{B}_0 = \cup \mathcal{R}^{-n}(\mathcal{B}_{\Upsilon_\pm})$, where $\mathcal{B}_{\Upsilon_\pm}$ are the bottoms of the primary stable tongues. By Proposition 9.7, $\mathcal{B}_0$ has full Lebesgue measure. Since $z \mapsto z^4$ preserves Lebesgue measure, the Poincaré Recurrence Theorem gives that the set of points $\mathcal{B}_0' \subset \mathcal{B}_0$ whose orbits return infinitely many times to $\mathcal{B}_0$ has full Lebesgue measure in $\mathcal{B}_0$. Hence it also has full measure in $\mathcal{B}$.

For any $x \in \mathcal{B}_0'$, there is an infinite sequence of times n_i for which $\mathcal{R}^{n_i} x \in \mathcal{B}_0$, implying $\gamma_{n_i}(x) \subset \mathcal{W}^s(\mathcal{B})$. Then, because of the invariance $\mathcal{R}(\gamma_n(x)) \subset \gamma_{n+1}(x)$, we have $\gamma_n(x) \subset \mathcal{W}^s(\mathcal{B})$ for all n.

Therefore, it suffices to prove that almost every point of $\mathcal{W}^s(\mathcal{B})$ is in $\bigcup_{x \in \mathcal{B}_0'} \gamma_0(x)$. However, this follows since $\mathcal{F}^c$ is C^∞ on $\mathcal{W}^s(\mathcal{B})$ and that $\mathcal{B}_0'$ has full Lebesgue measure in $\mathcal{B}$. $\square$

12.4. **Convergence.** Given a metric space M, the *Hausdorff metric* on the space of closed subsets of M is defined as follows: $\mathrm{dist}_H(X, Y)$ is the infimum of $\epsilon > 0$ such that X is contained in the ϵ-neighborhood of Y, and the other way around.

We say that a sequence of strictly vertical laminations $\mathcal{F}^n$ converges to a lamination $\mathcal{F}$ if they converge in the Hausdorff metric on subsets of the space $C^0[0,1]$. Convergence is called exponentially with rate $r \in (0,1)$ if there exists $C > 0$ such that

$$\mathrm{dist}_H(\mathcal{F}^n, \mathcal{F}) \leq C r^n.$$

Theorem 12.4. *For any strictly vertical lamination $\mathcal{F}$, the pullbacks $\mathcal{F}^n := (\mathcal{R}^n)^* \mathcal{F}$ converge exponentially (with the same rate) to the central foliation $\mathcal{F}^c$ of $\mathcal{R}$.*

Proof. Given any $\gamma_0 \in \mathcal{F}^c$, let γ_i be the curve from $\mathcal{F}^c$ containing $\mathcal{R}^i\gamma_0$ for $i = 1, \dots, n$. Let η_n be any leaf from $\mathcal{F}$. We choose a sequence of iterated preimages $\eta_{n-1}, \dots, \eta_0$, so that η_{i-1} is obtained from η_i under the same inverse branch of $\mathcal{R}$ as γ_{i-1} is obtained from γ_i. Let h_0 be the shorter of the two segments on $\mathcal{B}$ connecting from γ_0 to η_0. By construction, $l^h(h_0) \le \pi/4^n$.

For any $t \in [0,1]$, let h_t be the genuinely horizontal curve from $\gamma_0(t)$ to $\eta_0(t)$ chosen so that $\gamma_0[0,t] \cup h_t \cup \eta_0[0,t]$ and h_0 are homotopic (relative to their endpoints) on $\mathcal{C}$. Then, their iterates under $\mathcal{R}^n$ are also homotopic. The horizontal length of any vertical curve in $\mathcal{C}$ is bounded by some constant K. Therefore, since $\mathcal{R}^n\gamma_0[0,t]$ and $\mathcal{R}^n\eta_0[0,t]$ are vertical, we have that $l^h(\mathcal{R}^n h_t) \le l^h(\mathcal{R}^n h_0) + 2K \le \pi + 2K$.

Using horizontal expansion, Theorem 8.1, we have $l^h(h_t) \le C\lambda^{-n} \cdot l^h(\mathcal{R}^n h_t)$ for appropriate $C > 0$ and $\lambda > 1$. Since $l^h(\mathcal{R}^n h_t) \le \pi + 2K$, we have $l^h(h_t) \le D\lambda^{-n}$. Therefore, given any $\gamma_0 \in \mathcal{F}^c$ there is some $\eta_0 \in \mathcal{F}^n$ that is $D\lambda^{-n}$ close within $C^0[0,1]$.

After switching the roles of γ_0 and η_0, the same proof shows that given any $\eta_0 \in \mathcal{F}^n$ there is some $\gamma_0 \in \mathcal{F}^c$ that is $D\lambda^{-n}$ close within $C^0[0,1]$. □

Corollary 12.5. *The sequence of Lee-Yang loci $\mathcal{S}_n$ converges exponentially to the central foliation $\mathcal{F}^c$.*

We have a better convergence at low temperatures. For any $\epsilon > 0$ let $\mathcal{F}^n_\epsilon$ and $\mathcal{F}^c_\epsilon$ the truncations of $\mathcal{F}^n$ and $\mathcal{F}^c$ to the cylinder $\mathbb{T} \times [0, t_c - \epsilon]$.

Proposition 12.6. *For any $\epsilon > 0$ we have exponential convergence*

$$\operatorname{dist}^1_H(\mathcal{F}^n_\epsilon, \mathcal{F}^c_\epsilon) \le C(\epsilon)\lambda^n$$

where dist^1_H denotes the Hausdorff metric on subsets of $C^1[0, t_c - \epsilon]$.

Proof. Given any $\gamma \in \mathcal{F}^c$, let $\eta \in \mathcal{F}^n$ be the curve constructed in the proof of Theorem 12.4. Let $t \in [0, t_c - \epsilon]$. By Lemma 7.3, $\eta'(t)$ is exponentially close to $\mathcal{L}^c(\eta(t))$. Since $\gamma(t)$ and $\eta(t)$ are exponentially close and $\mathcal{L}^c$ is Hölder in $\mathcal{W}^s(\mathcal{B})$ (see Proposition 9.13) $\gamma'(t) = \mathcal{L}^c(\gamma(t))$ and $\mathcal{L}^c(\eta(t))$ are also exponentially close. □

12.5. **Holonomy and the transverse measure.**

12.5.1. *Regularity of the holonomy.*

Proposition 12.7. *All holonomy transformations g_t, $0 \le t < 1$, are uniformly $\frac{1}{2}$-Hölder continuous homeomorphisms. If $y = g_t(x) \in \mathcal{W}^s(\mathcal{B})$ then g_t is a C^∞ local diffeomorphism near x.*

Proof. Let us take an interval $J \subset \mathcal{B}$, and let $J_t = g_t(J)$. Since $\mathcal{R}(z) = z^4$ on $\mathcal{B}$, there is an $n \in \mathbb{N}$ such that $\mathcal{R}^n(J)$ covers $\mathbb{T}$ at least once, but no more than four times. Then the same is true for $\mathcal{R}^n(J_t)$. Hence the horizontal length of both intervals $\mathcal{R}^n(J)$ and $\mathcal{R}^n(J_t)$ is squeezed in between 2π and 8π. It follows that $l(J) \asymp 4^{-n}$, while $l(J_t) = O(2^{-n})$ with from Theorem 8.1. Hence $l(J_t) = O(l(J)^\sigma)$ with $\sigma = \frac{1}{2}$.

Since the $g_t : \mathbb{T} \to \mathbb{T}_t$ are continuous bijections on a compact space, they are homeomorphisms. The last assertion follows from C^∞ smoothness of the foliation $\mathcal{F}^c | \mathcal{W}^s(\mathcal{B}) = \mathcal{F}^s(\mathcal{B})$ (Proposition 9.13). □

At the top of the cylinder, the holonomy degenerates to the *Devil Staircase*:

Proposition 12.8. *As $t \to 1$, the holonomy maps g_t uniformly converge to the map g_1 that collapse the bottoms $\mathcal{B}_{\Lambda(\alpha)}$ of the central tongues to their tips $\alpha \in \mathcal{A}$.*

Proof. Indeed, all the leaves that begin on the bottom $\mathcal{B}_{\Lambda(\alpha)}$ of the central tongue $\Lambda(\alpha)$ merge at its tip $\alpha \in \mathcal{T}$. □

12.5.2. *Balanced transverse measure.* Recall also that a *transverse invariant measure* μ for $\mathcal{F}$ is a family of measures μ_t, $t \in [0,1)$, such that $\mu_t = (g_t)_*(\mu_0)$. Obviously, it is uniquely determined by μ_0. It can be evaluated not only on genuinely horizontal sections but on all local transversals to $\mathcal{F}^c$, in particular, on all almost horizontal curves.

The transverse measure μ for which μ_0 is the normalized Lebesgue measure $d\phi/2\pi$ will be called *balanced*. For $t \in [0,1)$, we let

$$O_t = \{x = (\phi, t) \in \mathcal{C} : x \in \mathcal{W}^s(\mathcal{B})\}, \quad K_t = \mathbb{T} \smallsetminus O_t.$$

Remark that K_t has positive Lebesgue measure for any $t > t_c$ and that its measure tends to 2π as $t \to 1$. This follows from the fact that $\mathcal{W}^s(\mathcal{T}) \cap \mathbb{T}_t \subset K_t$ and Corollary 10.4. Meanwhile, Lemma 9.1 gives that O_t is a dense open set, hence K_t is nowhere dense.

Let us also use a special notation for the *horizontal expansion factor*:

$$\lambda^h(x) := \frac{\partial(\phi \circ \mathcal{R})}{\partial \phi}(x), \tag{12.1}$$

equal to the upper-left entry of the Jacobi matrix $D\mathcal{R}$. Note that the transverse measure $d\mu = \rho\, d\phi/2\pi$ is transformed by the rule

$$\frac{\partial(\mathcal{R}^*\mu)}{\partial\mu} = \lambda^h(x)\frac{\rho(\mathcal{R}x)}{\rho(x)}, \quad x \in \mathcal{W}^s(\mathcal{B}). \tag{12.2}$$

Proposition 12.9. *Let μ be the balanced transverse measure on the central foliation $\mathcal{F}^c$. For any $t \in [0,1)$, the measure μ_t is absolutely continuous and $\mu_t(O_t) = 1$. Its density ρ_t is positive and C^∞ on each component of O_t. Moreover, for any almost horizontal curve γ we have the transfer rule:*

$$\mu(R(\gamma)) = 4\mu(\gamma), \quad \text{or equivalently:} \quad 4\rho(x) = \lambda^h(x)\,\rho(\mathcal{R}x), \quad x \in \mathcal{W}^s(\mathcal{B}). \tag{12.3}$$

Proof. By Proposition 9.7 (v), μ is supported on the union of stable tongues $\Upsilon_k(\alpha)$, hence $\mu_t(O_t) = 1$. By the second assertion of Proposition 12.7, μ_t has a positive C^∞ density on O_t.

Property (12.3) is obviously satisfied for the Lebesgue measure on $\mathcal{B}$. By holonomy invariance of μ, it is satisfied for any almost horizontal curve. The equivalent formulation in terms of the density ρ comes from (12.2). □

Corollary 12.10. *The holonomy maps $g_t : \mathcal{B} \to \mathbb{T}_t$ are absolutely continuous, while the inverse maps g_t^{-1} are not for $t > t_c$.*

Proof. Absolute continuity of g_t is equivalent to absolute continuity of the push-forward measure $\mu_t = (g_t)_*\mu_0$, so Proposition 12.9 implies the first assertion.

On the other hand, by Theorem 10.3, the complement to the stable tongues on any section $\mathbb{T}_t$, $t > t_c$, has positive measure, while on the bottom $\mathcal{B}$, it has measure zero (by Proposition 9.7 (v)). This yields the second assertion. □

At the top, the transverse measure becomes purely atomic:

Proposition 12.11. *As $t \to 1$, the distributions μ_t weakly converge to the distribution μ_1 supported on the pre-indeterminacy set $\mathcal{A}$ that assigns to a point $\alpha \in \mathcal{A}^n$ weight $1/4^{n+1}$.*

Proof. This follows from Proposition 12.8, taking into account that $\mu_0(\mathcal{B}_{\Lambda(\alpha)}) = 1/4^{n+1}$. □

12.6. **High-temperature hairs and critical temperatures.** As usual, we parameterize each $\gamma \in \mathcal{F}^c$ by temperature t. Every $\gamma \in \mathcal{F}^c \smallsetminus \{\mathcal{I}_{\pm\pi/2}\}$ maps by $\mathcal{R}$ homeomorphically onto $\mathcal{R}(\gamma)$, so there are temperatures $0 < t_\gamma^- \le t_\gamma^+ \le 1$ so that

$$\begin{aligned} \gamma \cap \mathcal{W}^s(\mathcal{B}) &= \gamma[0, t_\gamma^-) \text{ and} \\ \gamma \cap \mathcal{W}^s(\mathcal{T}) &= \gamma(t_\gamma^+, 1] \text{ or } \gamma[t_\gamma^+, 1]. \end{aligned}$$

We call $h_\gamma := \gamma \cap \mathcal{W}^s(\mathcal{T})$ the *high-temperature hair* contained in γ. Recall the notion of Cantor bouquet introduced in §10.1.

Proposition 12.12. *$\mathcal{W}^s(\mathcal{T})$ is a Cantor bouquet containing all of the bouquets that were constructed in §10.1.*

We call $e_\gamma := \gamma(t_\gamma^+)$ the *endpoint of* h_γ and $c_\gamma := \gamma([t_\gamma^-, t_\gamma^+])$ the *critical temperatures of* γ. Let

$$\mathscr{E} := \bigcup_\gamma e_\gamma \text{ and } \qquad \mathscr{C} := \bigcup_\gamma c_\gamma$$

Each is an $\mathcal{R}$ invariant set.

Proposition 12.13. *The set of critical temperatures $\mathscr{C}$ has zero Lebesgue measure.*

Proof. By construction, the high temperature hair through any $x \in \mathcal{W}_0^{s,o}(\mathcal{T})$ extends below x. Therefore, $\mathscr{C}$ lies in the complement of $\mathcal{W}^s(\mathcal{B}) \cup \mathcal{W}_0^{s,o}(\mathcal{T})$ so that it has measure 0 by Theorem 11.1. □

Corollary 12.14. *The set of endpoints to the high-temperature hairs $\mathscr{E}$ has zero Lebesgue measure.*

12.7. **Summary of the proof of Main Theorem (dynamical version).** At this point we have completed the proof of the Main Theorem (dynamical version). Let us summarize: In §7 we proved that $\mathcal{R} : \mathcal{C}_1 \to \mathcal{C}_1$ admits a dominated splitting and in §8 we proved that $\mathcal{R} : \mathcal{C}_1 \to \mathcal{C}_1$ is horizontally expanding. Together, these properties give that $\mathcal{R} : \mathcal{C}_1 \to \mathcal{C}_1$ is partially hyperbolic. In §12.1 we proved that there is a unique central foliation $\mathcal{F}^c$. In §9.7 we proved that $\mathcal{F}^c$ is C^∞ in $\mathcal{W}^s(\mathcal{B})$. The assertion that $\mathcal{F}^c$ is not absolutely continuous on $\mathcal{W}^s(\mathcal{T})$ is proved in Corollary 12.10.

The assertion that the pullback $(\mathcal{R}^n)^*\gamma$ of a proper vertical curve γ on $\mathcal{C}$ consists of 4^n proper vertical curves was proved in Lemma 6.2 and that $(\mathcal{R}^n)^*\gamma$ converges exponentially fast to $\mathcal{F}^c$ was proved in §12.4.

Finally, the assertions about $\mathcal{W}^s(\mathcal{B})$ being open and dense in $\mathcal{C}$, $\mathcal{W}^s(\mathcal{T})$ being of positive area, and their union being of full area in $\mathcal{C}$ are proved in §9, §10, and §11, respectively. □

13. Lee-Yang Distributions, Critical exponents, and Local Rigidity

Now, having plowed hard in the RG dynamical cylinder, let us collect the physics harvest:

13.1. **Lee-Yang Distributions.** Recall from §1.3 that the Lee-Yang locus $\mathcal{S}_n$ of level n is equal to the pullback $(\mathcal{R}^n)^*\mathcal{S}_0$ of the principal LY locus $\mathcal{S} \equiv \mathcal{S}_0$ (3.8). By Theorem 12.4, these loci converge exponentially fast to the central foliation $\mathcal{F}^c$.

Moreover, on the bottom $\mathcal{B}$ the Lee-Yang zeros are obviously asymptotically equidistributed with respect to the Lebesgue measure μ_0. It follows that on the circle $\mathbb{T}_t$, they are asymptotically equidistributed with respect to the measure $(g_t)_*(\mu_0) = \mu_t$, which is the balanced transverse measure for $\mathcal{F}^c$.

13.2. **Critical Exponents.** Let $x = (z,t) \in \mathcal{C} \smallsetminus \mathcal{W}^s(\mathcal{B})$ (so that the density ρ_t of the transverse measure μ_t vanishes at x). If

$$\mu_t(J) \asymp |J|^{\sigma^h+1} \text{ for a horizontal interval } J \text{ containing } x \text{ on its boundary,} \tag{13.1}$$

then $\sigma^h = \sigma^h(x)$ is called the *horizontal critical exponent* of the transverse measure at x (on the left- or right-hand side of x, depending on J – if we do not specify the side, it means that the critical exponent exists on both sides).

Let additionally, x lie on the boundary point of $\mathcal{F}^c(x) \cap \mathcal{W}^s(\mathcal{B})$ (in other words, let all points on the central leaf $\mathcal{F}^c(x)$ below x converge to the bottom of the cylinder). If

$$\rho(y) \asymp \operatorname{dist}(x,y)^{\sigma^v} \text{ for } y \in \mathcal{F}^c(x) \text{ below } x, \tag{13.2}$$

then σ^v is called the *vertical critical exponent* of the transverse measure at x.

Let x be a periodic point for $\mathcal{R}$ of period p with multipliers $\lambda^u > \lambda^c$. Here the *unstable multiplier* λ^u corresponds to the eigenvector of $D\mathcal{R}^p_x$ in the horizontal cone $\mathcal{K}^h(x)$, while the *central multiplier* corresponds to the eigenvector tangent to the central leaf $\mathcal{F}^c(x)$. The inequality between the multipliers follows from the dominated splitting. Also, $\lambda^u > 1$ because of the horizontal expansion. The corresponding *characteristic exponents* at x are defined as

$$\chi^u(x) = \frac{1}{p}\log\lambda^u, \quad \chi^c(x) = \frac{1}{p}\log\lambda^c.$$

Proposition 13.1. *Let x be a periodic point for $\mathcal{R}$ of period p with the characteristic exponents χ^u and χ^c. Then*

$$\sigma^h(x) = \frac{\log 4}{\chi^u} - 1. \tag{13.3}$$

Moreover, if x is a boundary point of some component J of the basin O_t, then

$$\rho_t(y) \asymp \operatorname{dist}(x,y)^{\sigma^h}, \quad y \in J \text{ near } x. \tag{13.4}$$

If x is the boundary point of $\mathcal{F}^c(x) \cap \mathcal{W}^s(\mathcal{B})$ and $\chi^c(x) > 0$, then

$$\sigma^v(x) = \frac{\log 4 - \chi^u}{\chi^c}. \tag{13.5}$$

Proof. Given a horizontal interval $J \ni x$, let us apply to it an iterate $\mathcal{R}^n$ that stretches J to a horizontal curve that wraps around the cylinder at least once but at most four times. Then both $l^h(\mathcal{R}^n(J))$ and $\mu(\mathcal{R}^n(J))$ are comparable with 1.

On the other hand, $l^h(\mathcal{R}^n(J)) \asymp \exp(n\chi^u)|J|$ while $\mu(\mathcal{R}^n(J)) = 4^n\mu_t(J)$. Hence $\mu_t(J) \asymp |J|^{\sigma+1}$ with exponent $\sigma = \log 4/\chi^u - 1$. This proves (13.3).

Let us prove (13.4). Iterating the transfer rule (12.3), we obtain:

$$4^n\rho(y) = \lambda^h_n(y)\,\rho(\mathcal{R}^n y), \quad \text{where } \lambda^h_n(y) = \prod_{k=0}^{n-1}\lambda^h(\mathcal{R}^k y). \tag{13.6}$$

Let $y \in J$ be a point near x. Because of the horizontal expansion, we can find an iterate $\mathcal{R}^n$ such that $\mathrm{dist}^h(\mathcal{R}^n x, \mathcal{R}^n y) \asymp 1$. Then $\rho(\mathcal{R}^n y) \asymp 1$, while $\lambda_n^h(y) \asymp \exp(n\chi^u(x))$. Incorporating these into (13.6), we obtain:

$$\rho(y) \asymp \exp(n(\chi^u - \log 4)).$$

On the other hand, $\mathrm{dist}(x,y) \asymp \exp(-n\chi^u(x))$, and (13.4) follows.

To prove (13.5), take a point $y \in \mathcal{F}^c(x)$ below x. Then $y \in \mathcal{W}^s(\mathcal{B})$ since x lies on the boundary of $\mathcal{F}^c(x) \cap \mathcal{W}^s(\mathcal{B})$. Hence we can find an iterate $\mathcal{R}^n$ such that $\mathrm{dist}^c(\mathcal{R}^n x, \mathcal{R}^n y) \asymp 1$. However, in this case x repels y at exponential rate with exponent χ^c, so $\mathrm{dist}(x,y) \asymp \exp(-n\chi^c(x))$, and (13.5) follows. $\square$

Corollary 13.2. *The horizontal and vertical critical exponents at the fixed point $\beta_c = (0, t_c) \in \mathcal{I}_0$ are equal to $\sigma^h(\beta_c) = 0.0643\ldots$ and $\sigma^v(\beta_c) = 0.162\ldots$.*

Proof. $D\mathcal{R}(\beta_c)$ has eigenvalues $\lambda^u = \frac{4}{t_c^2+1}$ and $\lambda^c = \frac{8t_c(1-t_c^2)}{(t_c^2+1)^3}$, corresponding to the purely horizontal and vertical directions. The result then follows from (13.4), (13.5), and $t_c = 0.296\ldots$. $\square$

One can define the critical exponents at a point $x \in \mathcal{W}^s(\mathcal{T})$ in a *weak sense*:

$$\sigma^h(x) = \lim_{|J|\to 0} \frac{\log \mu(J)}{\log |J|} - 1, \quad \sigma^v(x) = \lim_{y \to x} \frac{\log \rho(y)}{\log \mathrm{dist}(x,y)}$$

(if the limits exist), where the meaning of J and $y \in \mathcal{F}^c(x) \cap \mathcal{W}^s(x)$ are the same as in formulas (13.1), (13.2). These critical exponents can be expressed in terms of the unstable and central Lyapunov exponents

$$\chi^h(x) = \lim_{n\to\infty} \frac{1}{n} \log \lambda_n^h(x), \quad \chi^c(x) = \lim_{n\to\infty} \frac{1}{n} \log \lambda_n^c(x)$$

by the same formulas as in the case of periodic points:

Proposition 13.3. *Let $x \in \mathcal{C} \smallsetminus \mathcal{W}^s(\mathcal{B})$ be a point with the unstable Lyapunov exponent χ^u. Then the horizontal critical exponent $\sigma^h(x)$ exists in the weak sense and is given by formula (13.3). Moreover, if x is a boundary point of some component J of the basin O_t, then*

$$\sigma^h(x) = \lim \frac{\log \rho_t(y)}{\log \mathrm{dist}(x,y)} \quad \text{as } y \to x,\ y \in J. \tag{13.7}$$

If x is the boundary point of $\mathcal{F}^c(x) \cap \mathcal{W}^s(\mathcal{B})$ and the central Lyapunov exponent $\chi^c(x)$ exists and positive, then the vertical critical exponent $\sigma^v(x)$ exists in the weak sense and is given by formula (13.5).

The proof mimics that of Proposition 13.1.

Corollary 13.4. *For every $x \in \mathcal{C} \smallsetminus \mathcal{W}^s(\mathcal{B})$ at which $\sigma^h(x)$ exists we have $\sigma^h(x) \le 1$. Moreover, equality holds at Lebesgue almost every such x.*

Proof. Theorem 8.1 gives that $\mathcal{R}$ is horizontally expanding with rate $\lambda = 2$, implying the upper bound. The second statement follows by combining Corollary 11.2 with Proposition 13.3. $\square$

13.3. **Summary of the proof of Main Theorem (physical version).** We have now completed the proof of the Main Theorem (physical version). Let us summarize: The assertion that the limiting distribution of Lee-Yang zeros $\mu_t(\phi)$ exists, is absolutely continuous with respect to the Lebesgue measure (for $0 \le t < 1$), and the properties of its density and its degeneration as $t \to 1$ follow from the discussion on §13.1 and the properties of the balanced transverse measure established in §12.5. Finally, the statement about the critical exponent $\sigma \equiv \sigma^h$ is established in Corollary 13.2. □

Remark 13.1. As mentioned in §2.6, existence of the limiting distribution μ_t (without absolute continuity or any of the other properties discussed above) follows directly from existence of the thermodynamic limit for the Ising Model on the DHL (Proposition 2.4) combined with Proposition 2.2.

13.4. **Local Rigidity.** Recall the notion of *local rigidity* introduced in §1.2.

Proposition 13.5. *The Lee-Yang zeros for the DHL are locally rigid at any point where the limiting density is positive $\rho(\phi, t) > 0$.*

Proof. The Lee-Yang zeros $\phi_k^n(t)$ of level n on $\mathbb{T}_t$ are the solutions to

$$h_t^n(\phi) := \frac{1}{2 \cdot 4^n} f_1 \circ \mathcal{R}^n(\phi, t) = \frac{\pi k}{4^n} + \frac{\pi}{2 \cdot 4^n}, \; k = 0, 1, \dots 2 \cdot 4^n - 1, \tag{13.8}$$

where $f_1 : \mathcal{C}_1 \to \mathbb{T}$ is the first coordinate of the mapping f given in (3.3). Notice that f_1 is a degree 2 map tangent to 2ϕ on $\mathcal{B}$.

Fix a point $x = (\phi_*, t) \in \mathcal{W}^s(\mathcal{B})$ and a closed horizontal interval $J_t \subset O_t$ containing x in its interior.

By Lemma 9.16, the maps h_t^n converge in the C^1 topology on J_t to the Böttcher coordinate $\Phi_t(\cdot) \equiv \Phi(\cdot, t)$. Since $\partial\Phi/\partial\phi \neq 0$, the maps h_t^n are invertible on J_t for sufficiently large n. Let $g_t^n : J_0 \to J_t$ be the inverse (where $J_0 := h_t^n(J_t) \subset \mathcal{B}$). Since the holonomy map g_t is the inverse of Φ_t, we conclude that $(g_t^n)' \to g_t'$ uniformly on J_0.

For each n, let ϕ_l^n be the Lee-Yang zero that is closest to ϕ_*. We will show that the rescaled Lee-Yang zeros

$$s_k^n = \frac{2 \cdot 4^n}{2\pi} \; \rho_{\rm t}(\phi_*) \left(\phi_{l+k}^n - \phi_l^n\right), \tag{13.9}$$

converge locally uniformly to the integer lattice $\mathbb{Z}$.

After fixing k, ϕ_{l+k}^n and ϕ_l^n will be in J_t for all n sufficiently large. The Mean Value Theorem gives

$$\phi_{l+k}^n - \phi_l^n = (g_t^n)'(\psi_k^n) \frac{2\pi k}{2 \cdot 4^n}, \tag{13.10}$$

where ψ_k^n is between $\frac{\pi l}{4^n} + \frac{\pi}{2 \cdot 4^n}$ and $\frac{\pi(l+k)}{4^n} + \frac{\pi}{2 \cdot 4^n}$.

Equation (13.9) becomes

$$s_k^n = k \cdot (g_t^{-1})'(\phi_*) \cdot (g_t^n)'(\psi_k^n), \tag{13.11}$$

since $\rho_t(\phi_*) = (g_t^{-1})'(\phi_*)$. For fixed k

$$\lim \psi_k^n = \Phi_t(\phi_*) = (g_t)^{-1}(\phi_*).$$

Thus, $(g_t^n)'(\psi_k^n) \to g_t'((g_t)^{-1}\phi_*)$ giving that $s_k^n \to k$. □

14. Periodic Leaves

Let us distinguish between two distinct types of leaves $\gamma \subset \mathcal{F}^c$ that are periodic under $\mathcal{R}$. We say that a periodic leaf γ is *regular* if $\gamma \cap \mathcal{T}$ is a periodic point. If γ is a periodic leaf that is not regular, then $\gamma \cap \mathcal{T}$ is in the preindeterminacy set $\mathcal{A}$.

Associated to any periodic point $x_1 \in \mathcal{T}$ is a regular periodic leaf meeting $\mathcal{T}$ at x_1. Meanwhile, associated to any periodic point $x_0 \in \mathcal{B}$ there is a periodic leaf meeting $\mathcal{B}$ at x_0, which need not be regular—since almost every point in $\mathcal{B}$ is in the union of the stable tongues, it is quite common to obtain a singular periodic leaf from a periodic $x_0 \in \mathcal{B}$.

The periodic points x_0 and x_1 at the bottom and top of a regular periodic leaf γ are horizontally repelling and vertically (super) attracting. They have real-analytic (super) stable manifolds $\mathcal{W}^s(x_0)$, $\mathcal{W}^s(x_1)$, which extend slightly below $\mathcal{B}$ and above $\mathcal{T}$, respectively. The high-temperature hair $h_\gamma = \mathcal{W}^s(x_1) \cap \mathcal{C}$ and the low-temperature hair $l_\gamma := \mathcal{W}^s(x_0) \cap \mathcal{C}$ are both real-analytic and non-trivial. Therefore, the smoothness of a regular periodic leaf γ is determined within its critical points $c_\gamma = \gamma \smallsetminus (h_\gamma \cup l_\gamma)$.

Proposition 14.1. *Suppose that $\gamma \in \mathcal{F}^c$ is a regular periodic leaf of prime period $k > 1$. Then, γ is not real-analytic.*

The assumption that $k > 1$ is necessary, since the vertical interval $\mathcal{I}_0$ is a regular periodic leaf of period 1. The assumption that γ is a regular periodic leaf is also necessary, because there are many periodic leaves contained entirely within $\mathcal{W}^s(\mathcal{B})$ that are real-analytic.

Proof. We suppose that γ is a regular periodic leaf, of prime period $k > 1$, that is real-analytic. Let x_0 and x_1 be the periodic points at the bottom and top of γ, respectively. We extend γ analytically slightly below x_0 and above x_1 and then take a complexification $\gamma_{\mathbb{C}}$, chosen sufficiently small so that it is an embedded complex disc.

Let $x_c = \gamma(t_\gamma^-)$ be the periodic point "at the bottom of c_γ". It has one-dimensional central direction and one-dimensional unstable direction, with multipliers $1 \leq \lambda_c < \lambda_u$. Any small piece of γ containing x_c in its interior will be a central manifold $\mathcal{W}^c_{\text{loc}}(x_c)$. Similarly, an open disc from $\gamma_{\mathbb{C}}$ containing x_c will form a complex analytic central manifold $\mathcal{W}^c_{\mathbb{C},\text{loc}}(x_c)$.

Consider the case that x_c is vertically repelling: $\lambda_c > 1$. Let $\rho_0 : \mathbb{D} \to \mathcal{W}^c_{\mathbb{C},\text{loc}}(x_c)$ be a local linearizing coordinate, i.e. $\rho_0(\lambda_c x) = \mathcal{R}^k(\rho_0(x))$. It can be globalized to form a non constant $\rho : \mathbb{C} \to \mathbb{CP}^2$ satisfying $\rho(\lambda_c x) = \mathcal{R}^k(\rho(x))$ that is given by

$$\rho(x) := \lim_{n \to \infty} \mathcal{R}^{n \cdot k}(\rho_0(x/\lambda_c^n)).$$

Suppose $\mathcal{R}^l(\rho_0(x_*/\lambda_c^n))$ lands on an indeterminacy point for some $x_* \in \mathbb{C}$. After taking appropriate blow-ups at the indeterminate point, $\mathcal{R}$ extends to some holomorphic $\tilde{\mathcal{R}}$. (See Appendix A.2.) The image under $\mathcal{R}^l(\rho_0(x/\lambda_c^n))$ of some complex disc D containing x_* lifts to the blown-up space via the proper transform, intersecting the exceptional divisor in a single point. We define the next iterate on this disc using $\tilde{\mathcal{R}}$. This definition coincides with $\mathcal{R}^{l+1}(\rho_0(x/\lambda_c^n))$ on $D \smallsetminus \{x_*\}$ and gives a holomorphic extension through x_*.

A global central "manifold" $\mathcal{W} \equiv \mathcal{W}^c_{\mathbb{C},\mathrm{glob}}(x_c)$, invariant under $\mathcal{R}^k$, is given by $\rho(\mathbb{C})$. A-priori, $\mathcal{W}$ can be wild, possibly accumulating on itself and/or intersecting itself countably many times. However, we will show that $\mathcal{W}$ can be compactified to form an algebraic curve.

Given $x, y \in \mathbb{C}$, let $x \sim y$ if $\rho(x) = \rho(y)$ and there exist neighborhoods N_x and N_y such that the images $\rho(N_x)$ and $\rho(N_y)$ coincide. If x and y are both regular points of ρ, this definition coincides with requiring that $\rho(x) = \rho(y)$ and that there be a biholomorphism $h : N_x \to N_y$ so that $\rho_{|N_x} = \rho_{|N_y} \circ h$. (In fact, $\sim$ is obtained by extending this description at regular points to all of $\mathbb{C}\times\mathbb{C}$ by taking the closure.) We will now show that $\hat{\mathcal{W}} = \mathbb{C}/\sim$ can naturally be given the structure of a Riemann surface in such a way that the map $\hat{\rho} : \hat{\mathcal{W}} \to \mathcal{W}$ that is induced by ρ and the projection $\pi : \mathbb{C} \to \hat{\mathcal{W}}$ are holomorphic.

We will use local sections of π to define charts on $\hat{\mathcal{W}}$. Let $S \subset \mathbb{C}$ be the set of critical points of ρ. Suppose $w_0 \in \hat{\mathcal{W}}$ and $r \in \pi^{-1}(w_0)\setminus S$. Then, the corresponding local section of π defines a chart in a neighborhood of w_0.

Now suppose $s \in \pi^{-1}(w_0) \cap S$. Let N_s be a small neighborhood of s such that $N_s \cap S = \{s\}$. Then there is a sequence of blowups over $\rho(s)$ so that ρ lifts to a mapping $\tilde{\rho} : N_s \to \widetilde{\mathbb{CP}^2}$ whose image $\tilde{\rho}(N_s)$ is a smooth holomorphic disc intersecting the exceptional divisor over $\rho(s)$ in a single point. Then, $\tilde{\rho} : N_s \to \tilde{\rho}(N_s)$ is given by $z \mapsto z^d$, for some integer $d \geq 0$, in suitable local coordinates on N_s and on $\tilde{\rho}(N_s)$. Therefore, the identifications made by $\sim$ in N_s are the same as those by $z \mapsto z^d$. We can now make N_s smaller, if necessary, so that N_s is a round disc in the z coordinate. We define a chart on the neighborhood $\pi(N_s)$ of w_0 by $w \mapsto ((\pi|_{N_s})^{-1}(w))^d$. By the previous discussion, the resulting map is well-defined and a homeomorphism from $\pi(N_s)$ to a disc.

Because of the description of $\sim$ away from critical points of ρ, any two such charts are holomorphically compatible away from at most two possible bad points, which would correspond to the centers of the discs $\pi(N_s)$ in the second construction. Since the charts differ by a homeomorphism, the result extends holomorphically across these points. This gives $\hat{\mathcal{W}}$ the structure of a Riemann surface such that π and the induced map $\hat{\rho}$ are holomorphic.

Because of the identifications we've made, the action of $\mathcal{R}^k : \mathcal{W} \to \mathcal{W}$ can be lifted (in the natural way) to $\hat{\mathcal{R}}^k : \hat{\mathcal{W}} \to \hat{\mathcal{W}}$. Notice that $\pi(0) \in \hat{\mathcal{W}}$ is a repelling fixed point under $\hat{\mathcal{R}}^k$ so that $\hat{\mathcal{W}}$ is non-hyperbolic.

Let $\hat{U} = \{x \in \hat{\mathcal{W}} : \hat{\rho}$ maps a neighborhood of x into $\gamma_{\mathbb{C}}\}$, which is non-empty since $\rho(\mathbb{D}) \subset \gamma_{\mathbb{C}}$. Let $U = \hat{\rho}(\hat{U}) \subset \gamma_{\mathbb{C}}$. The identifications we've made when forming $\hat{\mathcal{W}}$ imply that $\hat{\rho} : \hat{U} \to U$ is biholomorphic. Consider the larger Riemann surface $V := \hat{\mathcal{W}} \cup_{\hat{\rho}} \gamma_{\mathbb{C}}$, where $x \in \hat{\mathcal{W}}$ and $y \in \gamma_{\mathbb{C}}$ are identified if $\hat{\rho}(x) = y$ with some neighborhood of x in $\hat{\mathcal{W}}$ mapping by $\hat{\rho}$ into $\gamma_{\mathbb{C}}$. The natural inclusion $\iota : \hat{\mathcal{W}} \to V$ is holomorphic. Since $\hat{\mathcal{W}}$ is not hyperbolic, ι can omit at most two points of V. This implies that $\gamma_{\mathbb{C}} \setminus U$ consists of at most two points. In particular, there are no omitted points in $\gamma_{\mathbb{C}} \setminus \{x_0, x_1\}$ near x_0 and x_1. So, there are two punctured discs U_0 and $U_1 \subset \gamma_{\mathbb{C}}$ having x_0 and x_1 as their punctures, respectively, and two punctured discs $\hat{U}_0$ and $\hat{U}_1 \subset \hat{\mathcal{W}}$ mapped biholomorphically by $\hat{\rho}$ to U_0 and U_1, respectively.

Since β_0 is the only point in $\mathcal{L}_0$ having an iterated preimage under $\mathcal{R}$ outside of $\mathcal{L}_0$, it is the only point that can possibly be in $\mathcal{W} \cap \mathcal{L}_0$. By assumption, $x_0 \neq \beta_0$, so there is no point in $\hat{\mathcal{W}}$ mapping to x_0. Thus, the puncture in $\hat{U}_0$ corresponds to an actual puncture in $\hat{\mathcal{W}}$. Let $\hat{\mathcal{W}}_0 \equiv \hat{\mathcal{W}} \cup \{w_0\}$ be the Riemann surface obtained by filling in this puncture. Both $\hat{\rho}$ and $\hat{\mathcal{R}}^k$ extends to $\hat{\mathcal{W}}_0$, with $\hat{\rho}(w_0) = x_0$ and $\hat{\mathcal{R}}^k(w_0) = w_0$.

Since $\hat{\mathcal{W}}$ is non-hyperbolic with at least one puncture, $\hat{\mathcal{W}}_0$ is biholomorphic to either $\mathbb{C}$ or $\mathbb{CP}^1$. In either case, $\hat{\mathcal{R}}^k : \hat{\mathcal{W}}_0 \to \hat{\mathcal{W}}_0$ has a degree. (If $\hat{\mathcal{W}}_0 \cong \mathbb{C}$, the action of $\hat{\mathcal{R}}^k$ is polynomial, since any point has finitely many preimages under $\hat{\mathcal{R}}^k$.) Since w_0 is totally invariant under $\hat{\mathcal{R}}^k$, with a neighborhood mapped to itself with degree 2^k, we see that $\mathcal{R}^k : \hat{\mathcal{W}}_0 \to \hat{\mathcal{W}}_0$ has degree 2^k.

If there were a point $w_1 \in \hat{\mathcal{W}}_0$ filling the puncture in $\hat{U}_1$, it would satisfy $\hat{\rho}(w_1) = x_1$, and the local degree of $\hat{\mathcal{R}}^k$ at w_1 would be 2^k. However, if $x_1 \in \mathcal{W}$, there must be iterated preimages of x_1 under $\mathcal{R}^k$ in $\mathcal{W}$ converging to x_c, violating that the total degree of $\hat{\mathcal{R}}^k : \hat{\mathcal{W}}_0 \to \hat{\mathcal{W}}_0$ is 2^k. Therefore, the puncture in $\hat{U}_1$ corresponds to an actual puncture in $\hat{\mathcal{W}}_0$. We let $\hat{\mathcal{W}}_{0,1} \equiv \hat{\mathcal{W}}_0 \cup \{w_1\}$ be the Riemann surface obtained by filling this puncture. The parameterization $\hat{\rho}$ extends holomorphically to $\hat{\mathcal{W}}_{0,1}$.

Since $\hat{\mathcal{W}}$ is non-hyperbolic and has two punctures, it is biholomorphic to the twice punctured Riemann sphere. Therefore, $\hat{\mathcal{W}}_{0,1}$ is biholomorphic to the Riemann Sphere and $\overline{\mathcal{W}} = \hat{\rho}\left(\hat{\mathcal{W}}_{0,1}\right) = \mathcal{W} \cup \{x_0, x_1\}$ is a compact analytic curve. Chow's Theorem (see, e.g., [GH]) gives that $\overline{\mathcal{W}}$ is therefore algebraic. Since it is parameterized by a connected curve, $\overline{\mathcal{W}}$ is irreducible.

One local branch of $\overline{\mathcal{W}}$ at x_0 is $\mathcal{W}^s_{\mathbb{C},\mathrm{loc}}(x_0)$, which intersects $\mathcal{L}_0$ perpendicularly. If $\overline{\mathcal{W}}$ had degree 1, then it would intersect $\mathcal{T}$ at the single point $(\phi_0, 1)$, where $x_0 = (\phi_0, 0)$. Since ϕ_0 has prime period $k > 1$ under angle quadrupling, it has prime period $2k$ under angle doubling. This contradicts that $\overline{\mathcal{W}}$ intersects $\mathcal{T}$ at x_1, a point of prime period k.

Therefore, Bezout's Theorem gives a second intersection of $\overline{\mathcal{W}}$ with $\mathcal{L}_0$. It corresponds to some disc in $\hat{\mathcal{W}}$, disjoint from $\hat{U}_0$, whose image under $\hat{\rho}$ intersects $\mathcal{L}_0$. Such an intersection point must then have iterated preimages under $\mathcal{R}^k$ converging to x_c. However, the only point of $\mathcal{B}$ having iterated preimages outside of $\mathcal{B}$ is the fixed point β_0. We conclude that $\overline{\mathcal{W}}$ intersects $\mathcal{L}_0$ at β_0. Meanwhile, $\overline{\mathcal{W}}$ does not contain either of the invariant separatrices $\{z = 1\}$ and $\{t = 0\}$ since it is irreducible and contains the point x_c that is on neither separatrix. Therefore, the dynamics near β_0 would result in infinitely many branches, which is impossible.

Suppose that x_c is vertically neutral. Then, within $\mathcal{W}^c_{\mathrm{loc},\mathbb{C}}(x_c)$ is some repelling petal $\mathcal{P}$ for the parabolic point x_c. Then there is some open $H \subset \mathbb{C}$ containing a left half-space with Fatou coordinate $\rho_0 : H \to \mathcal{P}$ a conformal isomorphism that satisfies $\rho_0(x+1) = \mathcal{R}^k(\rho_0(x))$. We define $\rho : \mathbb{C} \to \mathbb{CP}^2$ by

$$\rho(x) = \lim_{n\to\infty} \mathcal{R}^{n\cdot k}(\rho_0(x-n)).$$

The composition extends through indeterminate points of $\mathcal{R}$ in the same way as in the repelling case.

Then, $\rho(\mathbb{C}) \subset \mathbb{CP}^2$ is forward invariant under $\mathcal{R}^k$ and contains $\mathcal{P}$, which is an open subset of $\gamma_{\mathbb{C}}$. As in the repelling case, one can compactify $\rho(\mathbb{C})$ to form a periodic algebraic curve. This again leads to an intersection with β_0, and a contradiction. $\square$

Remark 14.1. Artur Avila has shown that for almost all points $x = (\phi, 1)$ on the top $\mathcal{T}$, the leaf landing at x is not real analytic [Av].

Proposition 14.2. *Suppose that γ is a regular periodic leaf (of prime period $k > 1$) containing no vertically neutral periodic points. Then, γ has a finite degree of smoothness.*

Proof. Let $\gamma_m = \gamma([0, t_m])$ be the maximal real-analytic piece of γ extending from $\mathcal{B}$. By Proposition 14.1, $\gamma_m \subsetneq \gamma$, with $x_m := \gamma(t_m)$ a periodic point of prime period k.

By hypothesis, $\lambda_c(x_m) \neq 1$. Furthermore, x_m cannot be vertically attracting, since the stable manifold $\mathcal{W}^s(x_m)$ would be a real-analytic curve within γ that extends above and below x_m. Therefore, x_m is vertically repelling.

Suppose that x_m is linearizable. Then, $\mathcal{R}$ is conjugate to the linear map $(u, v) \mapsto (\lambda_u u, \lambda_c v)$. Any central invariant manifold has the form

$$u = \begin{cases} C_1 v^\alpha \text{ if } v \geq 0, \text{ and} \\ C_2 v^\alpha \text{ if } v < 0, \end{cases}$$

where $\alpha = \log(\lambda_u)/\log(\lambda_c)$. By the choice of x_m, the central manifold that is formed by γ is not analytic at x_m, therefore it does not correspond to $C_1 = C_2 = 0$ or, if $\alpha \in \mathbb{N}$, to $C_1 = C_2$. In the remaining cases, the central manifold is not of class C^r, where $r - 1 < \alpha \leq r$.

Since $\lambda_c, \lambda_u > 1$, we are in the Poincaré domain, with the only obstruction to linearization being a resonance of the form $\lambda_c^r = \lambda_u$ for some $r \in \mathbb{N}$. Thus, if x_m is not linearizable, the Poincaré-Dulac Theorem gives that in some neighborhood $U \subset \mathcal{C}$ of x_m, $\mathcal{R}$ is real-analytically conjugate to the normal form

$$(u, v) \mapsto (\lambda_u u + a v^r, \lambda_c v)$$

with $a \neq 0$. (See, e.g. [IY].)

Any central manifold is given by $u = g(v)$. Invariance gives:

$$a v^r = g(\lambda_c v) - \lambda_u g(v) = g(\lambda_c v) - \lambda_c^r g(v)$$

Differentiating r and $r + 1$ times, respectively, one finds

$$ar!/\lambda_c^r = g^{(r)}(\lambda_c v) - g^{(r)}(v), \text{ and} \tag{14.1}$$

$$0 = \lambda_c g^{(r+1)}(\lambda_c v) - g^{(r+1)}(v). \tag{14.2}$$

By (14.2), either $g^{(r+1)}(v) \equiv 0$, or it is undefined at $v = 0$. In the former case, substitution of $g^{(r)}(v) \equiv C$ into (14.1) gives a contradiction. Thus, the central manifold is not of class C^{r+1}. □

Since the leaves of $\mathcal{F}^c$ are obtained by integrating the continuous line field $\mathcal{L}^c(x)$, they are all at least C^1 smooth. In fact, the regular periodic leaves have a slightly better smoothness:

Proposition 14.3. *Any regular periodic leaf $\gamma \in \mathcal{F}^c$ is $C^{1+\delta}$ for some $\delta > 0$.*

Proof. It suffices to show that the line field $\mathcal{L}^c$ is Hölder on γ with exponent δ.

Replacing $\mathcal{R}$ with an iterate of itself (keeping the same notation) we can assume that γ is invariant. Below, the inverse map $\mathcal{R}^{-1}$ will stand for $(\mathcal{R}|\gamma)^{-1}$.

For any $x, y \in \gamma$ and two parallel tangent lines $X \in \mathcal{K}^v(x)$, $Y' \in \mathcal{K}^v(y)$ we have a Lipschitz estimate:

$$\operatorname{dist}_a(D\mathcal{R}^{-1}X, D\mathcal{R}^{-1}Y') \quad \leq \quad M \operatorname{dist}(x, y),$$

where dist_a denotes the angular distance.

On the other hand, Lemma 7.3 implies that there exists $\sigma \in (0,1)$ so that for any vertical lines $Y', Y \in \mathcal{K}^v(y)$ we have

$$\operatorname{dist}_a(D\mathcal{R}^{-1}Y', D\mathcal{R}^{-1}Y) \leq \sigma \operatorname{dist}_a(Y', Y)$$

Putting the last two estimates together, we obtain

$$\operatorname{dist}_a(D\mathcal{R}^{-1}X, D\mathcal{R}^{-1}Y) \leq \sigma \operatorname{dist}_a(X, Y) + M \operatorname{dist}(x, y)$$

for any $X \in \mathcal{K}^v(x)$, $Y \in \mathcal{K}^v(y)$. Iterating this estimate in the backward time we obtain[33]:

(14.3)

$$\operatorname{dist}_a(D\mathcal{R}^{-n}X, D\mathcal{R}^{-n}Y) \leq \frac{M d_n(x,y)}{1-\sigma}, \quad \text{where } d_n(x,y) = \max_{0 \leq k \leq n-1} \operatorname{dist}(\mathcal{R}^{-k}x, \mathcal{R}^{-k}y),$$

as long as $\operatorname{dist}_a(X, Y) < d/(1-\sigma)$. (We can always start with parallel X and Y).

Let now K be a Lipschitz constant for $\mathcal{R}|\gamma$, and let $K_1 > \max(K, 1)$. Take two nearby points $\alpha, \beta, \in \gamma$ and find n such that

$$K_1^{-n} \leq \operatorname{dist}(\alpha, \beta) < K_1^{-(n-1)}. \tag{14.4}$$

Letting $x = \mathcal{R}^n \alpha$, $y = \mathcal{R}^n \beta$, we obtain

$$d_n(x,y) \leq K^n \operatorname{dist}(\alpha, \beta) \leq K_1 \kappa^n, \quad \text{where } \kappa = K/K_1 < 1. \tag{14.5}$$

By (14.3), we have

$$\operatorname{dist}_a(D\mathcal{R}^{-n}X, D\mathcal{R}^{-n}Y) = O(\kappa^n).$$

But according to Proposition 7.1, $D\mathcal{R}^{-n}X$ is exponentially close to the tangent line $\mathcal{L}^v(\alpha)$ to γ at α, and likewise $D\mathcal{R}^{-n}Y$ is exponentially close to the tangent line $\mathcal{L}^v(\beta)$. Hence

$$\operatorname{dist}_a(\mathcal{L}^v(\alpha), \mathcal{L}^v(\beta)) = O(\eta^n) \quad \text{for some } \eta \in (0,1).$$

Together with (14.4), this implies that

$$\operatorname{dist}_a(\mathcal{L}^v(\alpha), \mathcal{L}^v(\beta)) \leq C \operatorname{dist}(\alpha, \beta)^\delta, \quad \text{with } \delta = \frac{\log K_1}{\log(1/\kappa)}.$$

□

Remark 14.2. The above argument applies to any vertical leaf whose forward orbit stays away from the points of indeterminacy $\alpha_\pm$. The problem with other leaves is that the Lipschitz estimate for $\mathcal{R}|\gamma$ may fail for leaves γ passing through α because of the big expansion near the $\alpha_\pm$.

Appendix A. Elements of complex geometry

We are primarily interested in rational maps between complex projective spaces in two dimensions. However, in order to understand the behavior near indeterminate points, we will need a discussion at somewhat greater generality. Much of the below material can be found in with greater detail in [Da, De, GH, Shaf].

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[33]The simplest way to see it is to notice that the iterates are bounded by the fixed point of the linear map $x \mapsto \sigma x + Md$.

A.1. **Projective varieties and rational maps.** Let $\pi : \mathbb{C}^{k+1} \smallsetminus \{0\} \to \mathbb{CP}^k$ denote the canonical projection. Given $z \in \mathbb{CP}^k$, any $\hat{z} \in \pi^{-1}(z)$ is called a *lift* of z. One calls $V \subset \mathbb{CP}^k$ a (projective) algebraic hypersurface if there is a homogeneous polynomial $\hat{p} : \mathbb{C}^{k+1} \to \mathbb{C}$ so that

$$V = z \in \mathbb{CP}^k : \quad \hat{p}(\hat{z}) = 0.$$

More generally, a (projective) algebraic variety is the locus satisfying a finite number projective polynomial equations. Any algebraic variety V has the structure of a smooth manifold away from a proper subvariety $V_{\text{sing}} \subset V$ and the dimension of $V \smallsetminus V_{\text{sing}}$ is called *the dimension of* V.

A rational map $R : \mathbb{CP}^k \to \mathbb{CP}^l$ is given by a homogeneous polynomial map $\hat{R} : \mathbb{C}^{k+1} \to \mathbb{C}^{l+1}$ for which we will assume the components have no common factors. One defines $R(z) := \pi(\hat{R}(\hat{z}))$ if $\hat{R}(\hat{z}) \neq 0$, and otherwise we say that z is an *indeterminacy point for* R. Since $\hat{R}$ is homogeneous, the above notions are well-defined. Because the components of $\hat{R}$ have no common factors, the set of indeterminate points $I(R)$ is a projective variety of codimension greater than or equal to two.

Given two projective varieties, $V \subset \mathbb{CP}^k$ and $W \subset \mathbb{CP}^l$, a *rational map* $R : V \to W$ is the restriction of a rational map $R : \mathbb{CP}^k \to \mathbb{CP}^l$ such that $R(V \smallsetminus I(R)) \subset W$. As above, $I(R) \subset V$ is a projective subvariety of codimension greater than or equal to two in V. If $I(R) = \emptyset$, we say that R is a (globally) *holomorphic (regular)* map.

A rational mapping $R : V \to W$ between non-singular varieties is *dominant* if there is a point $z \in V \smallsetminus I(R)$ such that $\operatorname{rank} DR(z) = \dim W$.

We will call a subvariety $U \subset V$ a *collapsing variety*[34] if $\dim(R(U)) < \dim(U)$.

Lemma A.1. *Let $R : V \to W$ be a dominant rational map between projective manifolds of the same dimension. If z is not an indeterminate point for R and not on any collapsing variety for R, then R is locally surjective at z.*

It is a consequence of the Weierstrass Preparation Theorem—see for example, [De, Ch. II, §4.2] or [GH, Ch. 0.1].

A.2. **Blow-ups.** A non-singular variety of dimension two is called a *projective surface*. Matters are simpler for maps $R : V \to W$ between projective surfaces since $I(R)$ is finite in this case. (We will refer to such maps as "rational surface maps".)

Given a pointed projective surface (V, p), the *blow-up* of V at p is another projective surface $\tilde{V}$ with a holomorphic projection $\pi : \tilde{V} \to V$ such that

- $L_{\text{exc}}(p) := \pi^{-1}(p)$ is a complex line $\mathbb{CP}^1$ called the *exceptional divisor*;
- $\pi : \tilde{V} \smallsetminus L_{\text{exc}}(p) \to V \smallsetminus \{p\}$ is a biholomorphic map.

See [GH, Shaf].

The construction has a local nature near p, so it is sufficient to provide it for $(\mathbb{C}^2, 0)$. The space of lines $l \subset \mathbb{C}^2$ passing through the origin is $\mathbb{CP}^1$ by definition. Then $\tilde{\mathbb{C}^2}$ is realized as the surface X in $\mathbb{C}^2 \times \mathbb{CP}^1$ given by equation $\{(u, v) \in l\}$ with the natural projection $(u, v, l) \mapsto (u, v)$. In this model, points of the exceptional divisor $L_{\text{exc}} = \{(0, 0, l) : \ l \in \mathbb{CP}^1\}$ get interpreted as the *directions* l at which the origin is approached.

[34] Algebraic geometers call such a variety "exceptional". However this term has a conflicting meaning in complex dynamics, where the "exceptional set of R" consists of the largest proper algebraic variety that is completely invariant under R.

Any line $l \subset \mathbb{C}^2$ naturally lifts to the "line" $\tilde{l} = \{(u,v,l) : (u,v) \in l\}$ in $\tilde{\mathbb{C}^2}$ crossing the exceptional divisor at $(0,0,l)$.[35] Moreover, $\tilde{\mathbb{C}^2} \smallsetminus \tilde{l}$ is isomorphic to $\mathbb{C}^2$. Indeed, let $\phi(u,v) = au + bv$ a linear functional that determines l. It is linearly independent with one of the coordinate functionals, say with v (so $a \neq 0$). Then

$$(u,v,l) \mapsto (\phi, \kappa := v/\phi)$$

is a local chart that provides a desired isomorphism. In particular, two charts corresponding to the coordinate axes in $\mathbb{C}^2$ provide us with local coordinates $(u, \kappa = v/u)$ and $(v, \kappa = u/v)$ which are usually used in calculations.

The value of this construction lies in the fact that it can be used to *resolve* the indeterminacy of a rational map, as follows:

Theorem A.2 (See [Shaf], Ch. IV, §3.3)**.** *Let $R : V \to W$ be a rational surface map. Then there exists a sequence of blow-ups $V_m \xrightarrow{\pi_m} \cdots \xrightarrow{\pi_2} V_1 \xrightarrow{\pi_1} V$ so that R lifts to a globally holomorphic map $\tilde{R} : V_m \to W$ making the following diagram commute*[36]

(A.1)
$$\begin{tikzcd} V_m \arrow[d, "\pi"] \arrow[dr, "\tilde{R}"] & \\ V \arrow[r, "R"] & W. \end{tikzcd}$$

Here, $\pi = \pi_1 \circ \cdots \circ \pi_m$.

Any analytic curve D on V lifts to an analytic curve $\tilde{D} := \overline{\pi^{-1}(D \smallsetminus \{p\})}$ on $\tilde{V}$, known as the *proper transform* of D, which tends to have milder singularities than D:

Theorem A.3. *Let D be an irreducible curve on a non-singular projective surface V. Then, there exist a sequence of blow-ups $V_m \xrightarrow{\pi_m} \cdots \xrightarrow{\pi_2} V_1 \xrightarrow{\pi_1} V$ so that the proper transform of $\tilde{D}$ of D is a non-singular curve on V_m.*

See [Shaf, Ch. IV, §4.1].

A.3. **Divisors.** Divisors are a generalization of algebraic hypersurfaces that behave naturally under dominant rational maps. We will present an adaptation of material from from [Da, Ch. 3], [F, §3], and [Shaf] suitable for our purposes.

A *divisor* D on a projective surface V is a collection of irreducible hypersurfaces $C_1, \dots, C_r$ with assigned integer multiplicities $k_1, \dots, k_r$. One writes D as a formal sum

$$D = k_1 C_1 + \cdots + k_r C_r. \tag{A.2}$$

Alternatively, D can be described by choosing an open cover $\{U_i\}$ of V and rational functions $g_i : U_i \to \mathbb{C}$ with the comparability property that g_i/g_j is a non-vanishing holomorphic function on $U_i \cap U_j \neq \emptyset$. Taking zeros and poles of the g_i counted with multiplicities, we obtain representation (A.2).

These two equivalent descriptions of divisors allow us to pull them back and push them forward under rational maps: If $f : V \to W$ is a dominant holomorphic

[35] This turns $\tilde{\mathbb{C}^2}$ into a line bundle over $\mathbb{CP}^1$ known as the *tautological* line bundle.

[36] As with diagram (3.1), commutativity is only at points where all maps are defined.

map, and $D = \{U_i, g_i\}$ is a divisor on W. The *pullback* f^*D is the divisor on V given by $\{f^{-1}U_i, f^*g_i\} \equiv \{f^{-1}U_i, g_i \circ f\}$.

If $f : V \to W$ is a proper holomorphic map and D is an irreducible curve on V we define its push-forward f_*D to be the divisor that assigns multiplicity[37] $\deg_{\text{top}}(f|D : D \to f(D))$ to the image curve $f(D)$. (If f collapses D to a point, we assign multiplicity 0.) This definition extends linearly to arbitrary divisors on V expressed in form (A.2).

If $R : V \to W$ is a rational map having indeterminacy, we use Theorem A.2 to define the pull-back and push forward of divisors by

$$R^*D_1 := \pi_*\tilde{R}^*D_1 \qquad R_*D_2 := \tilde{R}_*\pi^*D_2 \tag{A.3}$$

where D_1 and D_2 are divisors on W and V, respectively.

Alternatively, one can pull-back D under $R : V \smallsetminus I(R) \to W$. Since $I(R)$ is a finite collection of points, the result (in terms of local defining functions) can be extended trivially to obtain a divisor R^*D on all of V. Since the trivial extension of a divisor is unique, this alternative definition agrees with the previous one.

Any hypersurface C can be triangulated as a singular cycle, thus to any divisor D is an associated *fundamental class* $[D] \in H_2(V)$. Representing D by local defining functions allows us to associate a cohomology class $(D) \in H^2(V)$ called its *Chern class*; see [F]. Furthermore, $[D]$ is the Poincaré dual of (D).

These classes are natural satisfying $[f_*D_1] = f_*[D_1]$ and $(f^*D_2) = f^*(D_2)$ for any holomorphic map $f : V \to W$. For a rational map $R : V \to W$, we again have

$$[R_*D_1] = R_*[D_1] \text{ and } (R^*D_2) = R^*D_2, \tag{A.4}$$

using (A.3) at the level of homology and cohomology and also Poincaré duality.

A.4. **Composition of rational maps.** The *algebraic degree* of a rational map $R : \mathbb{CP}^k \to \mathbb{CP}^l$, denoted $\deg R$, is the degree of the coordinates of its lift $\hat{R} : \mathbb{C}^{k+1} \to \mathbb{C}^{l+1}$.

The following statement appears in [Si, Prop. 1.4.3]:

Lemma A.4. *Let $R : \mathbb{CP}^k \to \mathbb{CP}^l$ and $S : \mathbb{CP}^l \to \mathbb{CP}^m$ be rational maps. Then, $\deg(S \circ R) = \deg(S) \cdot \deg(R)$ if and only if there is no algebraic hypersurface $V \subset \mathbb{CP}^k$ that is collapsed by R to an indeterminate point of S.*

Remark A.1. To understand this phenomenon geometrically (for simplicity, in dimension two: k=l=m=2), let us consider the algebraic curve G to which the indeterminacy point γ blows up under S. Then any line L must intersect G, and hence $S^{-1}L$ passes through γ. It follows that $V \subset R^{-1}(S^{-1}L)$. On the other hand, $V \not\subset (S \circ R)^{-1}L$ (unless $L \supset G$, which may happen only for a special line). But according to Lemma A.5 below,

$$\deg S \cdot \deg R = \deg(S^{-1}(R^{-1}L)), \quad \deg(S \circ R) = \deg(S \circ R)^{-1}L$$

So, components of V, possibly with multiplicities, account for the degree deficit.

[37] Note that it is possible for a non-generic point of $f(V)$ to have more than $\deg_{\text{top}}(f : V \to W)$ preimages under $f|V$.

A.5. **Degree of divisors in $\mathbb{CP}^2$.** Associated to any homogeneous polynomial $p : \mathbb{C}^3 \to \mathbb{C}$ is a divisor D_p given by $\{U_i, p \circ \sigma_i\}$, where the $\{U_i\}$ form an open covering of $\mathbb{CP}^2$ that admits local sections $\sigma_i : U_i \to \mathbb{C}^3 \smallsetminus \{0\}$ of the canonical projection π. Furthermore, every divisor can be described as a difference $D = D_p - D_q$ for appropriate p and q. The following simple formula describes the pull-back:

$$R^* D_p = D_{\hat{R}^* p} \equiv D_{p \circ \hat{R}}. \tag{A.5}$$

The *degree* of a divisor $D = D_p - D_q$ is $\deg D = \deg p - \deg q$. Any complex projective line L on which p does not identically vanish intersects the divisor D_p exactly $\deg D_p$ times (counted with multiplicity), providing an alternative geometric definition of $\deg D_p$.

More generally, *Bezout's Theorem* asserts that two divisors D_1 and D_2 intersect $\deg D_1 \cdot \deg D_2$ times in $\mathbb{CP}^2$, counted with appropriate *intersection multiplicities*. Suppose that D_1 and D_2 are irreducible algebraic curves assigned multiplicity one. Then, an intersection point z is assigned multiplicity one if and only if both curves are non-singular at z, meeting transversally there. See [Shaf, Ch. IV].

There is an alternative, homological, definition for $\deg D$. Namely, any algebraic curve D represents a class $[D] \in H_2(\mathbb{CP}^2)$. Moreover, $H_2(\mathbb{CP}^2) = \mathbb{Z}$ and is generated by the class $[L]$ of any line L. Then we have

$$[D] = \deg D \cdot [L] \tag{A.6}$$

Lemma A.5. *Given a dominant rational map $R : \mathbb{CP}^2 \to \mathbb{CP}^2$ and any divisor D on $\mathbb{CP}^2$ we have:*

$$\deg(R^* D) = \deg R \cdot \deg D, \ \textit{and} \tag{A.7}$$

$$\deg(R_* D) = \deg R \cdot \deg D. \tag{A.8}$$

In particular, $\deg(R^* L) = \deg(R_* L) = \deg R$ *for any projective line* $L \subset \mathbb{CP}^2$.

Proof. Equation (A.7) follows immediately from (A.5). To obtain (A.8) we make use of the homological interpretation of degree (A.6). By (A.4), the push-forward of divisors R_* preserves homology, so it suffices to check (A.8) for any complex projective line L.

We choose L disjoint from $I(R)$ and we can assume that $[0 : 0 : 1] \notin R(L)$. Let $\iota : \mathbb{CP}^1 \to \mathbb{CP}^2$ be the inclusion of L into $\mathbb{CP}^2$ and $\mathrm{pr} : \mathbb{CP}^2 \to \mathbb{CP}^1$ the central projection onto the line at infinity L_∞ from the center $[0 : 0 : 1]$. Note that both i_* and pr_* induce isomorphisms on the second homology.

We consider the composition $\mathrm{pr} \circ R \circ \iota : \mathbb{CP}^1 \to \mathbb{CP}^1$. By Lemma A.4,

$$\deg(\mathrm{pr} \circ R \circ \iota) = \deg \mathrm{pr} \cdot \deg R \cdot \deg \iota = \deg R,$$

since the image of ι avoids $I(R)$ and the image of $R \circ \iota$ avoids $I(\mathrm{pr}) = \{[0 : 0 : 1]\}$.

Thus,

$$\mathrm{pr}_* \circ R_*[L] = \mathrm{pr}_* \circ R_* \circ \iota_*[\mathbb{CP}^1] = \deg(\mathrm{pr} \circ R \circ \iota)[\mathbb{CP}^1] = \deg R \cdot [\mathbb{CP}^1]$$

Since $\mathrm{pr}_* : H_2(\mathbb{CP}^2) \to H_2(\mathbb{CP}^1)$ is an isomorphism we find $[R_* L] = \deg R \cdot [L_\infty]$. □

Remark A.2. Formulas (A.7) and (A.8) generalize to other varieties using the fact that pull-back and push-forward (under suitable maps) preserve *linear equivalence of divisors,* see [Da, Ch. 3 §5.2] and [F, §3.3].

A.6. **Iteration of rational maps.** A rational mapping $R : \mathbb{CP}^2 \to \mathbb{CP}^2$ is called *algebraically stable* if there is no integer n and no collapsing hypersurface $V \subset \mathbb{CP}^2$ so that $R^n(V)$ is contained within the indeterminacy set of R, [Si, p. 109]. By Lemma A.4, R is algebraically stable if and only if $\deg R^n = (\deg R)^n$. Together with Lemma A.5, this yields:

Lemma A.6. *If R is a dominant algebraically stable map and D is any divisor on $\mathbb{CP}^2$ we have:*

$$\begin{aligned} \deg((R^n)^* D) &= (\deg R)^n \cdot \deg D \\ \deg((R^n)_* D) &= (\deg R)^n \cdot \deg D \end{aligned}$$

Appendix B. Renormalization near the indeterminacy points

B.1. **Blow-ups.** Below we calculate blow-ups for the renormalization in the affine coordinates (u, v) and in the angular coordinates (ϕ, t).

B.1.1. *Affine coordinates.* Let us represent R as $Q \circ g$ where

$$g : (u, w) \mapsto \left(\frac{u^2+1}{u+w}, \frac{w^2+1}{u+w} \right)$$

and $Q : (u, w) \mapsto (u^2, w^2)$. The indeterminacy points for R and g are the same, $a_\pm = \pm(i, -i)$. Because of the basic symmetry $(u, w) \mapsto (w, u)$, it is sufficient to carry the calculation at $a_+ = (i, -i)$. In coordinates $\xi = u - i$ and $\chi = (w+i)/(u-i)$, we obtain the following expression for the map $\tilde{g} : \tilde{\mathbb{CP}}^2 \to \mathbb{CP}^2$ near $L_{\text{exc}}(a_+)$:

$$u = \frac{\xi + 2i}{1+\chi}, \quad w = \frac{\chi^2 \xi - 2i\chi}{1+\chi}. \tag{B.1}$$

So $L_{\text{exc}}(a_+) = \{\xi = 0\}$ is mapped by $\tilde{g}$ biholomorphically onto the line

$$\left\{ u = \frac{2i}{1+\chi}, \ w = -\frac{2i\chi}{1+\chi} \right\} = \{u - w = 2i\}. \tag{B.2}$$

In other words, g blows up the indeterminacy point a_+ to line (B.2). Notice that this line connects a_+ to the low-temperature fixed point $b_0 = (1 : 0 : 1)$ at infinity. Its slice by the Hermitian plane $C = \{w = \bar{u}\}$ (corresponding to the cylinder $\mathcal{C}$) is the horizontal line $\{\operatorname{Im} u = 1\}$.

The lift $\tilde{R}$ of R to $L_{\text{exc}}(a_+)$ is given by $\tilde{R} = Q \circ \tilde{g}$, and hence obtained by squaring the expressions for u and w in (B.1). The image of $L_{\text{exc}}(a_+)$ under $\tilde{R}$ is given by $G := Q(\{u - w = 2i\}) = \{(u-w)^2 + 8(u+w) + 16 = 0\}$.

B.1.2. *Angular coordinates.* (compare [BZ3, p. 419]). We will now calculate the blow-up of $\mathcal{R}$ at the indeterminacy points $\alpha_\pm$ in the angular coordinates (ϕ, t) on $\mathcal{C}$. As before, it suffices to consider α_+. We let $\epsilon = \frac{\pi}{2} - \phi$ and $\tau = 1 - t$, and $\kappa = \tau/\epsilon$. In the blow-up coordinates (ϵ, κ) we find:

$$(z', t') = \left(\frac{-i + \kappa - \epsilon - \epsilon\kappa^2/2}{i + \kappa - \epsilon - \epsilon\kappa^2/2}, \ \frac{1 - 2\epsilon\kappa}{1 + \kappa^2 - \epsilon(2\kappa + \kappa^3)} \right) + O(|\epsilon|^2), \tag{B.3}$$

where the constant in the residual term depends on an upper bound on κ.
Thus

$$\phi' = -i \log(z') = -2 \operatorname{arcctg}(\kappa - \epsilon + \epsilon\kappa^2/2) + O(|\epsilon|^2). \tag{B.4}$$

Taking the limit as $\epsilon \to 0$, we find:

$$(\phi', t') = \left(-2 \operatorname{arcctg} \kappa,\ \frac{1}{1+\kappa^2} \right) \tag{B.5}$$

Letting ω be the (complexified) angle between the collapsing line $\mathcal{I}_{\pi/2}$ and the line with slope $\kappa = -\operatorname{ctg} \omega$, we come up with expression (3.4) for the blow-up locus $\mathcal{G}$. (Notice that the blow-up loci for the maps f and $\mathcal{R} = f \circ Q$ coincide, since Q is a local diffeomorphism.)

Recall that point $\alpha = (\pi, 1)$, which mapped by $\mathcal{R}$ to the high temperature fixed point β_1.

Lemma B.1. *Let $c > 0$. If $\zeta = (\epsilon, \tau) \in \mathbb{C}^2$ is sufficiently close to one of the indeterminacy points $\alpha_\pm$ with the slope $\kappa = \tau/\epsilon$ sufficiently small and $|\epsilon - \kappa| \geq c|\epsilon|$, then*

$$\operatorname{dist}^h(\mathcal{R}(x), \alpha) \asymp |\epsilon - \kappa|,$$

with the constant depending on c.

Proof. Indeed, under our assumptions, formula (B.4) implies $|\phi' - \pi| \asymp |\kappa - \epsilon|$. $\square$

B.2. **The differential $D\mathcal{R}$.** Formula (3.6) implies the following explicit expression for the differential $D\mathcal{R}$ at $x = (\phi, t) \in \mathcal{C}$:

$$D\mathcal{R} = \frac{4}{\zeta^2} \begin{pmatrix} \zeta & 0 \\ 0 & 1-t^2 \end{pmatrix} \begin{pmatrix} 1 + t^2 \cos 2\phi & -\sin 2\phi \\ -t^2(1-t^2)\sin 2\phi & (1+t^2)(1+\cos 2\phi) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \tag{B.6}$$

where $\zeta(x) = 1 + 2t^2 \cos 2\phi + t^4$.

Expanding it in $\tau = 1 - t$ near $\mathcal{T}$, we obtain:

$$D\mathcal{R} = \begin{pmatrix} 2 + O(\tau) & -2 \operatorname{tg} \phi + O(\tau) \\ -2\tau^2 \operatorname{tg} \phi \,(\cos\phi)^{-2} + O(\tau^3) & 2\tau (\cos \phi)^{-2} + O(\tau^2) \end{pmatrix}. \tag{B.7}$$

In the $\epsilon = \pm\pi/2 - \phi$ coordinate near an indeterminacy point $\alpha_\pm$ we obtain the following asymptotic expression for the differential $\mathcal{R} : (\epsilon, \tau) \mapsto (\phi', \tau')$:

$$D\mathcal{R} \sim \frac{2}{\sigma^4} \begin{pmatrix} (\epsilon^2 + \tau)\sigma^2 & -\epsilon\sigma^2 \\ -\epsilon\tau^2 & \tau\epsilon^2 \end{pmatrix} \tag{B.8}$$

where $\sigma = \sqrt{\epsilon^2 + \tau^2}$.

B.3. **Horizontal stretching near $\mathcal{T}$.** Let us define the *horizontal expansion factor* $\lambda^h_{\min}(x)$ at $x \in \mathcal{C} \smallsetminus \{\alpha_\pm\}$ as

$$\lambda^h_{\min}(x) = \inf_{v \in \mathcal{K}^h(x)} \frac{D(\pi \circ \mathcal{R})(v)}{D\pi(v)}, \tag{B.9}$$

where $\pi(\phi, t) = \phi$. Equivalently, consider an almost horizontal curve ξ through $x = (\phi, t)$ naturally parameterized by the angular coordinate (by means of $(\pi|\xi)^{-1}$). Let

$$\chi \equiv \chi_\xi = \pi \circ \mathcal{R} \circ (\pi|\xi)^{-1}.$$

Then

$$\lambda^h_{\min}(x) = \inf_\xi \chi'_\xi(\phi).$$

The *n-th horizontal expansion factor* $\lambda^h_{\min,n}(x)$ is defined similarly, by replacing $D(\pi \circ \mathcal{R})$ with $D(\pi \circ \mathcal{R}^n)$ in (B.9).

Lemma B.2. *There exists* $\lambda_0 > 0$ *so that for any* $x = (\epsilon, \tau) \in \mathcal{C} \setminus \{\alpha_\pm\}$ *near one of the indeterminacy points* $\alpha_\pm$ *we have* $\lambda^h_{\min}(x) \geq \lambda_0$.

Moreover, given a slope $\bar{\kappa} > 0$, *there exists* $\lambda_1 > 0$ *such that if* $x = (\epsilon, \tau)$ *also satisfies* $|\kappa| \equiv |\tau/\epsilon| \leq \bar{\kappa}$, *then*

$$\lambda^h_{\min}(x) \geq \lambda_1 \left| \frac{\kappa + \epsilon}{\epsilon} \right|.$$

Proof. Take a horizontal vector $v = (1, s) \in \mathcal{K}^h(x)$ with slope s. By definition of the horizontal cone field $\mathcal{K}^h$, $|s| \leq \max\{\sqrt{2\tau}, |\epsilon|/3\}$ (see §6.2, items (ii)-(iii) in the definition of $\mathcal{K}^h$). Applying the asymptotical expression (B.8), we obtain

$$\begin{aligned} \chi'(\phi) &= D(\pi \circ \mathcal{R})(1, s) = 2\frac{\epsilon^2 + \tau - s\epsilon}{|\sigma|^2} \\ &\geq \frac{2}{|\sigma|^2} \min\{\epsilon^2 + \tau - |\epsilon|\sqrt{2\tau},\ (2/3)\epsilon^2 + \tau\} \geq \lambda_0 \frac{\epsilon^2 + \tau}{\epsilon^2 + \tau^2} \geq \lambda_0. \end{aligned}$$

The second to last estimate follows from positive definitiveness of the quadratic form $\epsilon^2 + \tau - \epsilon\sqrt{2\tau}$ in ϵ and $\sqrt{\tau}$. Finally,

$$\frac{\epsilon^2 + \tau}{\epsilon^2 + \tau^2} = \frac{|\kappa + \epsilon|}{|\epsilon|(\kappa^2 + 1)} \geq \frac{1}{\bar{\kappa}^2 + 1} \left| \frac{\kappa + \epsilon}{\epsilon} \right|,$$

and the second estimate follows. □

Lemma B.3. *There exists* $\lambda_2 > 0$ *so that for any* $x \in \mathcal{C} \setminus \{\alpha_\pm\}$ *we have* $\lambda^h_{\min}(x) \geq \lambda_2$.

Proof. Away from $\{\alpha_\pm\}$ this is true since the horizontal cones are transverse to the critical lines $\mathcal{I}_{\pm\pi/2}$. Near $\{\alpha_\pm\}$, it follows from Lemma B.2. □

We now estimate horizontal expansion of vectors (and hence curves) taken with respect to the algebraic cone field $\mathcal{K}^{ah}$. It will be useful to consider both upper and lower bounds:

$$\lambda^{ah}_{\min}(x) = \inf_{v \in \mathcal{K}^{ah}(x)} \frac{D(\pi \circ \mathcal{R})(v)}{D\pi(v)}, \text{ and } \lambda^{ah}_{\max}(x) = \sup_{v \in \mathcal{K}^{ah}(x)} \frac{D(\pi \circ \mathcal{R})(v)}{D\pi(v)}.$$

The n-th expansion factors $\lambda^{ah}_{\min,n}(x)$ and $\lambda^{ah}_{\max,n}(x)$ are defined similarly. Note that $\lambda^{ah}_{\min}(x) \geq \lambda^h_{\min}(x)$, since $\mathcal{K}^{ah}(x) \subset \mathcal{K}^h(x)$. In particular the estimate of Lemma B.2 also applies to $\lambda^{ah}_{\min}(x)$.

Lemma B.4. *For any* $\delta > 0$ *there exist* $\eta > 0$ *and* $\bar{\tau} > 0$ *such that for any* $x \in \mathcal{V}'$ *we have:*

$$(2 - \delta) < \lambda^{ah}_{\min}(x) < \lambda^{ah}_{\max}(x) < (2 + \delta).$$

Proof. The slope a vector $v \in \mathcal{K}^{ah}(x)$ is bounded by $\sqrt{2\tau}$, where $\tau = \tau(x)$.

Near the indeterminacy points $\alpha_\pm$ we can use (B.8) to bound from below the horizontal stretching of the boundary tangent vector $v = (1, \pm\sqrt{2\tau})$:

$$2\,\frac{\epsilon^2 + \tau - |\epsilon|\sqrt{2\tau}}{\epsilon^2 + \tau^2} < D(\pi \circ \mathcal{R})(1, s) < 2\,\frac{\epsilon^2 + \tau + |\epsilon|\sqrt{2\tau}}{\epsilon^2 + \tau^2}.$$

Since $x \in \mathcal{V}'$, we have $\sqrt{2\tau} < \sqrt{2\eta}|\epsilon|$ which gives:

$$(1 - \sqrt{2\eta})(\epsilon^2 + \tau) \leq \epsilon^2 + \tau \pm |\epsilon|\sqrt{2\tau} \leq (1 + \sqrt{2\eta})(\epsilon^2 + \tau).$$

Hence

$$2(1-\sqrt{2\eta}) \le D(\pi\circ\mathcal{R})(1,s) \le 2(1+\sqrt{2\eta}),$$

which can be made arbitrary close to 2 if one's η is chosen sufficiently small.

Now suppose that v is based $\bar{\epsilon}$-away from $\alpha_\pm$. Then $\mathrm{tg}\,\phi \le 2/\bar{\epsilon}$ and (B.7) implies:

$$\text{(B.10)}\qquad 2(1-2\sqrt{2\tau}/\bar{\epsilon})(1+O(\tau)) \le D(\pi\circ\mathcal{R})(1,s) \le 2(1+2\sqrt{2\tau}/\bar{\epsilon})(1+O(\tau)),$$

which can be made arbitrary close to 2 by choosing τ small enough. □

Appendix C. Complex extension of the cone fields

C.1. **Terminology and notation.** Let $\mathcal{C}^c := (\mathbb{C}/2\pi)\times\mathbb{C}$ be the complexification of the full cylinder $(\mathbb{R}/2\pi)\times\mathbb{R}$.

A real horizontal cone field over $\mathcal{C}$ is given as

$$\text{(C.1)}\qquad \mathcal{K}(x) = \{v=(dt,d\phi)\in T_x\mathcal{C} : |dt| < s(x)\,|d\phi|\} \subset T_x\mathcal{C}$$

for an appropriate slope function $s(x)\ge 0$.

The cones $\mathcal{K}(x)$ can be *complexified* to the complex cones $\mathcal{K}^c(x)\subset T_x\mathcal{C}^c$ by means of the same formula (C.1) where $d\phi$ and dt are interpreted as complex coordinates in $T_x\mathcal{C}$. Notice that $\mathcal{K}^c(x)$ is obtained by rotating $\mathcal{K}(x)$ by multiplications $v\mapsto e^{i\theta}v$. Since $\mathcal{R}$ commutes with this action, invariance of a real cone field $\mathcal{K}(x)$ implies invariance of its complexification.

So, let us complexify the cone field $\mathcal{K}^{ah}(x)$ (for $x\in\mathcal{C}$). (We will typically omit the superscripts c from the complexification, to simplify the notation.) We can further extend this cone field to a neighborhood of $\mathcal{C}$ in $\mathcal{C}^c$ by extending continuously the slope function s. By continuity, the extension of $\mathcal{K}^{ah}$ is invariant away from the top. However, for the application to distortion control in §11, we will need an extension that is invariant on a suitable "pinched" neighborhoods of the $\alpha_\pm$.

C.2. **Complex extension of $\mathcal{K}^{ah}$.** Define an extension of $\mathcal{K}^{ah}\equiv\mathcal{K}^{ah,c}$ to $\mathcal{C}^c$ by letting $s^a(\zeta)=\sqrt{|1-t^2|}=\sqrt{|\tau(2-\tau)|}$. (When τ is real this coincides with (6.3).) For $\rho>0$, let

$$\text{(C.2)}\qquad \mathcal{C}^c_\rho := \{(\phi,t)\in\mathcal{C}^c \;:\; |\operatorname{Im}\phi|<\rho,\ -\rho<\operatorname{Re}t<1+\rho,\ \text{and}\ |\operatorname{Im}t|<\rho\}.$$

To ensure invariance of the cone field $\mathcal{K}^{ah}$, we will need to appropriately "pinch" $\mathcal{C}^c$ near the points of indeterminacy $\alpha_\pm$. Given $\theta>0$, let

$$\mathcal{N}(\theta) := \{(\epsilon,\tau) \;:\; |\arg\epsilon\,(\mathrm{mod}\,\pi)|<\theta \text{ and } |\arg\tau\,(\mathrm{mod}\,2\pi)|<\theta\}\cup\{|\epsilon|<\frac{1}{2}|\tau|)\}.$$

(Note that in the first set, ϵ is allowed to be negative, while τ is not.) Furthermore, let

$$\mathcal{C}^c_\rho(\theta) := \{\zeta=(\epsilon,\tau)\in\mathcal{C}^c_\rho \;:\; \zeta\in\mathcal{N}(\theta) \text{ whenever } |\epsilon|<\rho \text{ and } |\tau|<\rho\}.$$

Proposition C.1. *There exist $\rho>0$ and $\theta>0$ sufficiently small so that if $\zeta\in\mathcal{C}^c_\rho(\theta)$ then*

$$\text{(C.3)}\qquad D\mathcal{R}(\mathcal{K}^{ah}(\zeta))\subset\mathcal{K}^{ah}(\mathcal{R}\zeta).$$

Proof. Since $\mathcal{K}^{ah}(x)$ is invariant on the real cylinder $\mathcal{C}$ and non-degenerate on any $K\Subset\mathcal{C}_1$, the extension is invariant on a complex neighborhood of K. Thus, we need only find $\rho>0$ and $\theta>0$ sufficiently small so that (C.3) holds at points in $\mathcal{C}^c_\rho(\delta)$ with $|\tau|<\rho$.

For c slightly above $\sqrt{2}$, consider the following auxiliary complex conefield

$$\widehat{\mathcal{K}}(\zeta) = \{v = (dt, d\phi) \in T_\zeta \mathcal{C}^c : |dt| < c\sqrt{|\tau|}|d\phi|\} \subset T_\zeta \mathcal{C}^c. \tag{C.4}$$

If $\rho > 0$ is sufficiently small, then we have $\mathcal{K}^{ah}(\zeta) \subset \widehat{\mathcal{K}}(\zeta)$ for all $\zeta \in \mathcal{C}^c_\rho$. Hence it is sufficient to verify that if $\zeta \in \mathcal{C}^c_\rho(\theta)$ with $|\tau(\zeta)| < \rho$, then we have $\mathcal{R}(\widehat{\mathcal{K}}(\zeta)) \subset \mathcal{K}^{ah}(\mathcal{R}\zeta)$. It will be shown in Lemmas C.2 and C.3 below. By symmetry, it suffices to work near α_+.

Lemma C.2 (**Invariance near** $\alpha_\pm$**).** *There exist $\rho > 0$ and $\theta > 0$ with the following property. For any point $\zeta = \left(\frac{\pi}{2} - \epsilon, 1 - \tau\right)$ with $|\tau| < \rho$ and $|\epsilon| < \rho$ lying in the pinched neighborhood $\mathcal{N}(\theta)$ of α_+ we have: $D\mathcal{R}(\widehat{\mathcal{K}}(\zeta)) \Subset \mathcal{K}^{ah}(\mathcal{R}\zeta)$.*

Proof. According to the blow-up formula (B.5), if ρ is sufficiently small then ζ is mapped by $\mathcal{R}$ to a point with $\tau' \approx \dfrac{\kappa^2}{1+\kappa^2}$, where $\kappa = \tau/\epsilon$.

Let $v = (1, s)$ be a complex tangent vector based at (τ, ϵ) with $v \in \mathrm{cl}(\widehat{\mathcal{K}}(\zeta))$, i.e., $|s| \le c\sqrt{|\tau|}$. We want to show that the slope s' of the image vector $D\mathcal{R}(v)$ satisfies

$$|s'| < \sqrt{|\tau'(2-\tau')|} \approx \frac{|\kappa|}{|1+\kappa^2|}\sqrt{|2+\kappa^2|}.$$

Equation (B.8) from the Appendix B gives us the matrix $A := \frac{(\epsilon^2+\tau^2)^2}{2} D\mathcal{R}$ to lowest order terms in ϵ and τ:

$$Av = \begin{bmatrix} (\tau + \epsilon^2 - s\epsilon)(\tau^2+\epsilon^2) \\ -\epsilon\tau^2 + s\epsilon^2\tau \end{bmatrix}$$

Thus

$$s' \approx \frac{\epsilon\tau(s\epsilon - \tau)}{(\tau^2+\epsilon^2)(\tau+\epsilon^2-s\epsilon)} = \frac{|\kappa|}{|1+\kappa^2|}\frac{s\epsilon-\tau}{\tau+\epsilon^2-s\epsilon}$$

so that $D\mathcal{R}(\widehat{\mathcal{K}}(\zeta)) \Subset \mathcal{K}^{ah}(\mathcal{R}\zeta)$ is equivalent to:

$$\left|\frac{s\epsilon-\tau}{\tau+\epsilon^2-s\epsilon}\right| < \sqrt{|2+\kappa^2|} \quad \text{whenever } |s| \le c\sqrt{|\tau|}.$$

The condition $(\epsilon, \tau) \in \mathcal{N}(\theta)$ with θ sufficiently small implies $\sqrt{2} \le \sqrt{|2+\kappa^2|}$, so it suffices to show that

$$\left|\frac{s\epsilon-\tau}{\tau+\epsilon^2-s\epsilon}\right| < \sqrt{2} \quad \text{whenever } |s| \le c\sqrt{|\tau|}. \tag{C.5}$$

Because $D\mathcal{R}$ maps cones to cones, we need only check that the boundary of $\widehat{\mathcal{K}}(\zeta)$ is mapped into $\mathcal{K}^{ah}(\mathcal{R}\zeta)$. Thus, we substitute $s = e^{i\theta}c\sqrt{|\tau|}$ into (C.5) obtaining:

$$\left|e^{i\theta}c\sqrt{|\tau|}\epsilon - \tau\right| < \sqrt{2}\left|\tau + \epsilon^2 - e^{i\theta}c\sqrt{|\tau|}\epsilon\right| \quad \text{for all } \theta \in \mathbb{R}/2\pi\mathbb{Z}. \tag{C.6}$$

For real ϵ, $\tau > 0$, and $\theta = 0$, this inequality is equivalent to positivity of a certain quadratic forms in $(\epsilon, \sqrt{\tau})$, which is straightforward to check.

For complex variables, let us square both sides:

$$|\tau|^2 - 2c\sqrt{|\tau|}\mathrm{Re}(e^{i\theta}\overline{\tau}\epsilon) + c^2|\tau||\epsilon|^2 + 4\mathrm{Re}(\tau\overline{\epsilon}^2) - 4c\sqrt{|\tau|}|\epsilon|^2\mathrm{Re}(e^{i\theta}\overline{\epsilon}) + 2|\epsilon|^4 > 0.$$

The hypothesis $(\epsilon, \tau) \in \mathcal{N}(\theta)$ with θ sufficiently small gives $4\mathrm{Re}(\tau\overline{\epsilon}^2) > \gamma|\tau||\epsilon|^2$, where $\gamma < 4$ is arbitrary close to 4. The other two terms with real parts we can replace with absolute values obtaining:

$$|\tau|^2 - 2c|\tau|^{3/2}|\epsilon| + (c^2+\gamma)|\tau||\epsilon|^2 - 4c|\tau|^{1/2}|\epsilon|^3 + 2|\epsilon|^4 > 0$$

In the variables $u = |\tau|^{1/2}$ and $v = |\epsilon|$, this reduces to positivity of the real quartic form

$$u^4 - 2cu^3v + (c^2 + \gamma)u^2v^2 - 4cuv^3 + 2v^4 > 0. \tag{C.7}$$

When $\gamma = 4$ and $c = \sqrt{2}$ this becomes

$$v^4\left(x^2(x - \sqrt{2})^2 + (2x - \sqrt{2})^2\right) > 0,$$

where $x = \frac{u}{v}$, which is obviously true. Positivity of (C.7) for γ close to 4 and c close to $\sqrt{2}$ then follows by continuity. $\square$

Lemma C.3 (**Invariance away from $\alpha_\pm$**). *There exist $\rho > 0$ and $\bar\tau > 0$ such that for any $\zeta = (\phi, 1 - \tau)$ with $|\tau| < \bar\tau$ and $|\phi \pm \frac{\pi}{2}| \geq \rho$ we have $D\mathcal{R}(\widehat{\mathcal{K}}(\zeta)) \subset \mathcal{K}^{ah}(\mathcal{R}\zeta)$.*

Proof. We select $\rho > 0$ as in Lemma C.2. Then for $|\phi \pm \frac{\pi}{2}| \geq \rho$ formula (B.7) implies

$$D\mathcal{R} = \begin{pmatrix} 2 + O(\tau) & O(1) \\ O(\tau^2) & O(\tau) \end{pmatrix},$$

with the coefficients depending on ρ. Applied to a tangent vector $v = (1, s) \in T_\zeta\mathcal{C}^c$ with $|s| = c\sqrt{|\tau|}$ we find that $D\mathcal{R}(v)$ has slope $|s'| = O(|\tau|^{3/2}) = o(|\tau|) = o(\sqrt{|\tau'|})$, which is less than $\sqrt{|\tau'(2 - \tau')|}$ for τ (and hence τ') sufficiently small. $\square$

Lemmas C.2 and C.3 complete to proof of Proposition C.1. $\square$

C.3. **Stretching of horizontal holomorphic curves.** We now consider how holomorphic curves ξ that are horizontal with respect to $\mathcal{K}^{ah}$ are stretched under $\mathcal{R}$.

Let $\pi(z, t) = z$. If ξ is a horizontal holomorphic curve with $\pi(\xi)$ a round disc $\mathbb{D}(z, r)$ centered at z, we will say that $r^h(\xi, x) = r$. More generally, we let $r^h_{\min}(\xi, x)$ stand for the supremum of the radii of the disks $\mathbb{D}(z, r)$ centered at z that can be inscribed into $\pi(\xi) \subset \mathbb{C}$ and $r^h_{\max}(\xi, x)$ the infimum of the discs $\mathbb{D}(z, r)$ that can be circumscribed about $\pi(\xi)$. They measure the "horizontal size" of ξ at x.

Proposition C.4. *Fix any $\bar\kappa > 0$. There is a $C > 0$ so that for any sufficiently small $a > 0$ and any $x \in \mathcal{C} \smallsetminus \{\alpha_\pm\}$ sufficiently close to $\alpha_\pm$ and satisfying $|\kappa(x)| \leq \bar\kappa$, there is an iterate $N \equiv N(x)$ so that, if ξ is any horizontal holomorphic curve based at x with*

$$r^h(\xi, x) = a|\epsilon(x)|,$$

then the subdisc $\xi_0 \subset \xi$ of radius $r^h(\xi_0, x) = a|\epsilon(x)|/2$ will have $\mathcal{R}^i\xi_0$ in the domain of definition of $\mathcal{K}^{ah}$ for $i = 1, \cdots, N$ and $r^h_{\min}(\mathcal{R}^N\xi_0, \mathcal{R}^N x) \;\geq\; C \cdot a$.

The proof of Proposition C.4 will be the consequence of several lemmas.

Throughout the following lemmas we will suppose that $a > 0$ is sufficiently small so that a holomorphic disc ξ centered at x with $r^h(\xi, x) = a|\epsilon(x)|$ is entirely within the domain of definition for $\mathcal{K}^{ah}$. This is possible by Proposition C.1.

Lemma C.5. *If ξ is any horizontal holomorphic curve centered at $x \in \mathcal{C} \smallsetminus \{\alpha_\pm\}$ of radius $r^h(\xi, x) = a|\epsilon(x)|$, then the restriction of $\mathcal{R}$ to the subdisc $\xi_0 \subset \xi$ of radius $r^h(\xi_0, x) = a|\epsilon(x)|/2$ has bounded horizontal distortion.*

Proof. Let $r = r^h(\xi_0, x)$. The function $\chi_{\xi_0} = \pi \circ \mathcal{R} \circ (\pi|\xi_0)^{-1}$ extends to a univalent function in the disk $\mathbb{D}(\phi, 2r)$, since ξ_0 is the restriction of ξ to half of its radius. The conclusion then follows from the Koebe Distortion Theorem. $\square$

In particular, we will have $r^h_{\min}(\mathcal{R}\xi_0, \mathcal{R}x) \asymp r^h_{\max}(\mathcal{R}\xi_0, \mathcal{R}x)$ and also there is a uniform constant $d \geq 1/4$ so that $r^h_{\min}(\mathcal{R}\xi_0, \mathcal{R}x) \geq d\ \lambda^h_{\min}(x) r^h(\xi_0, x)$.

Recall the point $\alpha = (\pi, 1)$, which is mapped by $\mathcal{R}$ to the high temperature fixed point $\beta_1 = (0, 1)$.

Lemma C.6. *Given $\underline{\kappa} > 0$ there exists $C_0 > 0$ so that for any $a > 0$ we have the following property. If ξ and $\xi_0 \subset \xi$ are as in Lemma C.5 and are based at a point $x \in \mathcal{C} \smallsetminus \{\alpha_\pm\}$ in an appropriately small neighborhood of $\alpha_\pm$ with a bounded slope: $|\kappa(x)| \leq \underline{\kappa}$, then*

$$r^h_{\min}(\mathcal{R}\xi_0, \mathcal{R}x) \geq C_0 \cdot a \operatorname{dist}^h(\mathcal{R}x, \alpha).$$

Proof. Let $x = (\epsilon, \tau)$.

Let us select $c = c(a)$ in such a way that any horizontal curve ξ of size $\geq a\epsilon$ centered in one of the parabolic sectors $\{|\kappa - \epsilon| \leq c|\epsilon|\}$ near $\alpha_\pm$ crosses the corresponding curve $\mathcal{S}_\pm$. The image $\mathcal{R}\xi$ of such a curve crosses $\mathcal{I}_\pi$, so that $r^h_{\max}(\mathcal{R}\xi, \mathcal{R}x) \geq \operatorname{dist}^h(\mathcal{R}x, \alpha)$. This is sufficient, since $r^h_{\min}(\mathcal{R}\xi_0, \mathcal{R}x) \asymp r^h_{\max}(\mathcal{R}\xi_0, \mathcal{R}x)$.

On the other hand, if $|\kappa - \epsilon| \geq c|\epsilon|$ then $\operatorname{dist}^h(\mathcal{R}x, \alpha) \asymp |\kappa - \epsilon|$ by Lemma B.1. Then, Lemmas B.2 and C.5 imply that

$$r^h_{\min}(\mathcal{R}\xi_0, \mathcal{R}x) \geq d\lambda^h_{\min}(x) r^h(\xi_0, x) \geq \tilde{C}_0 \cdot a\, |\kappa - \epsilon| \asymp \operatorname{dist}^h(\mathcal{R}x, \alpha). \tag{C.8}$$

$\square$

We now set up a complex neighborhood of β_1 designed so that suitable holomorphic curves ξ near β_1 can regrow to definite size: Let

$$\begin{aligned} \mathcal{V}^c \equiv \mathcal{V}^c_{\bar\tau} &:= \{(\phi, t) \in \mathcal{C}^c_\rho \,:\, |\tau| \leq \bar\tau,\ |\operatorname{Im}\phi| < \bar\tau\} \text{ and} \\ \mathcal{U}^c \equiv \mathcal{U}^c_{\bar\epsilon} &:= \{(\phi, t) \in \mathcal{V}^c \,:\, |\epsilon(\phi)| < \bar\epsilon\}. \end{aligned}$$

They are complex versions of the regions $\mathcal{V}$ and $\mathcal{U}$ from §6.2. We will take $\bar\tau$ sufficiently small relative to $\bar\epsilon > 0$ so that $\mathcal{V}^c \smallsetminus \mathcal{U}^c$ lies in the domain of definition of the complex extension of $\mathcal{K}^{ah}$.

Choosing $\bar\tau$ sufficiently small compared to $\bar\epsilon$, we can ensure that $\mathcal{R}$ is uniformly horizontally expanding on any horizontal holomorphic curve $\xi \subset \mathcal{V}^c \smallsetminus \mathcal{U}^c$. (It follows by continuity from (B.10).)

Lemma C.7. *Given $0 < r_{\min} < r_{\max}$ there exists $C_1 > 0$ so that for any $b > 0$ and any $x \in \mathcal{V} \smallsetminus \mathcal{U} \subset \mathcal{C}$ there is an iterate $n(x)$ with the following property. If $\eta \subset \mathcal{V}^c \smallsetminus \mathcal{U}^c$ is a horizontal holomorphic curve centered at x with*

$$r_{\min} \leq r^h_{\min}(\eta, x) < r^h_{\max}(\eta, x) < r_{\max} \qquad \text{and} \qquad r^h_{\min}(\eta, x) \geq b \operatorname{dist}^h(x, \beta_1)$$

then $\mathcal{R}^i\eta \subset \mathcal{V}^c \smallsetminus \mathcal{U}^c$ for $0 \leq i \leq n(x)$ and $r^h_{\min}(\mathcal{R}^{n(x)}\eta, \mathcal{R}^{n(x)}x) \geq C_1 \cdot b$. Moreover, C_1 depends only on $r_{\max}/r_{\min}$.

Proof. The standard distortion estimates near the hyperbolic fixed point β_1 show that the discs $\mathcal{R}^i\eta$ grow at the same rate as the horizontal distances between any point of $\mathcal{R}^i\eta$ and β_1. The iterate $n(x)$ is chosen as the maximum of those integers k for which $\mathcal{R}^j\eta \subset \mathcal{V}^c \smallsetminus \mathcal{U}^c$ for all $j \leq k$. $\square$

Proof of Proposition C.4: Equation (B.8) gives that $D\mathcal{R}$ expands horizontal vectors based at such points by at most $O(1/|\epsilon(\zeta)|)$. So, we can assume that a is sufficiently small so that $\mathcal{R}\xi$ is in the domain of definition of $\mathcal{K}^{ah}$. Moreover, using (B.3) we can choose $\underline{\kappa} \le \overline{\kappa}$ so that if ξ is centered at x with $|\kappa(x)| \le \underline{\kappa}$ then $\mathcal{R}^i\xi \subset \mathcal{V}^c \smallsetminus \mathcal{U}^c$ for $i = 1, 2$.

Lemma C.5 gives that $\mathcal{R}$ has bounded horizontal distortion on the subdisc $\xi_0 \subset \xi$ of radius $a|\epsilon(x)|/2$.

If $|\kappa(x)| \ge \underline{\kappa}$ then Lemma B.2 implies that $\lambda^h_{\min}(x) \ge K/|\epsilon(x)|$. Together with bounded distortion, this is sufficient to give that $r^h_{\min}(\mathcal{R}\xi_0, \mathcal{R}x) \ge C \cdot a$.

If $|\kappa(x)| \le \underline{\kappa}$ then Lemma C.6 gives that $r^h_{\min}(\mathcal{R}\xi_0, \mathcal{R}x) \ge C_0 \cdot a \operatorname{dist}^h(\mathcal{R}x, \alpha)$. There is some $\tilde{C}_0 > 0$ so that after one further iterate we have

$$r^h_{\min}(\mathcal{R}^2\xi_0, \mathcal{R}^2x) \ge \tilde{C}_0 \cdot a \operatorname{dist}^h(\mathcal{R}^2x, \beta_1).$$

Since the horizontal distortion of $\mathcal{R}^2$ is also bounded on ξ_0, there is a uniform bound on $r^h_{\max}(\mathcal{R}^2\xi_0, \mathcal{R}^2x)/r^h_{\min}(\mathcal{R}^2\xi_0, \mathcal{R}^2x)$. Lemma C.7 then gives M further iterates so that $\mathcal{R}^{M+2}\xi \subset \mathcal{V}^c \smallsetminus \mathcal{U}^c$ and

$$r^h_{\min}(\mathcal{R}^{M+2}\xi_0, \mathcal{R}^{M+2}x) \ge C \cdot a.$$

□

Appendix D. Critical locus and Whitney folds

D.1. **Critical locus.**

D.1.1. *Six lines and a conic.* The Jacobian of $\hat{R} : \mathbb{C}^3 \to \mathbb{C}^3$ (2.18) is equal to

$$\det D\hat{R} = 32\, V\, (UW - V^2)\, (U + W)^2\, (U^2 + V^2)\, (W^2 + V^2),$$

so its critical locus comprises 6 complex lines (where we count $\{U + W = 0\}$ only once, without multiplicity) and the cone $\{UW = V^2\}$. They descend to 6 complex lines and one conic on $\mathbb{CP}^2$:

$$\begin{aligned}
L_0 &:= \{V = 0\} = \text{line at infinity},\\
L_1 &:= \{UW = V^2\} = \text{conic } \{uw = 1\},\\
L_2 &:= \{U = -W\} = \{u = -w\},\\
L_3^\pm &:= \{U = \pm iV\} = \{u = \pm i\},\\
L_4^\pm &:= \{W = \pm iV\} = \{w = \pm i\}.
\end{aligned}$$

(Here the curves are written in the homogeneous coordinates $(U : V : W)$ and in the affine ones, $(u = U/V, w = W/V)$.) The configuration of these curves is shown in Figure 4.1.

The following general lemma shows that this coincides with the critical locus of R on $\mathbb{CP}^2$. It is a consequence of Euler's Theorem for Homogeneous Functions.

Lemma D.1. *Let $\hat{R} : \mathbb{C}^{m+1} \to \mathbb{C}^{m+1}$ be a homogeneous polynomial, and let $R : \mathbb{CP}^m \to \mathbb{CP}^m$ be the corresponding rational map of the projective space. Let $\hat{z} \in \mathbb{C}^{m+1}$ be such that $\hat{R}(\hat{z}) \ne 0$ (so that $z := \pi(\hat{z})$ is not a point of indeterminacy of R). Then $\hat{z}$ is a critical for $\hat{R}$ iff z is critical for R.*

D.1.2. *Critical points on the exceptional divisor.* To complete the picture, we need to take an account of the critical points hidden inside the points of indeterminacy, $a_\pm$. To make them visible, we blow up $\mathbb{CP}^2$ at $a_\pm$ and lift R to a holomorphic map $\tilde{R} = Q \circ \tilde{g} : \tilde{\mathbb{CP}}^2 \to \mathbb{CP}^2$.

By symmetry, it is enough to analyze the blow-up of a_+. Jacobian of $\tilde{g}$ (B.1) on the exceptional divisor $\{\xi = 0\}$ is equal to $2i(\chi - 1)/(\chi + 1)^2$, so $\tilde{g}$ has two critical points on it, $\chi = -1$ and $\chi = 1$, which are the intersections of the exceptional divisor with the collapsing line $\tilde{L}_2$ and the separatrix $\tilde{L}_1$ respectively (recall that $\tilde{L}$ stands for the lift of L to $\tilde{\mathbb{CP}}^2$). The first critical point is mapped by $\tilde{R}$ to the low temperature fixed point b_0, while the second one is mapped to $(-1, -1)$, which is a preimage of the high temperature fixed point b_1.

Also, points $\chi = \infty$ and $\chi = 0$ are mapped by $\tilde{g}$ to the coordinate lines $\{u = 0\}$ and $\{w = 0\}$ respectively which are critical for the squaring map Q. It creates two more critical points on the exceptional divisor, its intersections with the critical lines $\tilde{L}_3^+$ and $\tilde{L}_4^-$.

D.2. **Complex Whitney folds.** To simplify calculations near the critical points, it is convenient to bring R to a normal form. A complex Whitney fold is a generic and the simplest one (see [AGV]). Let $R : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be a germ of holomorphic map with a critical point at 0. The map R (and the corresponding critical set) is called a *complex Whitney fold* if

- (W1) The critical set L is a non-singular curve near 0;
- (W2) $DR(0)$ has rank 1 and $\operatorname{Ker} DR(0)$ is transverse to L;
- (W3) The second differential $D^2R(0)$ is not vanishing in the direction of $\operatorname{Ker} DR(0)$.

Lemma D.2. *A Whitney fold can be locally brought to a normal form $(u, w) \mapsto (u, w^2)$ in holomorphic coordinates.*

Proof. Properties (W1) and (W2) imply that $L' = R(L)$ also a non-singular curve near 0, so we can select local coordinates in such a way that both L and L' coincide with the axis $\{w = 0\}$. Let $(u', w') = R(u, w)$. Property (W2) implies $\frac{\partial u'}{\partial u}(0) \neq 0$, allowing us to replace the local coordinates in the domain of R with (u', w). This brings R to a fibered map $u' = u,\ w' = \psi(u, w)$ for some ψ.

Since $\{w = 0\}$ is in the critical locus of R,

$$\psi(u, w) = p_2(u)w^2 + p_3(u)w^3 + \dots .$$

Property (W3) gives $p_2(0) \neq 0$, so we can select a local branch of the square root $w\sqrt{p_2(u) + p_3(u)w + \dots}$ as a local coordinate replacing w. This brings R to the desired form. □

Lemma D.3. *All critical points of $\tilde{R}$ except the fixed points e, e', the collapsing line $\tilde{L}_2$, and two points $\{\pm(i, i)\} = \tilde{L}_3 \cap \tilde{L}_4$, are Whitney folds.*

Proof. Let us treat the components of the critical locus one by one.

Separatrix L_0. In affine coordinates $\xi = W/U$, $\eta = V/U$, the map R near $L_0 = \{\eta = 0\}$ looks like this:

$$\xi' = \left(\frac{\xi^2 + \eta^2}{1 + \eta^2}\right)^2 = \xi^4(1 + O(\eta^2)),$$

$$\eta' = \eta^2 \left(\frac{1+\xi}{1+\eta^2} \right)^2 = \eta^2 (1+\xi)^2 (1 + O(\eta^2)),$$

which shows that L_0 is a fold outside the fixed point e ($\xi = 0$) and the collapsing line L_2 ($\xi = -1$). (The other fixed point e' lies at infinity, $\xi = \infty$.)

Separatrix $\tilde{L}_1$. In local coordinates $(u, \tau = uw - 1)$, the map R near $L_1 = \{\tau = 0\}$ looks like this:

$$u' = \left(\frac{u^2+1}{u + (\tau+1)/u} \right)^2 = u^2 \left(1 + O\left(\frac{\tau}{u^2+1} \right) \right),$$

$$\tau' = \frac{\tau^2}{(u+w)^2} \left(-2 + \frac{\tau^2}{(u+w)^2} \right), \tag{D.1}$$

which shows that L_1 is a fold outside the indeterminacy points $\{u = \pm i\} = L_1 \cap \{u + w = 0\}$ (and outside the fixed points e and e' at infinity).

Let us now analyze the intersection $\tilde{a}_+ = (u = i, \chi = 1)$ of $\tilde{L}_1$ with the exceptional divisor L^+_{exc} (the intersection with L^-_{exc} is symmetric). Let us use local coordinates ($\xi = u - i, \lambda = \chi - 1$) near $\tilde{a}^+$. Representation (B.1) of $\tilde{g}$ near L^+_{exc} gives:

$$u = i + \frac{1}{2}(\xi - i\lambda) + \frac{i}{4}\lambda^2 - \frac{1}{4}\xi\lambda + \dots,$$
$$w = -i + \frac{1}{2}(\xi - i\lambda) + \frac{i}{4}\lambda^2 + \frac{3}{4}\xi\lambda + \dots,$$

so the vanishing direction for $D\tilde{g}(\tilde{a}_+)$ is $d\xi = id\lambda$. On the other hand, in these coordinates the (proper transform of the) separatrix $\{uw = 1\}$ assumes the form $\xi + i\lambda + \xi^2\lambda = 0$, so it is a non-singular curve tangent to $\{d\xi = -i\, d\lambda\}$ at $\tilde{a}_+$. This yields conditions (W1) and (W2). Moreover, at the kernel direction $d\xi = i\, d\lambda$, the second differential assumes the form $(0, i\, d\lambda^2)$, so it is non-vanishing.

Thus, $\tilde{a}^+$ is a Whitney fold for $\tilde{g}$. Since the squaring map Q is non-singular at $(i, -i)$, it is a a Whitney fold for $\tilde{R} = \tilde{g} \circ Q$ as well.

Lines $\tilde{L_3}^{\pm}$ *and* $\tilde{L_4}^{\pm}$. These lines intersect the line at infinity at the fixed points e and e', so we need to analyze only their affine parts. The map $\tilde{g}$ is non-singular on these lines (including their intersections with the exceptional divisors), and it maps them isomorphically onto the coordinate axes $\{u = 0\}$ and $\{w = 0\}$. Outside the origin $\mathbf{0}$, these axes are Whitney folds for the squaring map Q. Hence the lines $\tilde{L_3}^{\pm}$ and $\tilde{L_4}^{\pm}$ are folds for $\tilde{R}$ outside points $\{\pm(i, i)\} = \tilde{g}^{-1}(\mathbf{0}) = \tilde{L}_3 \cap \tilde{L}_4$. □

D.2.1. *Double points.* A *double point* of a holomorphic curve X is a point $a \in X$ such that the germ of X at a consists of two regular branches, X_1 and X_2, meeting at a. A double point is called *transverse* if the branches X_i intersect transversely at a. Otherwise, it is called *tangential*.

We say that a regular curve L intersects a curve X *transversely* at the double point $a \in X$ if it intersects transversely both branches X_i. Such intersection has multiplicity 2.

Lemma D.4. *Let R be a Whitney fold at a, and let X be a germ of regular holomorphic curve with the first order tangency to the critical value locus $R(L)$ at $R(a)$. Then the pullback R^*X has a transverse double point at a intersecting L transversally.*

Proof. In the normal coordinates, the pullback under R of a regular curve $w = cu^2(1+O(u))$, $c \neq 0$, tangent to $R(L) = \{w = 0\}$ is a pair of regular curves $w = \pm\sqrt{c}u(1+O(u))$. $\square$

Lemma D.5. *Let* $\pi : (\tilde{M}, \mathcal{E}_{\text{exc}}) \to (M, a)$ *be the blow-up of* M *at* a*, and let* $\tilde{p} \in \mathcal{E}_{\text{exc}}$*. Let* $\tilde{X} \subset \tilde{M}$ *be a holomorphic curve with a transverse double point at* $\tilde{p}$ *that intersects* $\mathcal{E}_{\text{exc}}$ *transversely. Then* $X = \pi(\tilde{X})$ *is a holomorphic curve in* M *with a (first order) tangential double point at* a*.*

Proof. Let us use the local coordinates $(u, v, m = v/u)$ from the definition of the the blow-up. In these coordinates, the pencil of lines $m - m_0 = \lambda u$, $\lambda \in \mathbb{CP}^1$ centered at $(0, 0, m_0) \in \mathcal{E}_{\text{exc}}$ projects to the pencil of parabolas $v = (\lambda u + m_0)u$ in M tangent to the line $v = m_0 u$. $\square$

Appendix E. Extra bits of stat mechanics

E.1. **The Lee-Yang Theorem.** To prove the Lee-Yang Theorem, we need to consider a more general, anisotropic, Ising model. It is convenient to assume that Γ is a complete graph without loops (connecting a vertex to itself), but to allow some of the coupling constants vanish. The model is parameterized by a symmetric matrix $\mathbf{J} = (J_{v,w})_{(v,w)\in\mathcal{E}}$ of couplings between the atoms and by a vector $\mathbf{h} = (h_v)_{v\in\mathcal{V}}$ of interaction strengths of the external field with the atoms. Then the energy of a spin configuration $\sigma : \mathcal{V} \to \{\pm 1\}$ assumes the form

$$-\mathrm{H}(\sigma) = <\mathbf{J}\sigma, \sigma> + <\mathbf{h}, \sigma> . \tag{E.1}$$

The original graph of interest corresponds to the subgraph $\Gamma' \subset \Gamma$ containing only the edges with $J_{v,w} \neq 0$. In the ferromagnetic model, $J_{v,w} \geq 0$.

Let us consider the "support" of a configuration σ,

$$\mathcal{V}(\sigma) = \{c \in \mathcal{V} : \ \sigma(v) = -1\},$$

and let

$$\mathcal{E}(\sigma) = \{(v, w) \in \mathcal{E} : \ v \in \mathcal{V}(\sigma), \ w \in \mathcal{V} \smallsetminus \mathcal{V}(\sigma)\}.$$

Let $l(\mathbf{J})$ and $l(\mathbf{h})$ be the the sums of all components of $\mathbf{J}$ and $\mathbf{h}$ respectively. We will work with a modified Hamiltonian

$$-\check{\mathrm{H}}(\sigma) = -\mathrm{H}(\sigma) - l(\mathbf{J}) - l(\mathbf{h}) = -2 \sum_{(v,w)\in\mathcal{E}(\sigma)} J_{v,w} - 2 \sum_{v\in\mathcal{V}(\sigma)} h_v.$$

Let us introduce the temperature-like and field-like variables:

$$t_{v,w} = e^{-2J_{v,w}/T} \quad \text{and} \quad \zeta_v = e^{-2h_v/T}.$$

Given subsets $X \subset \mathcal{V}$ and $Y \subset \mathcal{E}$, we will use notation

$$\zeta^X = \prod_{v\in X} \zeta_v, \quad t^Y = \prod_{(v,w)\in Y} t_{v,w}.$$

In this notation, we obtain the following expression for the modified Gibbs weights:

$$\check{W}(\sigma) = \exp(-\check{\mathrm{H}}(\sigma)/T) = W(\sigma)\, t^{\mathcal{E}/2}\, \zeta^{\mathcal{V}/2} = t^{\mathcal{E}(\sigma)}\, \zeta^{\mathcal{V}(\sigma)}$$

and for the modified partition function:

$$\check{\mathsf{Z}} = t^{\mathcal{E}/2}\, \zeta^{\mathcal{V}/2}\, \mathsf{Z} = \sum_{\sigma} t^{\mathcal{E}(\sigma)}\, \zeta^{\mathcal{V}(\sigma)}. \tag{E.2}$$

Obviously, the modification does not affect the roots of the partition function (modulo clearing up the denominator), so we can work with $\check{\mathsf{Z}}$ instead of Z.

Remark also that Lemma 2.1 on multiplicativity of the partition function naturally extends to this level of generality.

Lemma E.1. *Fix arbitrary $t_{v,w} \in [-1, 1]$ and let $\Gamma' \subset \Gamma$ be the subgraph containing only the edges with $t_{v,w} \neq \pm 1$. Suppose Γ' is connected. If $\check{\mathsf{Z}}(\zeta_1, \dots, \zeta_n) = 0$ and $|\zeta_i| \geq 1$ for $i = 1, \dots, n-1$, then $|\zeta_n| \leq 1$, where the inequality is strict unless $|\zeta_i| = 1$ for $i = 1, \dots, n-1$.*

Proof. To simplify notation and to emphasize dependence on $n = |\mathcal{V}|$, we let $P_n \equiv \check{\mathsf{Z}}$. We will carry induction in n.

For $n = 2$, we have

$$P_2(z_1, z_2) = 1 + t_{1,2}\zeta_1 + t_{2,1}\zeta_2 + \zeta_1\zeta_2,$$

which implies

$$\zeta_2 = -\frac{1 + t_{1,2}\zeta_1}{t_{2,1} + \zeta_1}.$$

The assumption that Γ' is connected, implies that $t_{1,2} = t_{2,1} \in (-1, 1)$. Therefore, this is a Möbius map sending $\mathbb{C} \smallsetminus \overline{\mathbb{D}}$ to $\mathbb{D}$.

We now pass from n to $n+1$. After relabeling the vertices, we can suppose that $\Gamma' \setminus \{v_{n+1}\}$ is connected. Observe that

$$P_{n+1}(\zeta_1, \dots, \zeta_{n+1}) = P_n(u_1, \dots, u_n) + \zeta_1 \dots \zeta_{n+1} \, P_n(v_1, \dots, v_n),$$

where $u_i = t_{i,n+1}\zeta_i$, $v_i = t_{i,n+1}/\zeta_i$.

Remark E.1. This formula is directly related to the Basic Symmetry of the Ising model which is ultimately responsible for the Lee-Yang Theorem.

If $P_{n+1} = 0$ then

$$\zeta_{n+1} = -\frac{1}{\zeta_1 \dots \zeta_n} \frac{P_n(u_1, \dots, u_n)}{P_n(v_1, \dots, v_n)}. \tag{E.3}$$

If $\zeta_i \in \hat{\mathbb{C}} \smallsetminus \bar{\mathbb{D}}$ for $i = 1, \dots, n$, then $|v_i| \leq 1$. Since $\Gamma' \setminus \{v_{n+1}\}$ is connected, the Induction Assumption gives that $P_n(v_1, \dots, v_n) \neq 0$. Hence the right-hand side of (E.3) is a well-defined holomorphic function in the polydisk $\Delta^n := (\hat{\mathbb{C}} \smallsetminus \bar{\mathbb{D}})^n$. On its Shilov boundary

$$\partial^s \Delta^n = \mathbb{T}^n \equiv \{|\zeta_i| = 1, \quad i = 1 \dots, n\}\}$$

we have $u_i = \bar{v}_i$. Since P_n has real coefficients, we conclude that $|\zeta_{n+1}| = 1$ on $\mathbb{T}^n$. By the Maximum Principle, $|\zeta_{n+1}| \leq 1$ in Δ^n, with equality only on $\mathbb{T}^n$, and we are done. □

General Lee-Yang Theorem ([YL, LY]). *Fix $J_{v,w} \in [0, +\infty]$, and assume $h_v/h_w > 0$. Then, all zeros of the partition function $\mathsf{Z}(\zeta_1, \dots, \zeta_n)$ lie on the unit torus $\mathbb{T}^n$.*

Proof. Recall that $\Gamma' \subset \Gamma$ is the subgraph containing only the edges with $J_{v,w} \neq 0$. By Lemma 2.1, the partition function $\mathsf{Z}(\zeta_1, \dots, \zeta_n)$ is the product of partition functions associated to each of the connected components of Γ'. Therefore, it suffices to prove the result in the case that Γ' is connected.

Because $h_v/h_w > 0$ for all v and w, we have that

(1) $|\zeta_v| = |e^{-2h_v/T}| = 1$ for all v,

(2) $|\zeta_v| = |e^{-2h_v/T}| < 1$ for all v, or
(3) $|\zeta_v| = |e^{-2h_v/T}| > 1$ for all v.

Lemma E.1 rules out (2) and (3). This lemma is applicable since $t_{v,w} = e^{-2J_{v,w}/T} \in [0,1]$, with $t_{v,w} = 1$ iff $J_{v,w} = 0$. □

To obtain the classical Lee-Yang Theorem, corresponding to Hamiltonian with magnetic moment M given by (1.1), one sets $h_v \equiv h$. To get the result for Hamiltonian with magnetic moment M given by (2.1), obtained by summing over magnetic moments of edges, one sets $h_v = h \cdot |v|/2$, where $|v|$ is the valence of the vertex v.

Many extensions and new proofs of this theorem have appeared since the 1950s: see Asano [A], Suzuki and Fisher [SF], Heilmann and Lieb [HL], Ruelle [R2, R4], Newman [N], Lieb and Sokal [LS], Borcea and Brändén [BB] and further references therein. The proof given above is based upon the original idea of Lee and Yang [LY], compare [R3, Thm 5.1.2].

E.2. **The Lee-Yang Theorem with Boundary conditions.** The Lee-Yang Theorem with Boundary Conditions, stated in §2.1, is a consequence of the following more general statement:

General Lee-Yang Theorem with Boundary Conditions. *Fix $J_{v,w} \in [0,+\infty]$, assume that the subgraph $\Gamma' \subset \Gamma$ containing only the edges with $J_{v,w} > 0$ is connected, and assume $h_v/h_w > 0$ for all v and w.*

Suppose $\mathcal{U} \subset V$ is given by $\{m+1,\dots,n\}$ with $1 < m < n$ and let $\sigma_{\mathcal{U}} \equiv +1$. Then all of the zeros of the conditional partition function $\mathsf{Z}^+(\zeta_1,\dots,\zeta_m)$ lie in the open polydisc $\Delta^m = (\hat{\mathbb{C}} \smallsetminus \overline{\mathbb{D}})^m$.

Proof. Let $\eta_i = \zeta_i \prod_{j=m+1}^{n} t_{i,j}$ for $1 \le i \le m$, so that $|\eta_i| \le |\zeta_i|$. Then, the (modified) conditional partition function

$$\check{\mathsf{Z}}_{\Gamma|\,\sigma_{\mathcal{U}}}(\zeta_1,\dots,\zeta_n) = t^{\mathcal{E}/2}\zeta^{(\mathcal{V}\smallsetminus\mathcal{U})/2}\ \mathsf{Z}_{\Gamma|\,\sigma_{\mathcal{U}}}(\zeta_1,\dots,\zeta_n)$$

satisfies

$$\check{\mathsf{Z}}_{\Gamma|\,\sigma_{\mathcal{U}}}(\zeta_1,\dots,\zeta_n) = \check{\mathsf{Z}}(\eta_1,\dots,\eta_m), \tag{E.4}$$

where $\check{\mathsf{Z}}$ is the (modified) partition function corresponding to $\Gamma\smallsetminus\mathcal{U}$. (Since $\sigma_{\mathcal{U}} \equiv +1$ no ζ_i appears on the left hand side of (E.4) for $m \le i \le n$.)

Lemma 2.1 gives that $\check{\mathsf{Z}}(\eta_1,\dots,\eta_m)$ factors into a product of (modified) partition functions of each of the connected components of $\Gamma' \setminus \mathcal{U}$. We will check that each of these factors is non-vanishing if $|\zeta_i| \le 1$ for any $1 \le i \le m$.

Let Υ be one of the connected components of $\Gamma' \setminus \mathcal{U}$. Without loss of generality, we can suppose Υ has vertices $\{1,\dots,j\}$ with $1 \le j \le m$, so that Υ corresponds to a factor $\check{\mathsf{Z}}_{\Upsilon}(\eta_1,\dots,\eta_j)$ of $\check{\mathsf{Z}}(\eta_1,\dots,\eta_m)$. Since Γ' is connected, one of the vertices of Υ (say vertex 1) is connected in Γ' to one of the vertices (say vertex n) from $\mathcal{U}$. Therefore, $t_{1,n} < 1$, implying $|\eta_1| < |\zeta_1|$.

Since $h_v/h_w > 0$ for all v and w, if $|\zeta_i| \le 1$ for any $1 \le i \le m$, then $|\zeta_i| \le 1$ for all $1 \le i \le m$. In this case, $|\eta_i| \le 1$ for $1 \le i \le m$ and $|\eta_1| < 1$ so that Lemma E.1 gives $\check{\mathsf{Z}}_{\Upsilon}(\eta_1,\dots,\eta_j) \ne 0$.

□

Remark E.2. The same statement holds if we assign $\sigma_{\mathcal{U}} \equiv \sigma_0 > 0$ and the classical Lee-Yang Theorem can be obtained from it by taking a limit $\sigma_0 \to 0$.

E.3. **The Lee-Yang zeros in the 1D Ising model.** By means of the well-known "Transfer Matrix technique" (see [Ba]), one can find explicitly the Lee-Yang zeros of the 1D Ising model. Let Γ_n be the linear chain with $n+1$ vertices $\{0, 1, \ldots, n\}$. For simplicity, we will consider periodic boundary conditions: $\sigma(n) = \sigma(0)$ (so that our graph is the circle $\mathbb{Z}/n\mathbb{Z}$). This assumption does not affect the thermodynamical limit.

The Hamiltonian of this lattice is:

$$\mathrm{H}_n(\sigma) = -\sum_{i=0}^{n-1} \{J\sigma(i)\sigma(i+1) + \frac{h}{2}(\sigma(i) + \sigma(i+1)\}.$$

Two neighboring spins $\{\sigma(i), \sigma(i+1)\}$ contribute the following factor to the Gibbs weight (compare (2.16)):

$$W(++) = e^{(J+h)/T} = \frac{1}{\mathrm{t}z}, \quad W(+-) = W(-+) = e^{-J/T} = \mathrm{t}, \quad W(--) = e^{(J-h)/T} = \frac{z}{\mathrm{t}},$$

which can be organized into the *Transfer Matrix*

$$W = \begin{pmatrix} (\mathrm{t}z)^{-1} & \mathrm{t} \\ \mathrm{t} & \mathrm{t}^{-1}z \end{pmatrix}.$$

We see that the partition function $\mathsf{Z}_n = \sum_\sigma \exp(-\mathrm{H}_n(\sigma)/T)$ can be expressed as

$$\mathsf{Z}_n = \sum_\sigma W(\sigma_1, \sigma_2) \cdot W(\sigma_2, \sigma_3) \cdots W(\sigma_n, \sigma_1) = \operatorname{tr} W^n = \tau_1^n + \tau_2^n,$$

where $\tau_{1,2}$ are the eigenvalues of W.[38]

Thus, the zeros of the partition functions are solutions of the equation

$$\tau_1^n + \tau_2^n = 0,$$

or

$$\text{(E.5)} \qquad \frac{\tau_2}{\tau_1} = e^{i\alpha_k}, \quad \alpha_k = \frac{\pi}{n} + \frac{2\pi k}{n}; \qquad k = 0, 1, \ldots, n-1.$$

The eigenvalues $\tau_{1,2}$ are the roots of the quadratic equation

$$\tau^2 - p\tau + q = 0, \quad \text{where } p = \frac{1}{\mathrm{t}}(z + \frac{1}{z}) = \frac{2}{\mathrm{t}}\cos\phi, \quad q = \frac{1}{\mathrm{t}^2} - \mathrm{t}^2.$$

This gives

$$2\cos\alpha_k = \frac{\tau_2}{\tau_1} + \frac{\tau_1}{\tau_2} = \frac{p^2}{q} - 2 = \frac{4\cos^2\phi_k}{1-\mathrm{t}^4} - 2,$$

so

$$\text{(E.6)} \qquad (1-\mathrm{t}^4)\cos^2\frac{\alpha_k}{2} = \cos^2\phi_k, \quad \text{where } z_k = e^{i\phi_k}.$$

Together with (E.5), this gives expression (1.3) for the zeros of the partition function.

Since the angles α_k are equidistributed with respect to the Lebesgue measure $d\alpha/2\pi$ on the circle, the distribution $\rho_t d\phi$ of the Lee-Yang zeros is obtained by pushing this measure to the interval $[-1, 1]$ by cos,[39] scaling it by $\sqrt{1-\mathrm{t}^4}$, and then pulling it back to the circle by cos. The calculation gives expression (1.4):

$$\rho_t(\phi) = \frac{1}{2\pi}\left|\frac{d\alpha}{d\phi}\right| = \frac{1}{2\pi}\left|\frac{d}{d\phi}\arccos\left(\frac{\cos\phi}{\sqrt{1-\mathrm{t}^4}}\right)\right| = \frac{|\sin\phi|}{2\pi\sqrt{1-\mathrm{t}^4-\cos^2\phi}}.$$

[38] Different boundary conditions would result in $\mathsf{Z}_n = a\tau_1^n + b\tau_2^n$ for appropriate a and b.

[39] which gives the "Chebyshev measure" $dx/\sqrt{1-x^2}$ on $[-1, 1]$

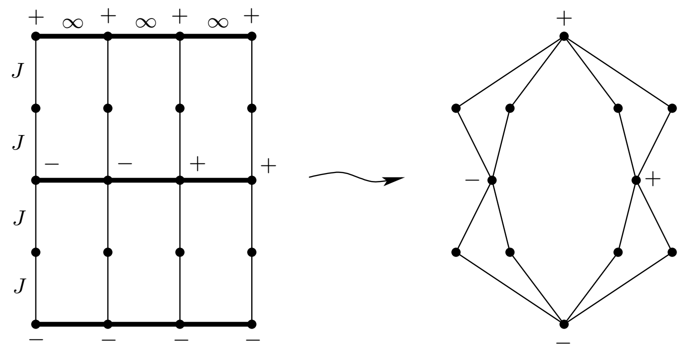

Figure E.1. The Migdal interaction.

E.4. **Diamond model as anisotropic regular 2D lattice model.** The diamond model can be viewed as the Ising model on the regular 2D lattice with a special anisotropic choice of the interaction parameters. Namely, let us consider a $2^n \times (2^n+1)$ rectangle

$$\Delta_n = \{(i,j) \in \mathbb{Z}^2 :\ 0 \le i \le 2^n - 1,\ 0 \le j \le 2^n\}$$

in $\mathbb{Z}^2$, with the Hamiltonian

$$H_n(\sigma) = -\sum_{|x-y|=1} J(x,y)\sigma(x)\sigma(y) - \sum_{x\in\Delta_n} h(x)\sigma(x), \tag{E.7}$$

where

$$h(i,j) = \begin{cases} h & \text{if } 0 < j < 2^n, \\ \dfrac{h}{2} & \text{if } j = 0,\, 2^n. \end{cases}$$

The interaction parameters $J(x,y)$ are defined as follows.

Representing any $j = 0, 1, \dots, 2^n$ in the dyadic arithmetics,

$$j = \sum_{k=0}^{n} j_k 2^k, \quad j_k \in \{0,1\},$$

let

$$o(j) = \min\{k :\ j_k \ne 0\} \text{ if } j > 0; \quad o(0) = n.$$

Let us partition each horizontal level $\Delta_n(j) = \{(i,j) \in \Delta_n\}$ into $2^{n-o(j)}$ intervals $[s2^{o(j)}, (s+1)2^{o(j)})$ of length $2^{o(j)}$, $s = 0, 1, \dots, 2^{n-o(j)} - 1$. Let $\underset{j}{\sim}$ denote the corresponding equivalence relation.

Now we let

$$J(x,y) = \begin{cases} J & \text{if } y - x = \pm(0,1); \\ \infty & \text{if } y - x = \pm(1,0) \quad \text{and} \quad x \underset{j}{\sim} y. \\ 0 & \text{otherwise.} \end{cases}$$

In other words, within a horizontal level $\Delta_n(j)$, non-equivalent sites do not interact, while the equivalent neighbors interact with infinite strength. Infinite interaction $J(x,y)$ is interpreted as condition $\sigma(x) = \sigma(y)$, so that the equivalent sites x and y can be identified. This leads to the diamond graph Γ_n. (Figure E.1 illustrates the

4×5 rectangle, with the vertical solid lines representing interaction J and horizontal dash lines representing the infinite interaction.) Moreover, the Hamiltonian (E.7) takes the form of (1.2):

$$H_n(\sigma) = -\frac{J}{2}\sum_{(x,y)\in\mathcal{E}_n^v}\sigma(x)\sigma(y) - \frac{h}{2}\sum_{(x,y)\in\mathcal{E}_n^v}(\sigma(x)+\sigma(y)),$$

where $\mathcal{E}_n^v$ is the set of vertical edges.

Thus, we have obtained the diamond model.

E.5. **Gibbs states.** The thermodynamic limit of the Gibbs distributions for DHL was studied by Griffiths and Kaufman [GK] and by Bleher and Zalys [BZ2]. As noticed in [GK], there are uncountably many non-isomorphic injective limits of the hierarchical lattice Γ_n as $n\to\infty$, which give rise to uncountably many non-isomorphic infinite hierarchical lattices. In [BZ2], limit Gibbs states on the infinite hierarchical lattices are constructed. It is proven that for any non-degenerate infinite hierarchical lattice, if $T<T_c$ and $h=0$ then there exist exactly two pure infinite Gibbs states, in the sense of Dobrushin-Lanford-Ruelle, while if $T\geq T_c$ or $h\neq 0$ then the infinite Gibbs state is unique.

Appendix F. Open Problems

Problem F.1 (**Critical exponents for the low-temperature intervals**). A consequence of Theorem 11.1 and Corollary 11.2 is that unstable Lyapunov exponents χ^u exist at almost every point of $\mathcal{C}_1$. However, the union of all endpoints of the intervals from $O_t=\mathcal{W}^s(\mathcal{B})\cap\mathbb{T}_t$, taken over all $t\in[t_c,1)$, has measure zero. Thus, we do not know that "most endpoints" have Lyapunov exponents. Do Lyapunov exponents and hence, by Proposition 13.3, weak critical exponents exist at the endpoints of the intervals from O_t?

Problem F.2 (**Principal stable tongues**). Consider the principal stable tongues $\Upsilon_\pm$. Are $\Upsilon_\pm$ bounded by high-temperature hairs of some positive length?

If this is the case, the discussion from Problem F.1 gives that for high enough values of t, the critical exponents $\sigma^h=1$ at the endpoints of the intervals formed by $\Upsilon_\pm\cap\mathcal{T}_t$.

The question can be asked for any of the stable tongues.

Problem F.3 (**Endpoints of hairs**). Recall the set $\mathscr{E}$ of endpoints to the high-temperature hairs that constructed in §12.6. According to Corollary 12.14, $\mathscr{E}$ has Lebesgue measure zero.

(a) What is the Hausdorff dimension of $\mathscr{E}$?
(b) Do any of the high-temperature hairs contain their endpoints, i.e. is there any endpoint within $\mathcal{W}^s(\mathcal{T})$?
(c) Is $\mathcal{W}^s(\mathcal{T})$ a "straight hairy brush" in the sense of [AO]? One consequence would be that the endpoints of the high-temperature hairs must accumulate from both sides to every point on every high-temperature hair. In particular, this would give a negative solution to Problems F.2 and F.1(b).

These questions are partly motivated by the structure of the *Devaney hairs* for the exponential maps, see [DT, McM, Kar].

Problem F.4 (**Control of expansion**). Do we have

$$\limsup \frac{\log d(\phi \circ \mathcal{R}^n)(v)}{n} \leq 4 \tag{F.1}$$

for any $v \in \mathcal{K}(x)$ based at any $x \in \mathcal{C}_1$? This bound would give continuity in ϕ of the density $\rho_t(\phi)$ by an estimate similar to the proof of Propositions 13.1 and 13.3. Notice, however that it does not hold for the first iterate: One can see from (B.8) that if x approaches $\alpha_\pm$ at a definite slope $\tau/\epsilon = \bar{\kappa}$, then the horizontal expansion of vectors in $\mathcal{K}^h(x)$ blows up like $1/\tau$.

Problem F.5 (**Critical temperatures and regularity**). Given $\gamma \in \mathcal{F}^c$, there are $0 < t_c^-(\gamma) \leq t_c^+(\gamma) \leq 1$ so that points on γ below $t = t_c^-(\gamma)$ are in $\mathcal{W}^s(\mathcal{B})$ and points above $t = t_c^+(\gamma)$ are in $\mathcal{W}^s(\mathcal{T})$. We call the points on γ having $t_c^-(\gamma) \leq t \leq t_c^+(\gamma)$ the γ-*critical temperatures*.

(a) Is there a unique γ-critical temperature $t_c(\gamma) := t_c^-(\gamma) = t_c^+(\gamma)$ on each $\gamma \in \mathcal{F}^c$?
(b) The union of γ-critical temperatures over all $\gamma \in \mathcal{F}^c$ is invariant under $\mathcal{R}$. Is there a "natural" invariant measure ν_{crit} supported on this set? What is the entropy of this measure?
(c) It is a consequence of Propositions 9.3 and 10.1 that each leaf $\gamma \in \mathcal{F}^c$ is real analytic below $t_c^-(\gamma)$ and C^1 above $t_c^+(\gamma)$. Does γ have only finite smoothness within the range of γ-critical temperatures?
(d) Proposition 14.1 gives a partial answer to the previous question for periodic leaves. A natural open question here is whether a periodic leaf can contain a neutral periodic point?

Cylinder maps having property (a) on almost every leaf are constructed in [BM, §3]. To ask questions (a) and (c) for almost every leaf in our situation, one must first choose a transverse invariant measure on $\mathcal{F}^c$. With respect to μ_t, almost every leaf is in the union of stable tongues and the result is trivial. The question is more interesting with respect to the transverse measure induced on $\mathcal{F}^c$ by Lebesgue measure on $\mathcal{T}$.

Appendix G. Table of notation

In the course of this paper various objects appear in parallel in two coordinate systems: the "physical coordinates" (z, t) and the affine coordinates $(u, v) \mapsto [u : 1 : v]$.[40] They are related by the semi-conjugacy Ψ from §3. We have attempted (not fully consistently) to use similar notation for corresponding objects, roughly using calligraphic and Greek symbols in the physical coordinates and the corresponding non-calligraphic and Latin symbols in the affine coordinates. For reader's convenience, some of the notation is collected in the following table:

[40] If not to count homogeneous coordinates $(U : V : W)$ and angular coordinates (ϕ, t) as systems in their own right.

Object	Physical coordinates $(z,t) =$	Affine coordinates (u,w)
Renormalization map	$\mathcal{R}$	R
Invariant cylinder	$\mathcal{C} = \mathbb{T} \times [0,1]$	$C = \{w = \bar{u},\ \lvert u\rvert \geq 1\}$
Topless cylinder	$\mathcal{C}_1 = \mathbb{T} \times [0,1)$	$C_1 = \{w = \bar{u},\ \lvert u\rvert > 1\}$
Horizontal/vertical algebraic cone field	$\mathcal{K}^{ah/av}$	$K^{ah/av}$
Modified horizontal/vertical cone fields	$\mathcal{K}^{h/v}$	$K^{h/v}$
Strong separatrix	$\mathcal{L}_0 = \mathbb{C} \times \{t = 0\}$	$L_0 =$ line at infinity
Weak separatrix	$\mathcal{L}_1 = \mathbb{C} \times \{t = 1\}$	$L_1 = \{uw = 1\}$
Bottom of the cylinder	$\mathcal{B} = \mathbb{T} \times \{0\}$	$\mathrm{B} \subset L_0,\ \lvert w/u\rvert = 1$
Top of the cylinder	$\mathcal{T} = \mathbb{T} \times \{1\}$	$\mathrm{T} = \{w = \overline{u}\,,\ \lvert u\rvert = 1\}$
Main indeterminacy pts	$\alpha_\pm = (\pm i, 1)$	$a_\pm = \pm(i, -i)$
Accidental indeterminacy pts	$\gamma, \mathbf{0}$	none
Low temp fixed point	$\beta_0 = (1,0)$	$b_0 = [1:0:1] \in L_0$
Critical temp fixed point	$\beta_c \approx (1, 0.2956)$	$b_c \approx (3.3830, 3.3830)$
High temp fixed point	$\beta_1 = (1,1)$	$b_1 = (1,1)$
Attracting fixed points in $\mathbb{CP}^2$	$\eta = (0,1), \eta' = (\infty, 0)$	$e = (\infty, 0), e' = (0, \infty)$
Principal LY locus	$\mathcal{S} = \{z^2 + 2tz + 1 = 0\}$	$S = \{u + w = -2\}$
Blow-up locus	$\mathcal{G} = \{z^2 + 4zt - 2z + 1 = 0\}$ $= \{t = \sin^2 \phi/2\}$	$G = \{(u-w)^2 + 8(u+w) + 16 =$

The following is further notation specific to $\mathcal{C}$:

Object	Physical coordinates	Initially defined in
Topless cylinder	$\mathcal{C}_1$	§3.1
Bottomless cylinder	$\mathcal{C}_0$	§3.1
Low temperature cylinder	$\mathcal{C}_*$	§9.2
Primary stable tongues	$\Upsilon(\alpha_\pm)$	§9.5
Secondary stable tongues	$\Upsilon_k^n(\alpha)$	§9.5
Basins of attraction for $\mathcal{T}$ with prescribed control	$\mathcal{W}^s_\eta(\mathcal{T})$, $\mathcal{W}^s_0(\mathcal{T})$	§10.2, §11
Central foliation	$\mathcal{F}^c$	§12
Horizontal critical exponent	σ^h	§13.2
Vertical critical exponent	σ^v	§13.2

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