PULLING BACK SINGULARITIES OF CODIMENSION ONE OBJECTS

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Abstract. We prove that the preimage of a germ of a singular analytic hypersurface under a germ of a finite holomorphic map $g : (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}^n, \mathbf{0})$ is again singular. This provides a generalization of previous results of this nature by Ebenfelt-Rothschild [3], Lebl [7], and Denkowski [2]. The same statement is proved for pullbacks of singular codimension one holomorphic foliations.

1. Introduction

This note is devoted to proving the following two theorems:

Theorem A. *Let $g : (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}^n, \mathbf{0})$ be a germ of a finite holomorphic map. Let X be a germ of an analytic hypersurface through $\mathbf{0}$ that is singular at $\mathbf{0}$. Then, the germ of a hypersurface $g^{-1}(X)$ is singular at $\mathbf{0}$.*

Theorem B. *Let $g : (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}^n, \mathbf{0})$ be a germ of a finite holomorphic map. Let $\mathcal{F}$ be a germ of a codimension one holomorphic foliation that is singular at $\mathbf{0}$. Then, the pullback foliation $g^*\mathcal{F}$ is singular at $\mathbf{0}$.*

In fact, these two theorems are equivalent; see Proposition 3.1 below.

It is important to note that Theorem A is about the "reduced structure" of analytic hypersurfaces and about the "set-theoretic" preimage—no multiplicities are taken into consideration. This causes a subtlety that is the main challenge in the proofs. In fact, the equivalence between Theorems A and B plays an important conceptual role[1] in working with the set-theoretic preimage of a hypersurface. This is because the pullback of a singular codimension one holomorphic foliation is defined in a way that ignores the multiplicities arising from the critical hypersurfaces of g; see Section 2.3.

When X is a germ of an analytic subvariety of $\mathbb{C}^n$ having arbitrary dimension, a version of Theorem A has been proved by Ebenfelt-Rothschild [3, Theorem 2.1], Lebl [7, Section 4], and Denkowski [2, Theorem 1.2] under the additional hypothesis that the Jacobian $\det(Dg)$ does not vanish identically on $g^{-1}(X)$. However, this hypothesis has been removed in the case that $\dim(X) = 1$, see [3, Theorem 4.1] and [7, Section 4]. We refer the reader to the aforementioned papers for further details.

Surprisingly, until now, the statement of Theorem A seems to be open in the case that $\det(Dg)$ is allowed to vanish identically on $g^{-1}(X)$.

Note also that a version of Theorem B was proved in dimension 2 by Kaschner, Pérez, and the second author of the present paper in [6, Lemma 2.4].

In Section 2 we give background on finite holomorphic maps, analytic hypersurfaces, and singular codimension one holomorphic foliations. Section 3 is devoted to proving that

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Date: November 1, 2019.

1991 *Mathematics Subject Classification.* 32SXX,32S65,14J17.

[1]We use it in the proof of Proposition 4.1, which is the most technical step of the proof of Theorem A.

Theorems A and B are equivalent. We conclude the proof of both theorems by presenting a proof of Theorem A in Section 4.

Notational Conventions:
Germs and their representatives will often be used interchangeably as a minor abuse of notation. The notation $g : (U, 0) \to (V, 0)$ when used for g that is not a germ means that $g : U \to V$ is the map and $g(0) = 0$.

Acknowledgments:
We thank Javier Fernández de Bobadilla and Jenia Tevelev for helpful conversations. We thank the referee for doing a careful reading of our paper and for several helpful comments that have helped us to improve the writing. Part of this research was done while the first author was visiting the Department of Mathematics of Indiana University at Bloomington, supported by the "Salvador de Madariaga" grant no. PRX18/00404 funded by the Spanish Government. He wants to thank the Indiana University Math Department for their hospitality. He has also been partially supported by the research grants MTM2015-63612-P and PGC2018-098321-B-I00, of the Spanish Government. The second author was supported by NSF grant DMS-1348589.

2. Background

2.1. **Finite Maps.** A holomorphic germ $g : (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}^n, \mathbf{0})$ is *finite* if $g^{-1}(\mathbf{0}) \cap U = \{\mathbf{0}\}$ for a sufficiently small neighborhood U of $\mathbf{0}$ and some representative g. Note that this implies that if $\boldsymbol{y}$ is sufficiently close to $\mathbf{0}$ then $g^{-1}(\boldsymbol{y}) \cap U$ is a finite set. The following lemma is well-known:

Lemma 2.1. *Let $g : (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}^n, \mathbf{0})$ be a germ of a finite holomorphic map. Then, there are arbitrarily small neighborhoods U, W of $\mathbf{0}$ and a representative $g : (U, \mathbf{0}) \to (W, \mathbf{0})$ of the germ that is proper.*

We refer the reader to [9, Theorem 15.1.6] for a proof of Lemma 2.1. We also refer the reader to [5, p. 667-670] for various additional properties of finite maps.

2.2. **Hypersurfaces.** We refer the reader to the monograph by Milnor [8] for more details than will be presented here. Let U be an open neighborhood of $\mathbf{0}$ in $\mathbb{C}^n$. The term *hypersurface in U* will refer to a potentially reducible analytic subset of U, each of whose irreducible components has codimension one. Throughout this paper we will be interested in the "reduced structure" of a hypersurface, thinking of it as an analytic set rather than a divisor.

Note that if X is a reducible germ of a hypersurface at $\mathbf{0}$ then X is singular at $\mathbf{0}$. Moreover, reducibility is preserved under preimage by g. Therefore, it suffices to prove Theorem A for irreducible germs, and most of our attention will be focused on them.

Associated to any irreducible hypersurface X in U is a holomorphic function $\psi : U \to \mathbb{C}$ that is prime in the ring of germs of holomorphic functions $\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}$ such that

$$X = \{\mathbf{x} \in U \,:\, \psi(\mathbf{x}) = 0\} \quad \text{and} \quad d\psi \not\equiv 0 \text{ on } X. \tag{1}$$

The condition that $d\psi \not\equiv 0$ on X follows from ψ being a prime element of $\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}$ and it ensures that ψ vanishes to order one on the smooth part of X. In other words, the Cartier divisor (ψ) defines the reduced divisor associated to X.

Lemma 2.2. *Suppose X is an irreducible hypersurface passing through $\mathbf{0}$ that is given by (1). If X is singular at $\mathbf{0}$ then there is a smaller neighborhood $U' \subset U$ of $\mathbf{0}$ in which the singular locus of X is given by*

$$X_{\text{sing}} = \{\boldsymbol{x} \in U' \,:\, d\psi(\boldsymbol{x}) = 0\},$$

where d denotes the exterior derivative.

Proof. We have,

$$X_{\text{sing}} = \{\boldsymbol{x} \in U \,:\, \psi(x) = 0 \text{ and } d\psi(\boldsymbol{x}) = 0\}.$$

Let

$$S = \{\boldsymbol{x} \in U \,:\, d\psi(\boldsymbol{x}) = 0\}.$$

We can choose $U' \subset U$ to be sufficiently small so that each irreducible component of $S \cap U'$ passes through $\mathbf{0}$. For the remainder of the proof we will work entirely in U'.

It remains to show that $\psi|_S(\boldsymbol{x}) \equiv 0$. This is trivial if $S = \{\mathbf{0}\}$. Otherwise, let S_0 be an irreducible component of S. The smooth locus $S_{0,\text{reg}}$ is a submanifold of U' that is dense in S_0. Any two points $p, q \in S_{0,\text{reg}}$ can be connected by a smooth path $\gamma : [0,1] \to S_{0,\text{reg}}$ and

$$\psi(q) - \psi(p) = \int_\gamma d\psi = 0.$$

Therefore ψ is constant on $S_{0,\text{reg}}$ and hence on S_0. As S_0 passes through $\mathbf{0}$ and $\psi(\mathbf{0}) = 0$, ψ vanishes on S_0. □

2.3. **Singular codimension one holomorphic foliations.** We refer the reader to the survey paper by Cerveau [1] for more background than we will present here. A singular codimension one holomorphic foliation $\mathcal{F}$ in a sufficiently small neighborhood U of $\mathbf{0}$ in $\mathbb{C}^n$ can be described by a holomorphic one form ω having the following two properties:

(i) $\omega \wedge d\omega = 0$ (integrability), and
(ii) $\omega_{\text{sing}} := \{\boldsymbol{x} \in U \,:\, \omega(\boldsymbol{x}) = 0\}$ has codimension ≥ 2.

It is a consequence of the Frobenius Theorem that any $\boldsymbol{x} \in U \setminus \omega_{\text{sing}}$ has a neighborhood N in which there is a system of holomorphic coordinates $\boldsymbol{z} = (z_1, \ldots, z_n)$ on N such that $\omega = u(\boldsymbol{z})dz_1$ with $u(\boldsymbol{z}) \neq 0$ on N (i.e. u is a unit). In these coordinates, the tangent planes to $\text{Ker}(\omega)$ become the "vertical hyperplanes" $\{z_1 = \text{const}\}$. They are interpreted as *local leaves* of $\mathcal{F}$ in the local coordinate $\boldsymbol{z}$. By transporting them back to the original coordinate $\boldsymbol{y}$ and gluing, one obtains *global leaves* of $\mathcal{F}$ on $U \setminus \omega_{\text{sing}}$. One calls ω_{sing} the singular locus of $\mathcal{F}$ and denotes it by $\mathcal{F}_{\text{sing}}$.

Occasionally, and only when there is no ambiguity, we will use the term "foliation" to mean "singular codimension one holomorphic foliation".

Let $U, W \subset \mathbb{C}^n$ be open sets and let $g : U \to W$ be a finite holomorphic mapping. Suppose $\mathcal{F}$ is a singular codimension one holomorphic foliation that is defined on W by a holomorphic one form ω (satisfying (i) and (ii)). Then, one defines the *pullback* $g^*\mathcal{F}$ on U by a holomorphic one form $g^\#\omega$ on U obtained by dividing $g^*\omega$ by a suitable holomorphic function, in order that the zero locus of $g^\#\omega$ is of codimension two or larger.[2] Because pullback g^* respects wedge product and exterior derivatives, $g^*\omega$ still satisfies (i), and a simple calculation shows that (i) continues to hold after rescaling $g^*\omega$ to obtain $g^\#\omega$. Meanwhile, the rescaling was done so that $g^\#\omega$ satisfies (ii).

[2]The notation $g^\#\omega$ is non-standard and the reader should note that it is only well-defined up to multiplication by a non-vanishing holomorphic function.

Remark 2.3. It is possible that $g^*\mathcal{F}$ be singular at $\boldsymbol{x}$ even though $\mathcal{F}$ is regular at $g(\boldsymbol{x})$. For example, in dimension two, if $\omega = dy_1$ and $g(x_1, x_2) = (x_1^2 - x_2^2, x_2)$, then

$$g^*\omega = d(x_1^2 - x_2^2) = 2x_1 dx_1 - 2x_2 dx_2 = g^\# \omega,$$

which vanishes at $(0, 0)$.

It is immediate from the definition that in a sufficiently small neighborhood N of any regular point for g the pullback g^* sends local leaves of $\mathcal{F}$ in $g(N)$ to local leaves of $g^*\mathcal{F}$ in N. The following lemma justifies that this also happens more generally:

Lemma 2.4. *Suppose that $\boldsymbol{x}$ is a regular point for $g^*\mathcal{F}$ and $g(\boldsymbol{x})$ is a regular point for $\mathcal{F}$. Then, there exists a neighborhood N of $\boldsymbol{x}$ such that if L is the local leaf of $\mathcal{F}$ through $g(\boldsymbol{x})$ in $g(N)$ then $g^{-1}(L)$ is the local leaf of $g^*\mathcal{F}$ through $\boldsymbol{x}$ in N.*

Proof. Since $g^*\mathcal{F}$ is regular at $\boldsymbol{x}$ and $\mathcal{F}$ is regular at $g(\boldsymbol{x})$ we can choose a connected neighborhood N of $\boldsymbol{x}$ sufficiently small so that:

(a) there is a system of local coordinates $(z_1, \ldots, z_n)$ in N in which local leaves of $g^*\mathcal{F}$ are given by vertical hyperplanes $\{z_1 = \text{const}\}$, and
(b) there is a system of local coordinates $(w_1, \ldots, w_n)$ in $g(N)$ in which $\mathcal{F}$ is given by vertical hyperplanes $\{w_1 = \text{const}\}$.

Let $N_0 \subset N \setminus \text{crit}(g)$ be an open set. (Remark that since g is finite it is an open mapping [9, Theorem 15.1.6] implying that $\text{crit}(g)$ is a proper analytic subvariety of N.) Then, g sends local leaves of $g^*\mathcal{F}$ within N_0 to local leaves of $\mathcal{F}$ within $g(N_0)$. Therefore, if we write

$$(w_1, \ldots, w_n) = g(z_1, \ldots, z_n) = (g_1(z_1, \ldots, z_n), \ldots, g_n(z_1, \ldots, z_n)),$$

then within N_0 we have that $g_1(z_1, \ldots, z_n)$ is independent of $z_2, \ldots, z_n$, i.e.

$$g_1(z_1, \ldots, z_n) \equiv g_1(z_1).$$

Because N is connected, this property carries over to g within all of N. In particular, g sends every local leaf of $g^*\mathcal{F}$ in N to some local leaf of $\mathcal{F}$ in $g(N)$.

We can then choose a smaller neighborhood $N' \subset N$ so that the local leaf of $g^*\mathcal{F}$ through $\boldsymbol{x}$ is the only local leaf of $g^*\mathcal{F}$ within N' whose image contains $g(\boldsymbol{x})$. □

3. Equivalence of Theorems A and B

Proposition 3.1. *Theorem A holds if and only if Theorem B holds.*

Proof. It suffices to show that the following are equivalent:

(a) There is a germ of an irreducible hypersurface X that is singular at $\mathbf{0}$ such that $g^{-1}(X)$ is smooth, and
(b) There is a germ of a singular codimension one holomorphic foliation $\mathcal{F}$ that is singular at $\mathbf{0}$ such that $g^*(\mathcal{F})$ is smooth.

Suppose (a) in order to prove (b). Suppose W is a neighborhood of $\mathbf{0}$ in $\mathbb{C}^n$ and X is an irreducible hypersurface in W given by

$$X = \{\boldsymbol{y} \in W \,:\, \psi(\boldsymbol{y}) = 0\} \quad \text{and} \quad d\psi \not\equiv 0 \text{ on } X,$$

with $\mathbf{0} \in X_{\text{sing}}$. Lemma 2.2 allows us to shrink to a smaller neighborhood $W' \subset W$ of $\mathbf{0}$ so that

$$X_{\text{sing}} = \{\boldsymbol{y} \in W' \,:\, d\psi(\boldsymbol{y}) = 0\}.$$

Since $X_{\rm sing}$ has codimension two or larger in $\mathbb{C}^n$, the one form $d\psi$ defines a singular codimension one holomorphic foliation $\mathcal{F}$ on W' with $\mathbf{0} \in \mathcal{F}_{\rm sing} = X_{\rm sing}$.

By hypothesis, $g^{-1}(X)$ is smooth, so we can choose holomorphic coordinates $(x_1, \ldots, x_n)$ in a neighborhood U of $\mathbf{0}$ so that $g^{-1}(X) = \{x_1 = 0\}$. Therefore $\psi \circ g(\boldsymbol{x}) = x_1^m u(\boldsymbol{x})$ for some integer $m \geq 1$ and some unit (non-vanishing holomorphic function) $u(\boldsymbol{x})$. Hence,

$$g^*(d\psi) = d(g^*\psi) = d(\psi \circ g(\boldsymbol{x})) = d(x_1^m u(\boldsymbol{x})) = m x_1^{m-1} dx_1 u(\boldsymbol{x}) + x_1^m du(\boldsymbol{x}). \tag{2}$$

Recall that the pullback $g^*\mathcal{F}$ is defined in a neighborhood of $\mathbf{0}$ by any holomorphic one form $g^{\#}(d\psi)$ obtained by rescaling $g^*(d\psi)$ in order that the zero locus of $g^{\#}(d\psi)$ is of codimension two or larger. In this case, one can use

$$g^{\#}(d\psi) := \frac{1}{x_1^{m-1}} g^*(d\psi) = m\ u(\boldsymbol{x}) dx_1 + x_1 du(\boldsymbol{x}). \tag{3}$$

Note that $g^{\#}(d\psi)(\mathbf{0}) = m\ u(\mathbf{0}) dx_1 \neq 0$, so that $g^*\mathcal{F}$ is smooth in some neighborhood of $\mathbf{0}$, thus proving (b).

We now prove that (b) implies (a). Suppose $\mathcal{F}$ is a singular codimension one holomorphorphic foliation that is singular at $\mathbf{0}$ and that $g^*\mathcal{F}$ is smooth in some neighborhood of $\mathbf{0}$. We will use the hypothesis that $g^*\mathcal{F}$ is smooth to prove that $\mathcal{F}$ is locally given by level hypersurfaces $\{\psi(\boldsymbol{y}) = \text{const}\}$ for a suitable holomorphic function ψ. Supposing that $\psi(\mathbf{0}) = 0$, this will produce for us an irreducible singular hypersurface $X = \psi^{-1}(0)$ whose preimage under g is smooth.

More specifically, suppose that $\mathcal{F}$ is given in some neighborhood W_0 of $\mathbf{0}$ by the holomorphic one form ω satisfying (i) and (ii) from the definition of foliation. Because of the hypothesis that $g^*\mathcal{F}$ is smooth in a neighborhood of $\mathbf{0}$, we can do a local holomorphic change of variables so that in some neighborhood U_0 of $\mathbf{0}$ we have that $g^*\mathcal{F}$ is given by dx_1. In other words, $g^*\mathcal{F}$ is given by the family of vertical hyperplanes of the form $\{x_1 = \text{const}\}$ within U_0.

By Lemma 2.1 we can choose neighborhoods $U \subset U_0$ and $W \subset W_0$ of the origin in domain and codomain, respectively, so that $g : U \to W$ is a proper map and so that $g^{-1}(\mathbf{0}) \cap W = \mathbf{0}$. Let us also suppose that U and W are connected. Let δ denote the topological degree of this mapping (number of preimages of a point, when counted with multiplicity). Let $\pi_1 : U \to \mathbb{C}$ be projection onto the first coordinate, $\pi_1(\boldsymbol{x}) = x_1$. Since $g : U \to W$ is proper, we can take the pushforward $\psi(\boldsymbol{y}) := g_*\pi_1(\boldsymbol{y})$, which is a holomorphic function on W defined by

$$\psi(\boldsymbol{y}) := (g_*\pi_1)(\boldsymbol{y}) = \sum_{\boldsymbol{x} \in g^{-1}(\boldsymbol{y})} \pi_1(\boldsymbol{x}),$$

where preimages are counted with multiplicities. (Remark that in certain contexts from algebraic geometry the proper pushforward is called the "trace map"; see, for example, [5, p. 668].)

We claim that level hypersurfaces of ψ are tangent to leaves of $\mathcal{F}$ at those places where both are regular. This will hold if and only if

$$d\psi \wedge \omega = 0 \quad \text{on } W, \tag{4}$$

which is equivalent to the kernels of both $d\psi$ and ω being parallel. Because $d\psi \wedge \omega$ is a holomorphic two-form and W is connected, it suffices to check (4) on any open subset of W.

Let N be a simply-connected open subset of W that is disjoint from the critical value locus, so that the Inverse Function Theorem and Monodromy Theorem allow us to construct

δ holomorphic inverse branches $h_1, \ldots, h_\delta : N \to U$ of g. Because $g^*\mathcal{F}$ is given by dx_1, we have that

$$\pi_1 \circ h_\ell(\boldsymbol{y}) \quad \text{is constant on } \gamma \cap N$$

for any leaf γ of $\mathcal{F}$ and any $1 \leq \ell \leq \delta$. This implies that ψ is constant on $\gamma \cap N$ and thus that $d\psi \wedge \omega = 0$ on the open set $N \subset W$.

It is possible from the definition of ψ that $d\psi$ vanishes on some hypersurfaces. Let η be a holomorphic one-form obtained by dividing $d\psi$ by a suitable holomorphic function in order that the zero locus of η has codimension two or larger.

Let X be the connected[3] component of

$$\{\boldsymbol{y} \in W \,:\, \psi(\boldsymbol{y}) = 0\}$$

that contains $\mathbf{0}$. We claim that X is singular at $\mathbf{0}$ and $g^{-1}(X) = \{x_1 = 0\}$, so that $g^{-1}(X)$ is smooth.

Suppose for contradiction that X is smooth at $\mathbf{0}$. Then, possibly after restricting to a smaller neighborhood of $\mathbf{0}$, we can choose a suitable local coordinate $\boldsymbol{z}$ in which $X = \{z_1 = 0\}$. This implies that in this coordinate $\psi(\boldsymbol{z}) = z_1^n u(\boldsymbol{z})$ for some unit $u(\boldsymbol{z})$. The same calculation as in (2) and (3) can then be used to show that $\eta(\mathbf{0}) \neq 0$ and hence that $\mathcal{F}$ is smooth at $\mathbf{0}$, contrary to our hypothesis.

Meanwhile, the smooth locus X_{reg} is a leaf of $\mathcal{F}$. Since $g^*\mathcal{F}$ is given by the vertical leaves $\{x_1 = \text{const}\}$, Lemma 2.4 gives that each point $\boldsymbol{y} \in X_{\text{reg}}$ has a neighborhood in X_{reg} each of whose preimages under g is contained in one of the vertical leaves $\{x_1 = \text{const}\}$. However, since X_{reg} is dense in X and since $g^{-1}(\mathbf{0}) = \{\mathbf{0}\}$ we conclude that $g^{-1}(X) = \{x_1 = 0\}$. Finally, remark that if X were reducible, then $g^{-1}(X)$ would also be reducible, contradicting the conclusion of the previous sentence.

□

4. Proof of Theorems A and B

In Proposition 3.1 we saw that Theorems A and B are equivalent. Therefore, it suffices to prove Theorem A, and we will focus entirely on that for the remainder of the paper. Let us start with two propositions.

Proposition 4.1. *Let $g : (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}^n, \mathbf{0})$ be a germ of a finite holomorphic map. Let X be a germ of an analytic hypersurface that has an isolated singularity at $\mathbf{0}$. Then, the preimage $g^{-1}(X)$ is singular at $\mathbf{0}$.*

Remark 4.2. A version of this was proved for $\mathbb{C}^2$ in [6, Lemma 2.4] and the proof below extends what was done there to higher dimensions by using the algebraic technique of Koszul complexes.

Proof. If X is reducible at $\mathbf{0}$ then $g^{-1}(X)$ is also, and hence $g^{-1}(X)$ is singular at $\mathbf{0}$.

Now suppose for contradiction that X is an irreducible analytic hypersurface defined in a neighborhood W of $\mathbf{0}$ and singular only at $\mathbf{0}$, with the property that $g^{-1}(X)$ is smooth. In this one step of the proof of Theorem A, it will be convenient to use Proposition 3.1 to replace X with a singular codimension one holomorphic foliation $\mathcal{F}$ that is singular at $\mathbf{0}$ whose pullback $g^*\mathcal{F}$ is smooth. Moreover, in the proof of Proposition 3.1 we saw that the resulting foliation satisfies $\mathcal{F}_{\text{sing}} = X_{\text{sing}}$, so that $\mathbf{0}$ is an isolated singularity for $\mathcal{F}$.

[3] A priori, X could be a reducible hypersurface. We'll rule this out in the last paragraph of the proof.

Let us write g in coordinates as $g(x_1, \dots, x_n) = (g_1(x_1, \dots, x_n), \dots, g_n(x_1, \dots, x_n))$. Suppose $\mathcal{F}$ is given in a neighborhood of $\mathbf{0}$ by the holomorphic one form

$$\omega = \sum_{i=1}^{n} a_i(\boldsymbol{y}) dy_i.$$

Remark that the assumption that $\mathcal{F}$ has an isolated singularity at $\mathbf{0}$ implies that the common zero set of the $a_i(\boldsymbol{y})$ is the origin, in other words the $a_i(\boldsymbol{y})$ for $1 \le i \le n$ form a *regular sequence* in the ring of germs of holomorphic functions $\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}$.

Suppose for contradiction that $g^*\mathcal{F}$ is smooth in a neighborhood of $\mathbf{0}$. Then, after a suitable change of variables in the domain, we can suppose that $g^*\mathcal{F}$ is given by $g^{\#}\omega = dx_1$. In other words, if we write

$$g^*\omega = \sum_{\ell=1}^{n} b_\ell(\boldsymbol{x}) dx_\ell,$$

then

$$b_\ell(\boldsymbol{x}) = \sum_{i=1}^{n} a_i(g(\boldsymbol{x})) \frac{\partial g_i}{\partial x_\ell} \equiv 0 \quad \text{for } 2 \le \ell \le n. \tag{5}$$

(Remark that one could have $b_1(\boldsymbol{x}) \not\equiv 1$ as pull-back of foliations allows one to eliminate common factors.)

Consider the holomorphic one form

$$\eta = \sum_{i=1}^{n} a_i(g(\boldsymbol{x})) dx_i.$$

We will now interpret (5) in terms of the Koszul Complex obtained by wedging with η:

$$0 \longrightarrow \mathcal{O}_{\mathbb{C}^n,\mathbf{0}} \xrightarrow{\wedge\eta} \Omega^1_{\mathbb{C}^n,\mathbf{0}} \xrightarrow{\wedge\eta} \Omega^2_{\mathbb{C}^n,\mathbf{0}} \xrightarrow{\wedge\eta} \cdots \xrightarrow{\wedge\eta} \Omega^{n-2}_{\mathbb{C}^n,\mathbf{0}} \xrightarrow{\wedge\eta} \Omega^{n-1}_{\mathbb{C}^n,\mathbf{0}} \xrightarrow{\wedge\eta} \Omega^{n}_{\mathbb{C}^n,\mathbf{0}} \longrightarrow 0.$$

Since g is a finite map, the $a_i(g(\boldsymbol{x}))$ for $1 \le i \le n$ also form a regular sequence in $\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}$. Therefore, the homology of the Koszul Complex vanishes in the second to last place (at the term $\Omega^{n-1}_{\mathbb{C}^n,\mathbf{0}}$). See, for example, [4, Theorem A2.49].

For each $2 \le \ell \le n$ consider $\tau_\ell \in \Omega^{n-1}_{\mathbb{C}^n,\mathbf{0}}$ given by

$$\tau_\ell := \sum_{i=1}^{n} (-1)^{n-i} \frac{\partial g_i}{\partial x_\ell} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n. \tag{6}$$

Here, the hat on $\widehat{dx_i}$ denotes that the dx_i has been omitted from the monomial. Condition (5) implies that

$$\tau_\ell \wedge \eta = 0.$$

As the homology of the Koszul Complex vanishes in the second to last place, there exists $\alpha_\ell \in \Omega^{n-2}_{\mathbb{C}^n,\mathbf{0}}$ with

$$\alpha_\ell \wedge \eta = \tau_\ell.$$

Let us write

$$\alpha_\ell = \sum_{1 \le j < k \le n} c_\ell^{j,k}(\boldsymbol{x}) dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge \widehat{dx_k} \wedge \cdots \wedge dx_n,$$

where $c_\ell^{j,k} \in \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}$.

Computing $\alpha_\ell \wedge \eta$ and comparing the coefficients with (6) we find that

(7)

$$\frac{\partial g_1}{\partial x_\ell} = \sum_{k=2}^{n} (-1)^{n-k} c_\ell^{1,k}(\boldsymbol{x}) a_k(g(\boldsymbol{x})),$$

$$\frac{\partial g_i}{\partial x_\ell} = \sum_{j=1}^{i-1} (-1)^{n-1-j} c_\ell^{j,i}(\boldsymbol{x}) a_j(g(\boldsymbol{x})) + \sum_{k=i+1}^{n} (-1)^{n-k} c_\ell^{i,k}(\boldsymbol{x}) a_k(g(\boldsymbol{x})) \quad \text{for } 2 \leq i \leq n-1, \text{ and}$$

$$\frac{\partial g_n}{\partial x_\ell} = \sum_{j=1}^{n-1} (-1)^{n-1-j} c_\ell^{j,n}(\boldsymbol{x}) a_j(g(\boldsymbol{x})).$$

All we will need from (7) is that for each $2 \leq \ell \leq n$ and $1 \leq i \leq n$ there exist some $e_\ell^{i,m} \in \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}$ for $1 \leq m \leq n$ such that

$$\frac{\partial g_i}{\partial x_\ell} = \sum_{m=1}^{n} e_\ell^{i,m}(\boldsymbol{x}) a_m(g(\boldsymbol{x})). \tag{8}$$

We will obtain a contradiction from (8) and the naive idea behind it will be that the left hand side of (8) should vanish to lower order than the right hand side because of the partial derivative. We will now make this idea rigorous.

Because g is a finite map we have that $g_m(0, x_2, \ldots, x_n) \not\equiv 0$ for all but at most one coordinate m. Let $m_1, \ldots, m_p$ be precisely the coordinates for which this holds (either $p = n-1$ or $p = n$). For each $1 \leq q \leq p$ we can write

$$g_{m_q}(\boldsymbol{x}) = x_1 g_{m_q}^1(\boldsymbol{x}) + g_{m_q}^2(x_2, \ldots, x_m)$$

with $g_{m_q}^1 \in \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}$ and $g_{m_q}^2 \in \mathcal{O}_{\mathbb{C}^{n-1},\mathbf{0}}$ such that $g_{m_q}^2(x_2, \ldots, x_m) \not\equiv 0$. Looking at the power series expansion for each $g_{m_q}^2(x_2, \ldots, x_m)$ there is some smallest total degree d of all monomials appearing over all $1 \leq q \leq p$. Let $x_2^{d_2} \cdots x_n^{d_n}$ be one such monomial of this total degree. It may occur for multiple choices of q and, without loss of generality, we can assume it occurs for $q = 1$. Because the mapping $(x_1, \ldots, x_n) \mapsto (x_1^N, x_2, \ldots, x_n)$ preserves the foliation given by $dx_1 = 0$ we can replace g with $\widehat{g}(x_1, \ldots, x_n) := g(x_1^N, x_2, \ldots, x_n)$ and all of the above discussion also holds for $\widehat{g}$. In particular, we can choose $N > d$ so that every component of $\widehat{g}$ vanishes to order at least d and the component $\widehat{g}_{m_1}(x_1, \ldots, x_n)$ still has the term $x_2^{d_2} \cdots x_n^{d_n}$ appearing in its power series.

Let x_r be a coordinate appearing with positive exponent in the monomial $x_2^{d_2} \cdots x_n^{d_n}$. Then, for $i = m_1$ and $\ell = r$ the left hand side of (8) vanishes to order $d-1$ and the right hand side vanishes to order at least d. This gives the contradiction. □

Proposition 4.3. *Let $U, W \subset \mathbb{C}^n$ be neighborhoods of $\mathbf{0}$ and suppose $g : (U, \mathbf{0}) \to (W, \mathbf{0})$ is a finite holomorphic map, $X \subset W$ is an irreducible hypersurface that is singular at $\mathbf{0}$ and has singular locus X_{sing} of positive dimension (passing through the origin), and $g^{-1}(X)$ is smooth.*

Then there exists open sets $U' \subset U$ and $W' \subset W$ with $g(U') = W'$ and a hyperplane H in $\mathbb{C}^n$ such that:

1. *$X \cap H$ is a singular hypersurface within $H \cap W'$,*
2. *$g^{-1}(H)$ is smooth within U', and*
3. *$g^{-1}(X \cap H)$ is a smooth hypersurface in $g^{-1}(H) \cap U'$.*

Proof. There is a holomorphic $\psi : W \to \mathbb{C}$ such that

$$X = \{\boldsymbol{y} \in W \ : \ \psi(\boldsymbol{y}) = 0\} \quad \text{and} \quad d\psi \not\equiv 0 \text{ on } X. \tag{9}$$

Let $S \subset X$ be an irreducible component of X_{sing} of maximal dimension

$$1 \leq k := \dim(S) \leq n-2.$$

Let V be an irreducible component of $g^{-1}(S)$. Since g is a finite map $\dim(V) = k$ and $\dim(g^{-1}(S_{\text{sing}})) < k$. Therefore, $V_{\text{reg}} \setminus g^{-1}(S_{\text{sing}})$ is an open dense subset of V. We can therefore find a point $p_0 \in V_{\text{reg}} \setminus g^{-1}(S_{\text{sing}})$ and an open neighborhood U' of p_0 in $\mathbb{C}^n$ such that $U' \cap V$ is contained in V_{reg} and $W' := g(U')$ satisfies that $W' \cap X_{\text{sing}} \subset S_{\text{reg}}$. Remark that W' is an open neighborhood of $g(p_0)$ because g is also an open mapping [9, Theorem 15.1.6].

The remainder of the proof will take place within the neighborhoods U' and W'. By abuse of notation we will now refer to g as the surjective finite holomorphic mapping $g : U' \to W'$, refer to X as a singular hypersurface in W' whose singular locus $X_{\text{sing}} = S$ is a smooth manifold of dimension k in W', and refer to V as a smooth manifold of dimension k in U' satisfying $g(V) \subset S$.

For any non-constant holomorphic curve $\gamma : \mathbb{D} \to U'$ we have that $Dg(\gamma(t))\gamma'(t) \neq \mathbf{0}$ for generic $t \in \mathbb{D}$, since otherwise $g(\gamma(\mathbb{D}))$ would be a single point. Since V is a smooth manifold of dimension $k > 0$ we can suppose $\gamma(\mathbb{D}) \subset V$ and that allows us to find a point $p_1 \in V$ and a tangent vector $\boldsymbol{v}_1 \in T_{p_1}V$ such that $\boldsymbol{w}_1 := Dg(p_1)\boldsymbol{v}_1 \in T_{g(p_1)}S$ is non-zero.

Let H_0 be a hyperplane through $g(p_1)$ that is transverse to $\boldsymbol{w}_1$. By definition, this implies that H_0 is transverse to S and to the map g. Let L be the line through $g(p_1)$ in the direction of $\boldsymbol{w}_1$. Denote by H_t the one-parameter family of hyperplanes parallel to H_0, which we can parameterize by the intersection point $t \in H_t \cap L$. As transversality is an open condition, we can reduce the neighborhoods U' and $W' = g(U')$ so that each member of H_t is transverse to both S and g. Without loss of generality, we can suppose the parameter t varies over the unit disc $\mathbb{D}$.

We will now select the desired hyperplane H satisfying Properties (1-3) from this family H_t. Since each H_t is transverse to g we have that $g^{-1}(H_t)$ is a smooth submanifold of U' so that (2) holds for all choices of t.

We will now show that Property (1) holds for all parameters t outside of a zero measure subset of $\mathbb{D}$. Let $\tau : W' \to \mathbb{D}$ be the function that assigns to each $\boldsymbol{y}$ the value of the parameter t such that $\boldsymbol{y} \in H_t$. Sard's Theorem gives that the critical values of $\tau|_{X_{\text{reg}}}$ form a measure zero set $\mathbb{D}_0 \subset \mathbb{D}$. For any $t \in \mathbb{D} \setminus \mathbb{D}_0$ we have that H_t is transversal to X_{reg}. At each $\boldsymbol{y} \in X_{\text{reg}}$ we have

$$\text{Ker}(d\psi(\boldsymbol{y})) = T_{\boldsymbol{y}}X_{\text{reg}}.$$

This implies that if $t \in \mathbb{D} \setminus \mathbb{D}_0$ then $d(\psi|_{H_t}) \neq 0$ at every point of $X_{\text{reg}} \cap H_t$.

Meanwhile, $d\psi$ vanishes on X_{sing}. Since each H_t intersects $S = X_{\text{sing}}$ transversally for each parameter t and since X_{sing} has codimension at least two in $\mathbb{C}^n$, we have that $H_t \cap X_{\text{sing}}$ has codimension at least two in H_t. We conclude that for $t \in \mathbb{D} \setminus \mathbb{D}_0$ we have

$$X \cap H_t := \{\boldsymbol{y} \in H_t \ : \ \psi|_{H_t}(\boldsymbol{y}) = 0\}$$

with $d(\psi|_{H_t})$ vanishing only on the non-empty subset $H_t \cap S$ that has codimension two or higher within H_t. Therefore, if $t \in \mathbb{D} \setminus \mathbb{D}_0$ the hyperplane H_t satisfies Property (1).

We will now show that Property (3) holds for all parameters t outside of another zero measure subset of $\mathbb{D}$. Because $g^{-1}(H_t)$ is smooth for each $t \in \mathbb{D}$ we have that the family $g^{-1}(H_t)$ forms a smooth codimension one holomorphic foliation of U'. Parameterizing the leaves using a local transversal in U' we can again use Sard's Theorem to show that all but a zero measure subset of the leaves $g^{-1}(H_t)$ are transverse to the smooth manifold $g^{-1}(X)$. Such leaves satisfy that $g^{-1}(X) \cap g^{-1}(H_t)$ is smooth.

We therefore find the desired hyperplane H by choosing t outside of the union of the two measure zero subsets of $\mathbb{D}$ that were described in the previous three paragraphs. □

Proof of Theorem A. The proof will be by induction on the dimension n of the ambient space $\mathbb{C}^n$. When $n = 2$, the singular locus of X must be of dimension 0 and Theorem A follows from Proposition 4.1.

Now suppose that the statement of Theorem A holds when the ambient space has dimension $n \geq 2$ in order to prove it for when the ambient space has dimension $n + 1$. Let X be a germ of a hypersurface at the origin in $\mathbb{C}^{n+1}$ that is singular at $\mathbf{0}$. If $\mathbf{0}$ is an isolated singularity for X, then the result again follows from Proposition 4.1. Meanwhile, if X is reducible, then the preimage under any finite map is again reducible, and hence singular.

Otherwise, we suppose for contradiction that there does exist a finite holomorphic mapping $g : (U, \mathbf{0}) \to (W, \mathbf{0})$ with $U, W \subset \mathbb{C}^{n+1}$ open and an irreducible hypersurface X with positive dimensional singular locus X_{sing} passing through $\mathbf{0}$ such that $g^{-1}(X)$ is smooth in U. In this case, Proposition 4.3 will allow us to reduce the dimension of the ambient space by one, in order to contradict the induction hypothesis.

More specifically, Proposition 4.3 gives us open sets $U' \subset U$ and $W' \subset W$ such that $g(U') = W'$ and a hyperplane H such that $\widetilde{X} := X \cap H$ is again singular in $H \cap W'$ but with $g^{-1}(H)$ smooth in U' and $g^{-1}(\widetilde{X})$ smooth within $g^{-1}(H) \cap U'$. Choosing local coordinates on $g^{-1}(H)$ and H, respectively, we obtain a new germ of a finite holomorphic mapping $\widetilde{g} : (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}^n, \mathbf{0})$ such that the preimage of a singular hypersurface $\widetilde{X}$ is smooth. This violates the induction hypothesis.

We conclude that the statement of Theorem A holds for any dimension ambient space. □

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