COMPLEX PERSPECTIVE FOR THE PROJECTIVE HEAT MAP ACTING ON PENTAGONS

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ABSTRACT. We place Schwartz's work on the real dynamics of the projective heat map H into the complex perspective by computing its first dynamical degree and gleaning some corollaries about the dynamics of H.

1. INTRODUCTION

Let $\mathcal{P}_N$ denote the space of projective equivalence classes of N-gons in $\mathbb{RP}^2$. The projective heat map is a self-mapping of $\mathcal{P}_N$ that was introduced by R. Schwartz in the monograph [16]. Suppose AB, BC, and CD are three consecutive edges of a polygon P. The *projective midpoint* of BC (with respect to the polygon P) is defined as $S := \overleftrightarrow{BC} \cap \overleftrightarrow{QR}$, where $Q = \overleftrightarrow{AB} \cap \overleftrightarrow{CD}$ and $R = \overleftrightarrow{AC} \cap \overleftrightarrow{BD}$. This is illustrated in Figure 1.

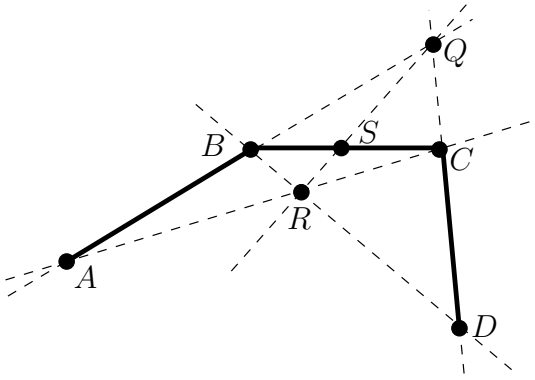

FIGURE 1. S is the projective midpoint of BC

The projective midpoint of BC typically does not coincide with the Euclidean midpoint of BC, however the construction is invariant under projective transformations. Remark also that the projective midpoint may not be defined for certain degenerate configurations of A, B, C, and D.

For any N-gon P let $H(P)$ be the N-gon whose vertices are the projective midpoints of the edges of P. Because the construction is invariant under projective transformations, H descends to a mapping $H : \mathcal{P}_N \dashrightarrow \mathcal{P}_N$ called the *projective heat map*. (We used a broken arrow to denote that H may not be defined at certain polygons for which a sequence of four consecutive vertices A, B, C, and D are not in general position.)

In the case of pentagons, the space of projective equivalence classes of polygons $\mathcal{P}_5$ is parameterized by a pair of real numbers (x, y) called the *flag invariants* of the (equivalence

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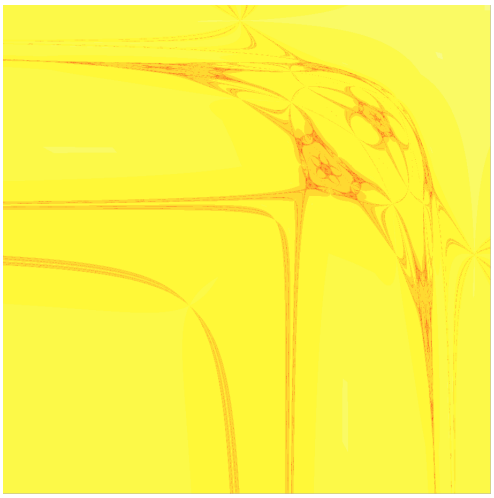

FIGURE 2. The region $(x, y) \in [-9, \phi^{-1}]^2$, where ϕ is the golden ratio. Points in the basin of attraction $\mathcal{W}^s((\phi^{-1}, \phi^{-1}))$ of the attracting fixed point (ϕ^{-1}, ϕ^{-1}) are colored yellow. The "Julia Set" $\mathcal{J} := \mathbb{R}^2 \setminus \mathcal{W}^s((\phi^{-1}, \phi^{-1}))$ is visible in red.

class) of the polygon; see [16, Section 3.6]. In these flag coordinates, the projective heat mapping becomes a rational mapping $H : \mathbb{R}^2 \dashrightarrow \mathbb{R}^2$ given by $(x', y') = H(x, y)$, where

$$x' = \frac{\left(xy^2 + 2\,xy - 3\right)\left(x^2y^2 - 6\,xy - x + 6\right)}{\left(xy^2 + 4\,xy + x - y - 5\right)\left(x^2y^2 - 6\,xy - y + 6\right)}, \quad \text{and}$$
$$y' = \frac{\left(x^2y + 2\,xy - 3\right)\left(x^2y^2 - 6\,xy - y + 6\right)}{\left(x^2y + 4\,xy - x + y - 5\right)\left(x^2y^2 - 6\,xy - x + 6\right)}. \tag{1}$$

This mapping has an obvious symmetry under the reflection $R(x, y) = (y, x)$. It also has a less-obvious symmetry under an action of the dihedral group D_5, corresponding to relabeling the vertices of the pentagon under rotations and reflections. Expressed as a group of birational mappings of the flag coordinates $(x, y) \in \mathbb{R}^2$ this action of D_5 is called the Gauss Group Γ; see [16, Section 3.8].

In the monograph [16] Schwartz uses computer-assisted proofs to provide a nearly complete description of the dynamics of $H : \mathbb{R}^2 \dashrightarrow \mathbb{R}^2$. Highlights of his work include proofs that

(1) Almost any projective equivalence class of a pentagon has orbit under H converging to the class of the regular pentagon. (In flag coordinates, this attracting fixed point is represented by (ϕ^{-1}, ϕ^{-1}), where ϕ is the golden ratio.)
(2) There is a repelling invariant Cantor set $\mathcal{JC}$ for H on which the dynamics of H is conjugate to the one-sided shift $\sigma : \Sigma_6 \to \Sigma_6$ on six symbols.
(3) H does not have an invariant fibration.

Moreover, much of Schwartz's work is dedicated to giving a topological description of the "Julia set" of H defined by $\mathcal{J} := \mathbb{R}^2 \setminus \mathcal{W}^s((\phi^{-1}, \phi^{-1}))$. A computer generated image of this Julia set is shown in Figure 2.

Schwartz often extends H to the compactification $\mathbb{RP}^1 \times \mathbb{RP}^1$ and sometimes to the surface $S_{\mathbb{R}}$ obtained by blowing up $\mathbb{RP}^1 \times \mathbb{RP}^1$ at the three points $(1, 1)$, $(\infty, 0)$, and $(0, \infty)$. The

surface $S_{\mathbb{R}}$ is well-adapted to the symmetries of H under the Gauss Group Γ, since Γ acts on $S_{\mathbb{R}}$ by diffeomorphisms; See [16, Section 10.5].

The formula for H naturally extends to a rational map $H : \mathbb{CP}^1 \times \mathbb{CP}^1 \dashrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ or even as a rational self-map $H : S \dashrightarrow S$, where S is the blow-up of $\mathbb{CP}^1 \times \mathbb{CP}^1$ at $(1,1)$, $(\infty, 0)$, and $(0, \infty)$. Starting in the 1990s, Briend-Duval, Fornaess-Sibony, Hubbard-Oberst-Vorth, Hubbard-Papadapol, Ueda, and many others used powerful tools from complex analysis and algebraic geometry to prove strong results about the dynamics of such mappings. *The purpose of this note is to place Schwartz's work on the real dynamics of the projective heat mapping H into this complex perspective and to glean some corollaries both about the dynamics of H.*

The coarsest invariants of the dynamics of a rational self-mapping $f : X \dashrightarrow X$ of a (complex) projective surface X are the dynamical degrees $\lambda_1(f)$ and $\lambda_2(f)$. The first dynamical degree $\lambda_1(f)$ is defined by

$$\lambda_1(f) := \lim_{n\to\infty} \left\| (f^n)^* : \mathrm{H}^{1,1}(X;\mathbb{C}) \to \mathrm{H}^{1,1}(X;\mathbb{C}) \right\|^{1/n},$$

where $\mathrm{H}^{1,1}(X;\mathbb{C})$ denotes the Dolbeault cohomology of bidegree $(1,1)$. (In the case considered in this note, the reader can replace it with singular cohomology $\mathrm{H}^2(X;\mathbb{C})$.) There are many subtleties to this definition, including the fact that the rational map f may not be continuous at a finite set of indeterminate points. Nevertheless, there is still a well-defined notion of pullback on cohomology. The caveat is that, unlike for continuous mappings, this pullback may not be functorial—one may have $(f^n)^* \neq (f^*)^n$. This often makes calculation of $\lambda_1(f)$ rather technical.

Since we are working in complex dimension two, the second dynamical degree is much simpler: $\lambda_2(f) = \deg_{\mathrm{top}}(f)$ is the the number of preimages of a generic point under f. We refer the reader to [1, 11, 15, 4, 10, 9] for further background on dynamical degrees. The facts we will need about them will be summarized in Section 2.

The two cases $\lambda_2(f) > \lambda_1(f)$ and $\lambda_1(f) > \lambda_2(f)$ correspond to quite different types of dynamics. Building on work of Briend-Duval [2, 3], Guedj and Dinh-Truong-Nguyen [13, 8] proved that the case $\lambda_2(f) > \lambda_1(f)$ corresponds to "predominantly repelling dynamics." Meanwhile, building from work of Bedford-Lyubich-Smillie, Diller-Dujardin-Guedj [5] proved that the case $\lambda_1(f) > \lambda_2(f)$ corresponds to "predominantly saddle-type dynamics." We refer the reader to the cited papers for details.

It was proved by Schwartz [16, Section 8] that $\lambda_2(H) = 6$. Our main technical result is

Theorem 1.1. $\lambda_1(H) = 4$.

Therefore, the projective heat map H has "large topological degree" and the work of Guedj and Dinh-Truong-Nguyen [13, 8] implies that $H : \mathbb{CP}^1 \times \mathbb{CP}^1 \dashrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ has a unique measure μ of maximal entropy, whose entropy is equal to $\log \lambda_2(H) = \log 6$. It also provides a lower bound on the Lyapunov exponents $\chi_{1,2}$ of μ by

$$\chi_{1,2} \geq \frac{1}{2} \log \frac{\lambda_2(H)}{\lambda_1(H)} = \frac{1}{2} \log \frac{3}{2}.$$

Meanwhile, the Bernoulli measure on Σ_6 is an invariant measure of entropy $\log 6$ for the one-sided full shift on six symbols $\sigma : \Sigma_6 \to \Sigma_6$. Pulling it back under the the conjugacy between $H : \mathcal{JC} \to \mathcal{JC}$ and $\sigma : \Sigma_6 \to \Sigma_6$ results in an invariant measure β for $H : \mathcal{JC} \to \mathcal{JC}$

which also has entropy $\log 6$. We will call β the "Bernoulli measure on $\mathcal{JC}$." We conclude that:

Corollary 1.2. *The Bernoulli measure β on $\mathcal{JC} \subset \mathbb{RP}^1 \times \mathbb{RP}^1$ is the unique measure of maximal entropy for $H : \mathbb{CP}^1 \times \mathbb{CP}^1 \dashrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$. Its Lyapunov exponents are bounded from below by $\frac{1}{2}\log\frac{3}{2}$.*

The dynamical degrees of rational maps having an invariant fibration were studied by Dinh-Nguyen and Dinh-Nguyen-Truong in [6, 7]. Their work implies that if a rational self-map of a surface has an invariant fibration, then $\lambda_1(f)$ divides $\lambda_2(f)$; see Lemma 2.4, below. We therefore also conclude that:

Corollary 1.3. *The projective heat map $H : \mathbb{CP}^1 \times \mathbb{CP}^1 \dashrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ has no invariant fibration.*

In particular, $H : \mathbb{RP}^1 \times \mathbb{RP}^1 \dashrightarrow \mathbb{RP}^1 \times \mathbb{RP}^1$ does not have an invariant fibration, thus giving a cohomological re-proof of one of Schwartz's results.

Remark 1.4. Computation of the first dynamical degree of a planar rational map is relatively standard for specialists in several variable complex dynamics. However, the purpose of this note is to relate the dynamical consequences of this calculation to Schwartz's work.

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2. Background on dynamical degrees and algebraic stability

In this section we will give the background needed for computing $\lambda_1(H)$. Let us focus on a somewhat restricted context suitable for this paper. Suppose X and Y are complex projective algebraic surfaces (e.g. a blow-up of $\mathbb{CP}^1 \times \mathbb{CP}^1$ at finitely many points) and $f : X \dashrightarrow Y$ is a rational map. We denote the indeterminacy locus of f by $I(f)$, which is a finite set of points. Throughout the section we will suppose f is *dominant*, meaning that $f(X \setminus I(f))$ is not contained in an algebraic hypersurface of Y.

2.1. **Pullback on cohomology.** It is a well-known fact [17, Ch. IV, §3.3] there is a finite sequence of blow-ups $\pi : \widetilde{X} \to X$ so that f lifts to a holomorphic map $\widetilde{f} : \widetilde{X} \to Y$, making the following diagram commute

(2)
$$\begin{array}{ccc} \widetilde{X} & & \\ \downarrow{\scriptstyle \pi} & \searrow{\scriptstyle \widetilde{f}} & \\ X & \overset{f}{\dashrightarrow} & Y, \end{array}$$

wherever $f \circ \pi$ is defined. Each of these blow-ups is done over a point of $I(f)$.

One uses (2) to define $f^* : \mathrm{H}^{1,1}(Y;\mathbb{C}) \to \mathrm{H}^{1,1}(X;\mathbb{C})$ by

$$f^*(\alpha) := \pi_*(\widetilde{f}^*\alpha) \tag{3}$$

for any $\alpha \in \mathrm{H}^{1,1}(Y;\mathbb{C})$. Here, $\pi_* : \mathrm{H}^{1,1}(\widetilde{X};\mathbb{C}) \to \mathrm{H}^{1,1}(X;\mathbb{C})$ is defined by

$$\pi_* := \mathrm{PD}_X^{-1} \circ \pi_\# \circ \mathrm{PD}_{\widetilde{X}},$$

with $\pi_\#$ denoting the push forward on homology and PD denoting the Poincaré duality isomorphism. The definition of the pullback f^* is well-defined, independent of the choice of resolution of indeterminacy (2); see, for example, [14, Lemma 3.1].

If $C \subset X$ is an irreducible algebraic curve, then, since C has real dimension two and is singular at only finitely many points, it has a well-defined fundamental homology class $\{C\}$ and cohomology class $[C] = \mathrm{PD}_X^{-1}(\{C\})$. From a more sophisticated point of view, C defines a locally principal divisor (C) and $[C]$ is its Chern class; [12]. The definition of f^* simplifies in this case:

Lemma 2.1. *Let $f : X \dashrightarrow Y$ be a dominant rational map between to projective surfaces. Suppose $C \subset Y$ is an irreducible algebraic curve. Then,*

$$f^*[C] = \sum_{\substack{D \subset f^{-1}(C) \\ \text{irreducible}}} m_D[D], \tag{4}$$

where $f^{-1}(C) = \overline{\left(f|_{X\setminus I(f)}\right)^{-1} C}$ and the multiplicity m_D is the order of vanishing of $\psi \circ f$ at any smooth point $p \in D \setminus I(f)$, with ψ being a local defining equation for C at $f(p)$ (chosen to vanish to order 1 at smooth points of C).

Proof. For a holomorphic map $h : X \to Y$ the pullback action on locally principal divisors and on cohomology are related by the commutative diagram:

$$\begin{array}{ccc} \mathrm{H}^1(Y, \mathcal{O}^*) & \xrightarrow{h^*} & \mathrm{H}^1(X, \mathcal{O}^*) \\ \downarrow c & & \downarrow c \\ \mathrm{H}^1(Y; \mathbb{C}) & \xrightarrow{h^*} & \mathrm{H}^1(X; \mathbb{C}), \end{array} \tag{5}$$

where $\mathcal{O}^*$ denotes the sheaf of germs of non-vanishing holomorphic functions and the vertical arrows denote taking the Chern class, see [12, p. 139]. Since (4) is a special case of the definition of how one pulls back locally principal divisors, this diagram justifies (4) in the case of a holomorphic map.

Now, suppose $f : X \dashrightarrow Y$ is rational, that we have the resolution of indeterminacy $\widetilde{f} : \widetilde{X} \to Y$ as in (2), and let $C \subset Y$ be an irreducible algebraic curve. By the discussion in the previous paragraph, (4) applies to $\widetilde{f} : \widetilde{X} \to Y$ giving

$$\widetilde{f}^*[C] = \sum_{\substack{A \subset \widetilde{f}^{-1}(C) \\ \text{irreducible}}} m_A[A] \tag{6}$$

with m_A being the order of vanishing of $\psi \circ \widetilde{f}$ at any smooth point of A. Therefore, we have

$$f^*[C] = \pi_* \widetilde{f}^*[C] = \sum_{\substack{A \subset \widetilde{f}^{-1}(C) \\ \text{irreducible}}} m_A \pi_*[A]. \tag{7}$$

If A is an irreducible component of $\widetilde{f}^{-1}(C)$ with $\pi(A)$ a single point, then by the definition of push forward on homology we have $\pi_*[A] = 0$. Otherwise, $\pi(A)$ is an irreducible algebraic curve in X. In this case, A intersects the exceptional divisors of π in finitely many points, so that $\pi : A \to \pi(A)$ is one-to-one away from finitely many points and hence $\pi_*[A] = [\pi(A)]$. We conclude that

$$f^*[C] = \sum_{\substack{A \subset \widetilde{f}^{-1}(C) \text{ irreducible} \\ \pi(A) \text{ not a point}}} m_A[\pi(A)]. \tag{8}$$

Suppose p is any smooth point of $\pi(A) \setminus I(f)$. Then, π is a biholomorphic map from a neighborhood U of $\pi^{-1}(p)$ to a neighborhood V of p. Therefore, if ψ is a local defining equation for C in a neighborhood of $f(p) = \widetilde{f}(\pi^{-1}(p))$ the order of vanishing of $\psi \circ f = \psi \circ \widetilde{f} \circ (\pi|_U)^{-1}$ at p is the same as that of $\psi \circ \widetilde{f}$ at $\pi^{-1}(p)$. This allows us to replace the m_A in (8) with $m_{\pi(A)}$. We obtain

$$f^*[C] = \sum_{\substack{D \subset \pi(\widetilde{f}^{-1}(C)) \text{ irreducible} \\ D \text{ not a point}}} m_D[D]. \tag{9}$$

Finally, note that each of the blow-ups done to achieve the resolution of indeterminacy (2) occurs over a point of $I(f)$. Therefore,

$$f^{-1}(C) = \overline{\left(f|_{X \setminus I(f)}\right)^{-1} C} \quad \text{and} \quad \pi \circ \widetilde{f}^{-1}(C)$$

differ by at most a subset of the finite set $I(f)$, giving a bijection between irreducible components of $f^{-1}(C)$ and irreducible components of $\pi(\widetilde{f}^{-1}(C))$ that are not a point. This gives (4) from (9).

□

2.2. **Algebraic stability and strategy for computing λ_1.**

Proposition 2.2. (Diller-Favre [4, Thm. 1.14]) *Let $f : X \dashrightarrow X$ by a rational self-map of a projective surface. Then, $(f^n)^* = (f^*)^n$ for all $n \geq 1$ if and only if there is no curve C and no iterate m such that $f^m(C \setminus I_{f^m}) \subset I_f$.*

When either of the two equivalent conditions stated in Proposition 2.2 hold, we will call $f : X \dashrightarrow X$ *algebraically stable*. (Readers who prefer to see a proof of Proposition 2.2 that is based on algebraic geometry instead of analysis can refer to [14, Prop. 1.4].)

If C is a curve such that $f(C \setminus I_f)$ is a point, we will say that C is *collapsed by* f. Since I_f is a finite set of points, the second equivalent condition for algebraic stability (from Proposition 2.2) asserts that there is no curve C that is collapsed by an iterate of f into I_f. In particular, if f does not collapse any curve, then f is algebraically stable.

Proposition 2.3. (Dinh-Sibony [10, Cor. 7]) *Suppose $f : X \dashrightarrow X$ and $g : Y \dashrightarrow Y$ are rational maps of projective surfaces that are conjugate by means of a birational map $\pi : X \dashrightarrow Y$. Then, $\lambda_1(f) = \lambda_1(g)$.*

(Proposition 2.3 actually holds considerably greater generality, including in arbitrary dimensions, but we will only need the simplest form, as stated here.)

Based on these two propositions, there is a clear strategy for computing the first dynamical degree of a rational map $f : X \dashrightarrow X$. One should try to do a sequence of blow-ups $\pi : \widetilde{X} \to X$ in order to make the lifted map $\widetilde{f} : \widetilde{X} \dashrightarrow \widetilde{X}$ satisfy the second equivalent condition from Proposition 2.2, and hence be algebraically stable. Then

$$\lambda_1(f) = \lambda_1(\widetilde{f}) = \text{spectral radius} \left(\widetilde{f}^* : \mathrm{H}^{1,1}(\widetilde{X}) \to \mathrm{H}^{1,1}(\widetilde{X}) \right), \tag{10}$$

where the first equality follows from applying Proposition 2.3 to the conjugacy π between f and $\widetilde{f}$, and the second equality follows from Proposition 2.2. Finally, if the cohomology $\mathrm{H}^{1,1}(\widetilde{X})$ is generated by the fundamental classes of algebraic curves, one can use Proposition 2.1 to compute the spectral radius on the right hand side of (10).

2.3. **Ruling out invariant fibrations by means of dynamical degrees.**

Lemma 2.4. *Suppose $f : X \dashrightarrow X$ is a dominant rational map of a projective surface that has an invariant fibration. Then, $\lambda_1(f)$ divides $\lambda_2(f)$.*

Proof. Suppose that Y is a one-dimensional projective curve and that $f : X \dashrightarrow X$ is semi-conjugate to a rational map $g : Y \dashrightarrow Y$, by a rational mapping $\pi : X \dashrightarrow Y$. The formula from [6, 7] gives that:

$$\lambda_1(f) = \max\Big(\lambda_1(g)\lambda_0(f|\pi), \lambda_0(g)\lambda_1(f|\pi)\Big), \quad \text{and}$$
$$\lambda_2(f) = \lambda_1(g)\lambda_1(f|\pi).$$

Here, the fiber-wise dynamical degrees are defined as

$$\lambda_i(f|\pi) := \lim_{n\to\infty} \|(f^n)^*(\omega_X^i) \wedge \pi^*(\omega_Y)\|^{1/n} \tag{11}$$

for $i = 0, 1$, where ω_X and ω_Y are the Fubini-Study forms on X and Y respectively and $\|\cdot\|$ denotes the mass of a current. (The exponent on ω_Y is 1 since $\dim(Y) = 1$.) It is always true that the 0-th dynamical degree of a mapping is 1, so that $\lambda_0(g) = 1$. (This is why we didn't mention 0-th dynamical degrees in the introduction). Meanwhile, it follows immediately from (11) that $\lambda_0(f|\pi) = 1$. Therefore, $\lambda_1(f) = \max(\lambda_1(g), \lambda_1(f|\pi))$. In either case, the result follows from $\lambda_2(f) = \lambda_1(g)\lambda_1(f|\pi)$. □

Corollary 1.3 follows.

3. Algebraic stability on a suitable blow-up of $\mathbb{CP}^1 \times \mathbb{CP}^1$.

We use multi-homogeneous coordinates $([X : U], [Y : V])$ on $\mathbb{CP}^1 \times \mathbb{CP}^1$, where the affine coordinates from (1) correspond to $(x, y) = (X/U, Y/V)$. There are three other "standard" choices of local coordinates on $\mathbb{CP}^1 \times \mathbb{CP}^1$ given by

$$(u, y) := (U/X, Y/V), \qquad (x, v) := (X/U, V/Y), \qquad \text{and} \qquad (u, v) := (U/X, V/Y).$$

In the (x, y) local coordinates, the critical set of $H : \mathbb{C}^2 \dashrightarrow \mathbb{C}^2$ consists of the following five irreducible curves:

$$C_1 = \{xy - 1 = 0\},$$
$$C_2 = \{2xy + x + y - 4 = 0\},$$
$$C_3 = \{x^2y^2 - 6xy - y + 6 = 0\},$$
$$C_4 = \{x^2y^2 - 6xy - x + 6 = 0\}, \text{and}$$
$$\begin{aligned} C_5 = \{&x^6y^6 - 10x^5y^5 - x^6y^3 + 2x^5y^4 + 2x^4y^5 - x^3y^6 - 4x^5y^3 + 39x^4y^4 - 4x^3y^5 + 3x^5y^2 - 12x^4y^3 \\ &- 12x^3y^4 + 3x^2y^5 + 10x^4y^2 - 47x^3y^3 + 10x^2y^4 - 3x^4y^2 + 22x^3y^2 + 22x^2y^3 - 3xy^4 - 12x^3y - 2x^2y^2 \\ &- 12xy^3 - 6x^2y - 6xy^2 + 9x^2 + 21xy + 9y^2 - 9x - 9y = 0\}. \end{aligned}$$

Lemma 3.1. *The only curves collapsed by $H : \mathbb{CP}^1 \times \mathbb{CP}^1 \dashrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ are $C_1, \ldots, C_4$.*

- *C_1 and C_2 are collapsed by H to $p_1 := \{(x, y) = (1, 1)\}$,*
- *C_3 is collapsed by H to $p_2 := \{(x, y) = (\infty, 0)\} = \{(u, y) = (0, 0)\}$, and*
- *C_4 is collapsed by H to $p_3 := \{(x, y) = (0, \infty)\} = \{(x, v) = (0, 0)\}$.*

The curve C_5 is not collapsed by H.

Proof. Any curve that is collapsed by H is in the critical set, and it is also easy to verify in (u, v) coordinates that the lines at infinity, $\{u = 0\}$ and $\{v = 0\}$, are not collapsed. Thus, we need only consider $C_1, \ldots, C_5$.

The variable x occurs at most linearly in the defining equations for C_1 and C_2. One can therefore solve for x as a function of y. Substituting into the defining equation for H (1) yields that each curve C_1 and C_2 collapses to p_1.

To see that C_3 is collapsed to p_2, we use (x, y) coordinates in the domain and $(u, y) = (1/x, y)$ in codomain:

$$u' = 1/x' = \frac{\left(xy^2 + 4\,xy + x - y - 5\right)\left(x^2y^2 - 6\,xy - y + 6\right)}{\left(xy^2 + 2\,xy - 3\right)\left(x^2y^2 - 6\,xy - x + 6\right)}, \quad \text{and} \tag{12}$$

$$y' = \frac{\left(x^2y + 2\,xy - 3\right)\left(x^2y^2 - 6\,xy - y + 6\right)}{\left(x^2y + 4\,xy - x + y - 5\right)\left(x^2y^2 - 6\,xy - x + 6\right)}.$$

Since the defining equation of C_3 appears in the numerator of both u' and y', it follows that C_3 collapses to $(u, y) = (0, 0)$. The fact that C_4 is collapsed by H to p_3 follows by symmetry under R, or by using a similar calculation.

Meanwhile, $(x, y) = (0, 0)$ and $(x, y) = (0, 1)$ are on C_5 and they map by H to different points. □

Let $\mathcal{X}$ be the blow-up of $\mathbb{CP}^1 \times \mathbb{CP}^1$ at p_1, p_2, and p_3, let E_1, E_2, and E_3 be the resulting exceptional divisors, respectively, and let $\pi\colon \mathcal{X} \to \mathbb{CP}^1 \times \mathbb{CP}^1$ be the canonical projection. We let $\mathcal{H} : \mathcal{X} \dashrightarrow \mathcal{X}$ be the lift of H to this space such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{X} & \overset{\mathcal{H}}{\dashrightarrow} & \mathcal{X} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathbb{CP}^1 \times \mathbb{CP}^1 & \overset{H}{\dashrightarrow} & \mathbb{CP}^1 \times \mathbb{CP}^1. \end{array} \tag{13}$$

For any algebraic curve $C \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ let

$$\widetilde{C} = \overline{\pi^{-1}(C \setminus \{p_1, p_2, p_3\})}$$

denote the *proper transform* of C under π.

The local coordinates $(x, y), (u, y), (x, v)$ and (u, v) on $\mathbb{P}^1 \times \mathbb{P}^1$ continue to serve as local coordinates on $\mathcal{X}$ away from the exceptional divisors E_1, E_2, and E_3. We now set up some standard choices of local coordinates on $\mathcal{X}$ near these exceptional divisors. Consider the local coordinates $(a, b) = (x - 1, y - 1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ centered at p_1. One can describe a neighborhood of E_1 in $\mathcal{X}$ using two systems of local coordinates. The local coordinates $(a, m_1) = (a, b/a)$ describe all directions of approach to p_1, except along the line $\{x = 1\}$, which corresponds to $m_1 = \infty$. (We actually won't need to other system of coordinates in this paper.) Similarly, the local coordinates $(u, m_2) = (u, y/u)$ describe a neighborhood of all but one point of the exceptional divisor E_2, and the local coordinates $(v, m_3) = (v, x/v)$ describe a neighborhood of all but one point of the exceptional divisor E_3. See Figure 3.

Lemma 3.2. *$\mathcal{H} : \mathcal{X} \dashrightarrow \mathcal{X}$ does not collapse any curves and is therefore algebraically stable. In particular,*

- *$\mathcal{H}$ maps $\widetilde{C_1}$ and $\widetilde{C_2}$ onto all of E_1*
- *$\mathcal{H}$ maps $\widetilde{C_3}$ onto all of E_2, and*
- *$\mathcal{H}$ maps $\widetilde{C_4}$ onto all of E_3.*

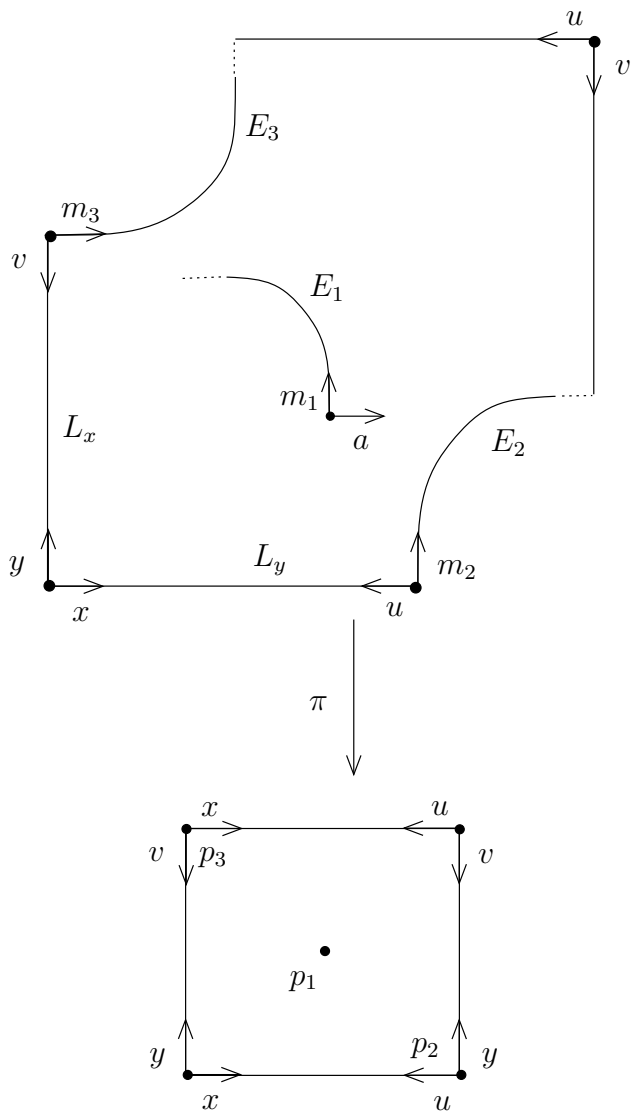

FIGURE 3. On the bottom is $\mathbb{CP}^1 \times \mathbb{CP}^1$, with the four sets of local coordinates labeled. On the top is $\mathcal{X}$, the blow-up of $\mathbb{CP}^1 \times \mathbb{CP}^1$ at p_1, p_2, and p_3, with the respective exceptional divisors E_1, E_2, and E_3.

Proof. If $\mathcal{H}$ collapses a curve A that is not one of the exceptional divisors E_i of π, then $\pi(A)$ is a curve that is collapsed by H. Therefore, since $C_1, \ldots, C_4$ are the only curves collapsed by H, it suffices to check that $\mathcal{H}$ does not collapse $E_1, E_2, E_3, \widetilde{C_1}, \ldots, \widetilde{C_3}$, or $\widetilde{C_4}$.

We will first check that $\mathcal{H}$ does not collapse the exceptional divisors E_1, E_2, and E_3. If we express $\mathcal{H}$ in using (a, m_1) coordinates in the domain and (x, y) in the codomain, we

find

$$x' = \frac{\left(a^2{m_1}^2 + a{m_1}^2 + 4\,am_1 + 4\,m_1 + 3\right)\left(a^3{m_1}^2 + 2\,a^2{m_1}^2 + 2\,a^2m_1 + a{m_1}^2 - 2\,am_1 + a - 4\,m_1 - 5\right)}{\left(a^2{m_1}^2 + a{m_1}^2 + 6\,am_1 + 5\,m_1 + 6\right)\left(a^3{m_1}^2 + 2\,a^2{m_1}^2 + 2\,a^2m_1 + a{m_1}^2 - 2\,am_1 + a - 5\,m_1 - 4\right)}$$

$$y' = \frac{\left(a^2m_1 + 4\,am_1 + a + 3\,m_1 + 4\right)\left(a^3{m_1}^2 + 2\,a^2{m_1}^2 + 2\,a^2m_1 + a{m_1}^2 - 2\,am_1 + a - 5\,m_1 - 4\right)}{\left(a^2m_1 + 6\,am_1 + a + 6\,m_1 + 5\right)\left(a^3{m_1}^2 + 2\,a^2{m_1}^2 + 2\,a^2m_1 + a{m_1}^2 - 2\,am_1 + a - 4\,m_1 - 5\right)}$$

The image of E_1 is obtained by setting $a = 0$:

$$(x', y') = \left(\frac{(4\,m_1 + 3)\,(4\,m_1 + 5)}{(5\,m_1 + 6)\,(5\,m_1 + 4)}, \frac{(3\,m_1 + 4)\,(5\,m_1 + 4)}{(6\,m_1 + 5)\,(4\,m_1 + 5)}\right).$$

Since these functions are not constant functions of m_1, we conclude that $\mathcal{H}$ does not collapse E_1.

If we use the local coordinates (u, m_2) in a neighborhood of generic points of E_2 and coordinates (x, y) in the codomain, we find

$$x' = \frac{\left({m_2}^2u + 2\,m_2 - 3\right)\left({m_2}^2u - 6\,m_2u + 6\,u - 1\right)}{\left({m_2}^2u^2 - m_2u^2 + 4\,m_2u - 5\,u + 1\right)\left({m_2}^2 - m_2u - 6\,m_2 + 6\right)} \tag{14}$$

$$y' = \frac{u\left(2\,m_2u + m_2 - 3\,u\right)\left({m_2}^2 - m_2u - 6\,m_2 + 6\right)}{\left(m_2u^2 + 4\,m_2u + m_2 - 5\,u - 1\right)\left({m_2}^2u - 6\,m_2u + 6\,u - 1\right)}.$$

The image of E_2 is obtained by setting $u = 0$:

$$(x', y') = \left(\frac{3 - 2\,m_2}{{m_2}^2 - 6\,m_2 + 6}, 0\right). \tag{15}$$

Again, since these functions are not constant functions of m_2, we conclude that $\mathcal{H}$ does not collapse E_2. The fact that $\mathcal{H}$ does not collapse E_3 follows by symmetry under R.

It remains to show that the curves $\widetilde{C}_1$, $\widetilde{C}_2$, $\widetilde{C}_3$, and $\widetilde{C}_4$ whose projections by π were collapsed by H are not collapsed by $\mathcal{H}$. We check $\widetilde{C}_1$ and $\widetilde{C}_2$ by expressing $\mathcal{H}$ using the local coordinates (x, y) in the domain and (a, m_1) in the codomain:

$$a' = -\frac{(2\,xy + x + y - 4)\,(xy - y - 3)\,(xy - 1)}{\left(xy^2 + 4\,xy + x - y - 5\right)\left(x^2y^2 - 6\,xy - y + 6\right)} \tag{16}$$

$${m_1}' = \frac{\left(x^2y^2 - 6\,xy - y + 6\right)\left(xy^2 + 4\,xy + x - y - 5\right)(xy - x - 3)}{(xy - y - 3)\left(x^2y + 4\,xy - x + y - 5\right)\left(x^2y^2 - 6\,xy - x + 6\right)}. \tag{17}$$

Recall that $\widetilde{C}_1 = \{xy - 1 = 0\}$, and substituting $y = 1/x$ into the equation for ${m_1}'$, we have

$${m_1}' = \frac{x(x+2)}{2x+1},$$

a non-constant function. Thus, ${m_1}'$ varies while traversing $\widetilde{C}_1$, so $\mathcal{H}$ does not collapse $\widetilde{C}_1$.

Similarly for $\widetilde{C}_2 = \{2xy + x + y - 4 = 0\}$, substituting $y = \frac{4-x}{2x+1}$ into ${m_1}'$, we have

$${m_1}' = \frac{9(x^2 + x + 1)}{(x^2 + x + 7)\,(2\,x + 1)}.$$

This is also non-constant, so $\mathcal{H}$ does not collapse $\widetilde{C}_2$. The defining equations for $\widetilde{C}_1$ and $\widetilde{C}_2$ appear in the numerator of a' so that they are both mapped by $\mathcal{H}$ to $E_1 = \{a = 0\}$. Since they are not collpased by $\mathcal{H}$ and E_1 is irreducible, we conclude that $\widetilde{C}_1$ and $\widetilde{C}_2$ are mapped onto all of E_1.

To check that $\widetilde{C}_3$ is mapped onto all of E_2, we use local coordinates (x, y) in the domain and (u, m_2) in the codomain:

$$u' = \frac{\left(xy^2 + 4\,xy + x - y - 5\right)\left(x^2y^2 - 6\,xy - y + 6\right)}{\left(xy^2 + 2\,xy - 3\right)\left(x^2y^2 - 6\,xy - x + 6\right)}$$
$$m_2{}' = \frac{\left(xy^2 + 2\,xy - 3\right)\left(x^2y + 2\,xy - 3\right)}{\left(xy^2 + 4\,xy + x - y - 5\right)\left(x^2y + 4\,xy - x + y - 5\right)}.$$

Since $(x, y) = (0, 6)$ and $(x, y) = (1, 6)$ are on $\widetilde{C}_3$ and are mapped by $\mathcal{H}$ to different points on E_2, we conclude that $\widetilde{C}_3$ is not collapsed by $\mathcal{H}$. Meanwhile, the defining equation for $\widetilde{C_3}$ is a factor of the numerator of the equation for u', so we conclude that $\mathcal{H}$ maps $\widetilde{C_3}$ onto all of $E_2 = \{u = 0\}$. By the symmetry of $\mathcal{H}$ under R, or very similar calculations, one can also show that $\widetilde{C}_4$ is mapped onto all of E_3.

We conclude that $\mathcal{H}$ does not collapse any curve. Therefore, Proposition 2.2 gives that $\mathcal{H}$ is algebraically stable. □

4. Pullback on cohomology and computation of $\lambda_1(H)$.

At this point we have completed all but the last step of the strategy for computing $\lambda_1(H)$ that was presented in Section 2.2. We have proved in Lemma 3.2 that $\mathcal{H} : \mathcal{X} \dashrightarrow \mathcal{X}$ is algebraically stable so we have

$$\lambda_1(H) = \lambda_1(\mathcal{H}) = \text{spectral radius}\left(\mathcal{H}^* : \mathrm{H}^{1,1}(\mathcal{X}) \to \mathrm{H}^{1,1}(\mathcal{X})\right). \tag{18}$$

It remains to use Proposition 2.1 to compute $\mathcal{H}^* : \mathrm{H}^{1,1}(\mathcal{X}) \to \mathrm{H}^{1,1}(\mathcal{X})$. The first step is to choose a good basis. It is well-known that

$$\mathrm{H}^{1,1}(\mathbb{CP}^1 \times \mathbb{CP}^1) \cong \mathbb{Z}^2$$

and is generated by $[L_x] = [\{X = 0\}]$ and $[L_y] = [\{Y = 0\}]$. Therefore, by [12, P. 474],

$$\left[\widetilde{L_x}\right], \quad \left[\widetilde{L_y}\right], \quad [E_1], \quad [E_2], \quad \text{and} \quad [E_3] \tag{19}$$

is an ordered basis for $\mathrm{H}^{1,1}(\mathcal{X})$, where $\widetilde{L_x}$ and $\widetilde{L_y}$ are the proper transforms of L_x and L_y under π.

Lemma 4.1. *Let $C_6 = \{xy^2 + 2xy - 3 = 0\}$ and $C_7 = \{x^2y + 2xy - 3 = 0\}$. We have*

$$\mathcal{H}^*\left[\widetilde{L_x}\right] = \left[\widetilde{C}_6\right] + [E_3], \tag{20}$$
$$\mathcal{H}^*\left[\widetilde{L_y}\right] = \left[\widetilde{C}_7\right] + [E_2], \tag{21}$$
$$\mathcal{H}^*[E_1] = \left[\widetilde{C}_1\right] + \left[\widetilde{C}_2\right], \tag{22}$$
$$\mathcal{H}^*[E_2] = \left[\widetilde{C_3}\right], \quad \textit{and} \tag{23}$$
$$\mathcal{H}^*[E_3] = \left[\widetilde{C_4}\right]. \tag{24}$$

Proof. We'll start by computing $\mathcal{H}^*\left[\widetilde{L_y}\right]$ and then obtain $\mathcal{H}^*\left[\widetilde{L_x}\right]$ by symmetry under R.

One sees from (1) that the only curves mapped by H to L_y are C_3 and C_7. After performing the blowups, we saw in Lemma 3.2 that $\mathcal{H}$ maps $\widetilde{C_3}$ onto E_2. Therefore, $\widetilde{C_3}$ does not contribute to $\mathcal{H}^*\left[\widetilde{L_y}\right]$. We also saw in (15) that $\mathcal{H}$ maps E_2 onto $\widetilde{L_y}$. Thus, $\mathcal{H}^*\left[\widetilde{L_y}\right]$

is a sum of $\left[\widetilde{C_7}\right]$ and $[E_2]$ with suitable multiplicities. The defining equation for C_7 occurs with multiplicity one in the equation for y' in (1), so that $\left[\widetilde{C_7}\right]$ is assigned multiplicity one in $\mathcal{H}^*\left[\widetilde{L_y}\right]$. (Had the defining equation of C_7 appeared with an exponent larger than one in the expression for y', the coefficient on homology would correspond to that exponent.) Meanwhile, E_2 is $\{u = 0\}$ in the (u, m_2) coordinates. Since u occurs with multiplicity one in the equation for y' in (14), we see that $[E_2]$ is also assigned multiplicity one in $\mathcal{H}^*\left[\widetilde{L_y}\right]$. This proves (21), and Equation (20) follows immediately under the symmetry R.

If we express $\mathcal{H}$ in local coordinates (x, y) in the domain and (a, m_1) in the codomain, as in (16), E_1 is locally $\{a = 0\}$. We see that the defining equations for both C_1 and C_2 appear exactly once in the numerator of the expression for a'. The only additional term in the product is $xy - x - 3$. However, this also appears in the denominator of the expression for ${m_1}'$, so the curve $\{xy - x - 3 = 0\}$ is mapped to a line at infinity. This proves (22).

The only curve mapping by $\mathcal{H}$ to E_2 is $\widetilde{C_3}$. A calculation in local coordinates like above shows that $\left[\widetilde{C_3}\right]$ occurs with multiplicity one in $\mathcal{H}^*[E_2]$, proving (23). Again, we get (24) by symmetry. □

We now need to re-express each of the cohomology classes on the right hand sides of (20-24) in terms of the ordered basis (19). We first express the cohomology classes $[C_1], \ldots, [C_4]$,$[C_6]$ and $[C_7]$ in the basis $[L_x]$ and $[L_y]$ for $\mathrm{H}^{1,1}(\mathbb{CP}^1 \times \mathbb{CP}^1)$:

Lemma 4.2. *In* $\mathrm{H}^{1,1}(\mathbb{CP}^2 \times \mathbb{CP}^1)$ *we have*

$$[C_1] = [L_x] + [L_y], \tag{25}$$

$$[C_2] = [L_x] + [L_y], \tag{26}$$

$$[C_3] = 2[L_x] + 2[L_y], \tag{27}$$

$$[C_4] = 2[L_x] + 2[L_y], \tag{28}$$

$$[C_6] = [L_x] + 2[L_y], \quad \textit{and} \tag{29}$$

$$[C_7] = 2[L_x] + [L_y]. \tag{30}$$

Proof. It is well-known that the intersection pairing on $\mathrm{H}^{1,1}(\mathbb{CP}^1 \times \mathbb{CP}^1)$ is given by the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

That is, $[L_x] \cup [L_x] = 0 = [L_y] \cup [L_y]$ and $[L_x] \cup [L_y] = 1$. We can express the fundamental class of an algebraic curve C in $\mathbb{CP}^1 \times \mathbb{CP}^1$ in terms of the basis $\{[L_x], [L_y]\}$ by taking the cup products with $[L_x]$ and $[L_y]$:

$$[C] \cup [L_x] = (a[L_x] + b[L_y]) \cup [L_x] = b\left[\mathbb{CP}^1 \times \mathbb{CP}^1\right], \text{and}$$
$$[C] \cup [L_y] = (a[L_x] + b[L_y]) \cup [L_y] = a\left[\mathbb{CP}^1 \times \mathbb{CP}^1\right].$$

Moreover, the coefficient b from computing $[C] \cup [L_x]$ is the sum of the intersection multiplicities at each point where C intersects L_x. After doing the analogous computation for a, we have $[C] = a[L_x] + b[L_y]$.

Note that L_x is cohomologous to $[\{x = 1\}]$, and $C_1 = \{xy - 1 = 0\}$ intersects $\{x = 1\}$ transversally at a single point, $(1, 1)$. Thus,

$$[C_1] \cup [L_x] = \left[\mathbb{CP}^1 \times \mathbb{CP}^1\right],$$

and by symmetry under R,

$$[C_1] \cup [L_y] = \left[\mathbb{CP}^1 \times \mathbb{CP}^1\right].$$

Equation (25) follows.

We can intersect $C_2 = \{2xy+x+y-4 = 0\}$ with a line $\{x = x_0\}$ away from $C_2 \cap \{v = 0\}$ (see Figure 3). Substituting $x = x_0$ into the defining equation for C_2, we can solve for a unique y value (independent of the choice of x_0). Since the equation for C_2 is symmetric under R, C_2 also intersects a generic line $\{y = y_0\}$ in a single point. Equation (26) follows.

Note that C_3 and $\{x = x_0\}$ (for a generic x_0) will not intersect $\{v = 0\}$ at the same place, so we can count intersections in $\mathbb{C}^2$. Then substituting x_0 into the defining equation for C_3, one can solve for exactly two values of y, both of which are simple zeros. Thus,

$$[C_3] \cup [L_x] = [C_3] \cup [\{x = x_0\}] = 2\left[\mathbb{CP}^1 \times \mathbb{CP}^1\right].$$

Similar reasoning with horizontal lines yields that (27) follows. Meanwhile, (28) follows by symmetry.

As with the other curves, we can intersect C_6 with a line $\{x = x_0\}$ that intersects $\{v = 0\}$ at a different place than where C_6 intersects $\{v = 0\}$. Substituting $x = x_0$ into the defining equation for C_6, we can solve for a two y values. Then intersecting with a line $\{y = y_0\}$ that intersects $\{u = 0\}$ at a different place than where C_6 intersects $\{u = 0\}$ and substituting $y = y_0$ into the defining equation for C_6, we can solve for a unique x value. Equation (29) follows, and Equation (30) follows by symmetry under R.

□

We now lift the results of Lemma 4.2 to the blown-up space $\mathcal{X}$:

Lemma 4.3. *In* $\mathrm{H}^{1,1}(\mathcal{X})$ *we have*

$$\left[\widetilde{C_1}\right] = \left[\widetilde{L_x}\right] + \left[\widetilde{L_y}\right] - [E_1], \tag{31}$$

$$\left[\widetilde{C_2}\right] = \left[\widetilde{L_x}\right] + \left[\widetilde{L_y}\right] - [E_1] + [E_2] + [E_3], \tag{32}$$

$$\left[\widetilde{C_3}\right] = 2\left[\widetilde{L_x}\right] + 2\left[\widetilde{L_y}\right] - [E_1] + [E_3], \tag{33}$$

$$\left[\widetilde{C_4}\right] = 2\left[\widetilde{L_x}\right] + 2\left[\widetilde{L_y}\right] - [E_1] + [E_2], \tag{34}$$

$$\left[\widetilde{C_6}\right] = \left[\widetilde{L_x}\right] + 2\left[\widetilde{L_y}\right] - [E_1] + [E_2], \quad \textit{and} \tag{35}$$

$$\left[\widetilde{C_7}\right] = 2\left[\widetilde{L_x}\right] + \left[\widetilde{L_y}\right] - [E_1] + [E_3]. \tag{36}$$

Proof. Since C_1 has multiplicity 1 at p_1, p_2, and p_3, and $[C_1] = [L_x] + [L_y]$ by Lemma 4.2, we can pull back both sides of the equation by π to see that

$$\left[\widetilde{C_1}\right] + [E_1] + [E_2] + [E_3] = \left[\widetilde{L_x}\right] + [E_3] + \left[\widetilde{L_y}\right] + [E_2].$$

Equation (31) follows. Equations (32-36) follow by similar calculations. The only subtlety is computing the local multiplicities, which we summarize here:

Curve	mult. at p_1	mult. at p_2	mult. at p_3
C_2	1	0	0
C_3	1	1	2
C_4	1	2	1
C_6	1	1	1
C_7	1	1	1

□

Lemmas 4.1 and 4.3 make it easy to express $\mathcal{H}^* : \mathrm{H}^{1,1}(\mathcal{X}) \to \mathrm{H}^{1,1}(\mathcal{X})$ in the ordered basis (19). For example,

$$\mathcal{H}^* \left[\widetilde{L_x}\right] = \left[\widetilde{C}_6\right] + [E_3] = \left[\widetilde{L_x}\right] + 2\left[\widetilde{L_y}\right] - [E_1] + [E_2] + [E_3]\,. \tag{37}$$

Computing all of the others, we find

$$\mathcal{H}^* = \begin{bmatrix} 1 & 2 & -1 & 1 & 1 \\ 2 & 1 & -1 & 1 & 1 \\ 2 & 2 & -2 & 1 & 1 \\ 2 & 2 & -1 & 0 & 1 \\ 2 & 2 & -1 & 1 & 0 \end{bmatrix}$$

This matrix has characteristic polynomial $(4-\lambda)(1+\lambda)^4$. The spectral radius is the largest root, which is 4. Using (18) we have that $\lambda_1(H) = \lambda_1(\mathcal{H}) = 4$. □(Theorem 1.1).

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