A degenerate Newton's Map in two complex variables: linking with currents

Roland K. W. Roeder

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Abstract

Little is known about the global structure of the basins of attraction of Newton's method in two or more complex variables. We make the first steps by focusing on the specific Newton mapping to solve for the common roots of $P(x, y) = x(1 - x)$ and $Q(x, y) = y^2 + Bxy - y$.

There are invariant circles S_0 and S_1 within the lines $x = 0$ and $x = 1$ which are superattracting in the x-direction and hyperbolically repelling within the vertical line. We show that S_0 and S_1 have local super-stable manifolds, which when pulled back under iterates of N form global super-stable spaces W_0 and W_1. By blowing-up the points of indeterminacy p and q of N and all of their inverse images under N we prove that W_0 and W_1 are real-analytic varieties.

We define linking between closed 1-cycles in W_i ($i = 0, 1$) and an appropriate closed 2 current providing a homomorphism $lk : H_1(W_i, \mathbb{Z}) \to \mathbb{Q}$. If W_i intersects the critical value locus of N, this homomorphism has dense image, proving that $H_1(W_i, \mathbb{Z})$ is infinitely generated. Using the Mayer-Vietoris exact sequence and an algebraic trick, we show that the same is true for the closures of the basins of the roots $\overline{W(r_i)}$.

Key Words. Complex dynamics, Newton's Method, homology, linking numbers, invariant currents.
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Newton's method is one of the fundamental algorithms of mathematics, so it is evidently important to understand its dynamics, particularly the structure of the basins of attraction of the roots. Even in one dimension, the topology of these basins can be complicated and there has been a good deal of research on this subject. In higher dimensions, next to nothing is known about the topology of the basins. In this paper we make the first steps at understanding their topology in two complex variables.

We focus on a specific system: the Newton's Method used to solve for the common roots of $P(x, y) = x(1 - x)$ and $Q(x, y) = y^2 + Bxy - y$. While this is one specific and relatively simple system, we believe that some of the techniques developed in this paper can be used to study more general systems.

Dynamical systems $g : \mathbb{C}^n \to \mathbb{C}^n$ are often classified in terms of: (1) The number of inverse images of a generic point by g, which is called the topological degree $d_t(g)$, and (2) Whether g has points of indeterminacy.

Mappings $g : \mathbb{P}^n \to \mathbb{P}^n$ with $d_t(g) > 1$, but without points of indeterminacy, have been studied by Bonifant and Dabija [10], Bonifant and Fornæss [11], Briend [14], Briend and Duval [15], Dinh and Sibony [20], Fornaess and Sibony [25, 27, 26], Hubbard and Papadopol [38], Jonnson [39], and Ueda [52].

Meanwhile, birational maps $g : \mathbb{P}^n \rightsquigarrow \mathbb{P}^n$ (rational maps with rational inverse) are examples of systems with points of indeterminacy, but with $d_t(g) = 1$. The famous Henon mappings $H : \mathbb{P}^2 \rightsquigarrow \mathbb{P}^2$ fall into this class. Such systems have been studied extensively by Bedford and Smillie [3, 4, 5, 6, 7, 9, 8], Bedford, Lyubich and Smillie [2], Devaney and Nitecki [18], Diller [19], Dinh

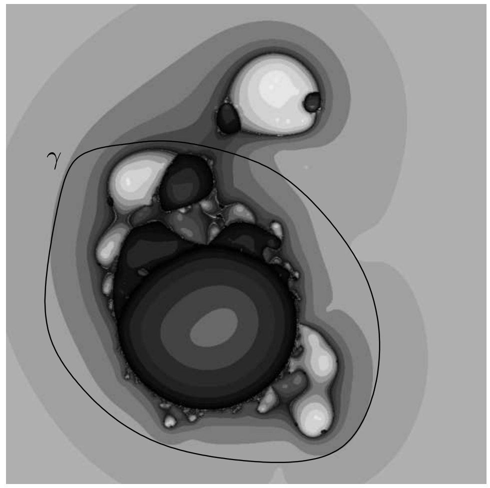

Figure 1: Is the loop γ trivial in the homology of it's basin, within $\mathbb{C}^2$?

and Sibony [21], Dujardin [22], Favre and Jonsson [23], Fornaess [24], Guedj [29], and Hubbard and Oberste-Vorth [34, 35, 36].

Not nearly as much is known about mappings $g : \mathbb{P}^n \rightsquigarrow \mathbb{P}^n$ with topological degree $d_t(g) > 1$ and with points of indeterminacy. The work of Russakovskii and Shiffman [46] considers a measure that is obtained by choosing a generic point, taking the each of its inverse images under $g^{\circ n}$ and giving them all equal weight in order to obtain a probability measure μ_n. Under appropriate conditions on g they show that the measures μ_n converge to a measure μ that is independent of the initial point. In [37], the authors present a proof by A. Douady that μ does not charge points in the line at infinity, a result not obtained in [46]. In a recent paper, Guedj [30] shows that if the topological degree $d_t(g)$ is sufficiently large, then μ does not charge the points of indeterminacy of g and does not charge any pluripolar set. He then uses these facts to establish ergodic properties of μ.

Many of the papers considering mappings with both $d_t(g) > 1$ and points of indeterminacy consider ergodic properties, invariant measures, and invariant currents, focusing less on topological properties. One paper that considers some topological properties is [37], by John Hubbard and Peter Papadopol, who consider the dynamics of the Newton Map N to solve for the zeros of two quadratic equations P and Q in two complex variables. The basins of attraction for this system show interesting topology: for example, when drawing intersections of a 1 complex-dimensional slice with the basins of attraction one often finds "bubbles" like the ones shown in Figure 1. It is natural to ask if a loop, such as the one labeled γ in the figure, corresponds to a non-trivial loop in the homology of its basin of attraction. Clearly $[\gamma]$ is non-trivial in the basin intersected with this slice, but it is much more difficult to determine if $[\gamma]$ is non-trivial when considered within the entire 2 complex-dimensional basin, which may reconnect in unusual ways outside of this slice.

Questions about the first homology of the basins are not answered by Hubbard and Papadopol. Using general principles they show that the basin of attraction for each of the four roots is path connected and, by resolving points of indeterminacy of N, they show that each basin is a Stein manifold. But, instead of addressing further questions about the topology of the individual basins, they focus on creating a compactification with manageable topology on which N is well-defined. Most questions about the topology and the detailed structure of these basins of attraction within their compactification of $\mathbb{C}^2$ remain as mysteries.

In order to develop tools to answer some detailed questions about the topology of basins of attraction for Newton maps $N : \mathbb{C}^2 \to \mathbb{C}^2$ to solve two quadratic equations $P(x, y) = 0$ and

$Q(x,y)=0$, we restrict our attention to the degenerate case when the four roots lie on two parallel lines. Normalizing, we study the Newton Map N for $P(x,y)=x(1-x)$ and $Q(x,y)=y^2+Bxy-y$. In this case, the first component of $N(x,y)$ depends only on x, while the second component depends on both x and y. Systems of this form are commonly referred to as *skew products* in the literature [1, 32, 39, 47, 50, 51] and they are often used as "test cases" when developing new techniques. While we rely upon the fact that N becomes a skew product in this degenerate case, we hope that some of the techniques developed here can eventually be adapted to non-degenerate cases.

Our approach is the following:

We compactify $\mathbb{C}^2$ obtaining a rational map $N : \mathbb{P}\times\mathbb{P} \to \mathbb{P}\times\mathbb{P}$ with four points of indeterminacy at $p=\left(\frac{1}{B},0\right)$, $q=\left(\frac{1}{2-B},\frac{1-B}{2-B}\right)$, (∞,∞), and $\left(\infty,\frac{B}{2}\right)$. There are three invariant subspaces $X_l := \{(x,y) : \mathrm{Re}(x) < 1/2 \text{ and } x \neq \infty\}$, $X_{1/2} := \{(x,y) : \mathrm{Re}(x) = 1/2 \text{ or } x=\infty\}$ and $X_r := \{(x,y) \in\ : \mathrm{Re}(x) > 1/2 \text{ and } x \neq \infty\}$. The common roots of P and Q are $r_1=(0,0)$, $r_2=(0,1)$, $r_3=(1,0)$, and $r_4=(1,1-B)$ with the basins of attraction $W(r_1)$ and $W(r_2)$ in X_l and the basins of attraction $W(r_3)$ and $W(r_4)$ in X_r. By restricting to parameters $B\in\Omega=\{B : |1-B|>1\}$ we can assume that both p and q are in X_l.

We will prove that within X_l and X_r there are "superstable separatrices" W_0 and W_1 consisting of the points that are attracted to invariant circles within the lines $x=0$ and $x=1$ respectively. By resolving the points of indeterminacy of N^k in X_l we obtain a modified space X_l^∞ in which all iterates of N are well-defined and in which W_0 is a real-analytic variety that provides a nice boundary between $W(r_1)$ and $W(r_2)$. Since $B\in\Omega$ there is no such problem in X_r: all iterates of N are already well defined on X_r and W_1 is a real-analytic variety in X_r.

In this paper we will study the topology of W_0 and W_1 in detail and we will use a Mayer-Vietoris decomposition to relate their homology to the homology of the basins of attraction of the four roots: $W(r_1)$, $W(r_2)$, $W(r_3)$, and $W(r_4)$ and the homology of X_l^∞ and X_r.

The major emphasis of this paper is to show that loops in W_0 and W_1 that are generated by intersections of W_0 or W_1 with the critical value locus C are actually homologically non-trivial. The essential difficulty is to choose a notion of linking that is well defined within the space X_l^∞, which is very topologically complicated as a result of the blow-ups.

We define linking between closed 1-cycles in W_i $(i=0,1)$ and an appropriate closed 2 current providing a homomorphism $lk : H_1(W_i,\mathbb{Z})\to\mathbb{Q}$. If W_i intersects the critical value locus of N, this homomorphism has dense image, proving that $H_1(W_i,\mathbb{Z})$ is infinitely generated. Using the Mayer-Vietoris exact sequence and an algebraic trick, we show that the same is true for the closures of the basins of the roots $\overline{W(r_i)}$.

Our work culminates to prove:

Theorem 0.1. *Let $\overline{W(r_1)}$ and $\overline{W(r_2)}$ be the closures in X_l^∞ of the basins of attraction of $r_1=(0,0)$ and $r_2=(0,1)$ under iteration of N and let $\overline{W(r_3)}$ and $\overline{W(r_4)}$ be the closures in X_r of the basins of attraction of $r_3=(1,0)$ and $r_4=(1,1-B)$.*

- *$H_1\left(\overline{W(r_1)}\right)$ and $H_1\left(\overline{W(r_2)}\right)$ are infinitely generated for every $B\in\Omega$.*
- *For $B\in\Omega$, if W_1 intersects the critical value parabola $C(x,y)=0$, then both $H_1\left(\overline{W(r_3)}\right)$ and $H_1\left(\overline{W(r_4)}\right)$ are infinitely generated, otherwise $H_1\left(\overline{W(r_3)}\right)$ and $H_1\left(\overline{W(r_4)}\right)$ are trivial.*

For $B\in\Omega_{\mathrm{reg}}$, the set of parameters for which the separatrices are genuine manifolds, the basins of the four roots and their closures in X_l^∞ and X_r have the some homotopy type. Hence:

Corollary 0.2. *For $B\in\Omega_{\mathrm{reg}}$, Theorem 0.1 remains true when replacing the closures of each of the basins with the basins themselves.*

1 Basic properties of N

In the first part of this section we summarize the basic results from [37].

Given two vector spaces V and W of the same dimension and a differentiable mapping $F : V \to W$, the associated Newton map $N_F : V \to V$ is given by the formula

$$N_F(\mathbf{x}) = \mathbf{x} - [DF(\mathbf{x})]^{-1}(F(\mathbf{x})). \tag{1}$$

If $DF(r_i)$ is invertible for each root r_i of F, then the roots of F correspond to super attracting fixed points of N_F. Conversely, every fixed point of N_F is a root of F. Since each fixed point r_i of N_F is super-attracting, there is some neighborhood U_i of r_i for which each initial guess $\mathbf{x_0} \in U_i$ will converge to r_i. An explicit lower bound on the size of U_i is given by Kantorovich's Theorem [40].

Proposition 1.1. **(Transformation rules)** *If $A : V \to V$ is affine, and invertible, and if $L : W \to W$ is linear and invertible, then:*

$$N_{L\circ F\circ A} = A^{-1} \circ N_F \circ A. \tag{2}$$

The proof is a careful use of the chain rule, see [37], Lemma 1.1.1.

Proposition 1.2. *Newton's Method to find the intersection of two quadratics depends only on the intersection points and not on the choice of curves.*

For the proof, see Corollary 1.5.2 from [37].

In this paper, we normalize so that the roots are at $\binom{0}{0}$, $\binom{1}{0}$, $\binom{0}{1}$, and $\binom{\alpha}{\beta}$. If we let $A = \frac{1-\alpha}{\beta}$ and $B = \frac{1-\beta}{\alpha}$, then $F\binom{x}{y} = \binom{x^2+Axy-x}{y^2+Bxy-y} = \binom{P(x,y)}{Q(x,y)}$ has these roots and the corresponding Newton Map is given by:

$$\begin{aligned} N_F\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} x \\ y \end{pmatrix} - \begin{bmatrix} 2x+Ay-1 & Ax \\ By & 2y+Bx-1 \end{bmatrix}^{-1} \begin{pmatrix} x^2+Axy-x \\ y^2+Bxy-y \end{pmatrix} \\ &= \frac{1}{\Delta}\begin{pmatrix} x(Bx^2+2xy+Ay^2-x-Ay) \\ y(Bx^2+2xy+Ay^2-Bx-y) \end{pmatrix}, \end{aligned} \tag{3}$$

where $\Delta = 2Bx^2 + 4xy + 2Ay^2 - (2+B)x - (2+A)y + 1$.

Proposition 1.3. *The critical value locus of N_F is the union of the two parabolas that go through the four roots of F.*

Proposition 1.4. *The Newton Map has topological degree 4.*

See [37] for a proof of Propositions 1.3 and 1.4.

It is a classical result that the dynamics of the Newton map $N(z)$ to solve for the roots of any quadratic polynomial $p(z)$ is conjugate to the map $z \mapsto z^2$. For the latter, the unit circle $\mathbb{S}^1$ forms the boundary between the basin of attraction of 0 and of ∞. If ϕ is the map conjugating $N(z)$ to $z \mapsto z^2$, then $\phi^{-1}(\mathbb{S}^1)$ is the line in $\mathbb{C}$ that is equidistant from the roots of p. This line forms the boundary between the basin of the two roots of $p(z)$ and the dynamics on this line (once you add a point at infinity) are conjugate to angle doubling on the unit circle.

Proposition 1.5. (**Invariant lines and invariant circles**) *The lines joining the roots of F are invariant under Newton Map N_F and on these lines N_F induces the dynamics of the one dimensional Newton Map to find the roots of a quadratic polynomial.*

Within each line is an invariant "circle," corresponding to the points of equal distance from the two roots in that line.

(See Proposition 1.5.3 in [37])

Proof: Given any pair of roots of F, there is an affine mapping taking them to $\binom{0}{0}$ and $\binom{1}{0}$ and a third root to $\binom{0}{1}$ The new system is also normalized, but with the chosen pair of roots on the x-axis. Using Proposition 1.1, if the x-axis is invariant, then we will have shown that the line connecting the chosen pair of roots is also invariant. But this is easy to see because there is a factor of y in the second coordinate of Equation 3 for N_F.

The dynamics on the x-axis correspond to taking the first coordinate of N_F in Equation 3 with $y = 0$. One finds $x \mapsto \frac{x(Bx^2-x)}{2Bx^2-(2+B)x+1} = \frac{x^2}{2x-1}$. This is the Newton's Method to solve $x(1-x) = 0$. Using the transformation rules from Proposition 1.1 one can show that the same is true for any other invariant line. □

1.1 The degenerate case: $A = 0$

The Newton map to find the common zeros of $P(x, y) = x(1-x)$ and $Q(x, y) = y^2 + Bxy - y$ is:

$$N\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} x(Bx^2 + 2xy - x) \\ y(Bx^2 + 2xy - Bx - y) \end{pmatrix} = \begin{pmatrix} \frac{x^2}{2x-1} \\ \frac{y(Bx^2+2xy-Bx-y)}{(2x-1)(Bx+2y-1)} \end{pmatrix} \tag{4}$$

with

$$\Delta = 2Bx^2 + 4xy - (2+B)x - 2y + 1 = (2x-1)(Bx + 2y - 1).$$

The fixed points of N are the four common roots of P and Q: $r_1 = (0, 0), r_2 = (0, 1), r_3 = (1, 0)$, and $r_4 = (1, 1 - B)$.

The critical value locus is the union of the two parabolas going through the four roots. One of these coincides with $P(x, y) = x(1-x) = 0$, while the other is the non-degenerate parabola given by

$$C(x, y) = y^2 + Bxy + \frac{B^2}{4}x^2 - \frac{B^2}{4}x - y = 0. \tag{5}$$

We will often refer to the locus $C(x, y) = 0$ as *the critical value parabola* and denote it by C. Figure 2 depicts the curves $P(x, y) = 0$ and $Q(x, y) = 0$, the critical value parabola C, and the four roots, r_1, r_2, r_3, and r_4.

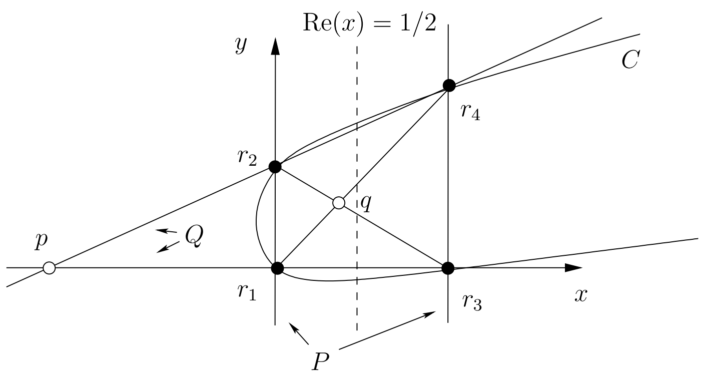

Figure 2: The degenerate case $A = 0$.

One can check directly from Equation 4 that N has topological degree 4, since every $x \neq 0, 1$ has two inverse images and the second component is an equation of degree two in y.

There are six invariant lines and, in this degenerate case, these lines have six points of intersection in $\mathbb{C}^2$. Four of these intersections correspond to the roots r_1, r_2, r_3, and r_4, while the remaining two correspond to points of indeterminacy. These are denoted p and q and are also shown in Figure 2.

The mapping governing the x coordinate is $x \mapsto \frac{x^2}{2x-1}$, which is itself the one variable Newton Map corresponding to the polynomial $x(x-1)$, with Julia set consisting of the line $\mathrm{Re}(x) = 1/2$. This simple dynamics in x is the main reason why the degenerate Newton map is much easier to understand than those considered in [37]: here all points in $\mathbb{C}^2$ with $\mathrm{Re}(x) < 1/2$ are super-attracted to the line $x = 0$ and all points with $\mathrm{Re}(x) > 1/2$ are super-attracted to the line $x = 1$. The vertical line at $x = m$ is mapped to the line at $x = m^2/(2m-1)$ by the second coordinate of (4), which is in fact a rational map of degree 2, except at those values of m where the numerator and the denominator in the second coordinate of (4) have a common factor. This occurs exactly when $x = 1/B, x = 1/(2-B)$, and $x = 1/2$. The first two correspond to the points of indeterminacy p and q.

There are three major invariant sets: $X_l := \{(x,y) : \mathrm{Re}(x) < 1/2 \text{ and } x \neq \infty\}$, $X_{1/2} := \{(x,y) : \mathrm{Re}(x) = 1/2 \text{ or } x = \infty\}$ and $X_r := \{(x,y) \in \ : \mathrm{Re}(x) > 1/2 \text{ and } x \neq \infty\}$. Figure 2 shows the case when both points of indeterminacy p and q are in X_l. The coordinates of p and q are $p = \left(\frac{1}{B}, 0\right)$ and $q = \left(\frac{1}{2-B}, \frac{1-B}{2-B}\right)$. It is easy to check that p and q either are both in X_l, both in the separator $X_{1/2}$, or both in X_r. Let $\Omega = \{B \in \mathbb{C} : |1-B| > 1\}$ so that if $B \in \Omega$ then both p and q are in X_l. Using the transformation Rules 1.1, one sees that systems with this restriction still represent every conjugacy class except for those corresponding to both $p, q \in X_{1/2}$.

Let S_0 and S_1 be the invariant circles in the fixed lines $x = 0$ and $x = 1$, respectively. Because the lines $x = 0$ and $x = 1$ are super-attracting in the x-direction, S_0 and S_1 are super-attracting in the x-direction, as well. In Section 3 we will show that these circles have local superstable manifolds W_0^{loc} and W_1^{loc}. Pulling W_0^{loc} and W_1^{loc} back under the Newton map we generate superstable spaces W_0 and W_1 that form the boundary between the basin $W(r_1)$ and $W(r_2)$ and between the basin $W(r_3)$ and $W(r_4)$, respectively. Figure 3 shows an illustration of these separatrices.

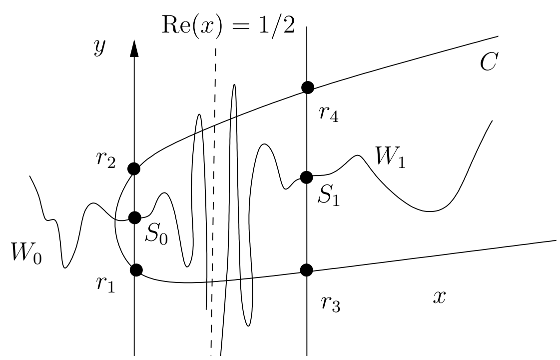

Figure 3: Superstable separatrices in the degenerate case, $A = 0$.

Proposition 1.6. (**Axis of symmetry**) *Let τ denote the vertical reflection about the line $Bx + 2y - 1 = 0$, that is: $\tau(x,y) = (x, 1 - Bx - y)$. Then, τ is a symmetry of N:*

$$\tau \circ N = N \circ \tau.$$

Furthermore, N maps this axis of symmetry to the line $y = \infty$.

Proof: The map τ is affine and interchanges r_1 with r_2 and r_3 with r_4. Let $F\binom{x}{y} = \binom{P(x,y)}{Q(x,y)}$ so that r_1, r_2, r_3, and r_4 are the roots of F. By Proposition 1.2, the Newton map $N_{F \circ \tau}$ for finding the roots of $F \circ \tau$ is the same as N_F, since they have the same roots. By the transformation rules of the Newton Map under affine coordinate changes, $N_{F\circ\tau} = \tau^{-1} \circ N \circ \tau$. Hence:

$$\tau \circ N = \tau \circ N_{F\circ\tau} = \tau \circ \tau^{-1} \circ N \circ \tau = N \circ \tau$$

The axis $Bx + 2y - 1 = 0$ is mapped to the line $y = \infty$ due to the factor $Bx + 2y - 1 = 0$ in the denominator the second component of N. □

2 Computer exploration of N

In this section we show computer images of the basins of attraction for the four common zeros of P and Q for the parameters $B = 0.7857 + 1.1161i$, and $B = -0.7902 + 1.7232i$. All of the computer images displayed in this paper were made using the wonderful program FractalAsm [44].

The separatrices W_0 and W_1 are clearly visible in these images, forming the smooth boundary between $W(r_1)$ and $W(r_2)$ and between $W(r_3)$ and $W(r_4)$, respectively. The boundary between $W(r_1) \cup W(r_2)$ and $W(r_3) \cup W(r_4)$, when visible, corresponds to points (x, y) with $\mathrm{Re}(x) = 1/2$.

Case 1: $B = 0.7857 + 1.1161i$

The first kind of slice that we consider is given by the critical value parabola C, which is parameterized by a single complex variable, the offset from the axis of C. Figure 4 shows part of this slice on the left while the image on the right shows a zoomed in view corresponding to the region enclosed in the small rectangle from the image on the left. The center of the symmetry τ is the center of the image on the left of Figure 4, but is outside of the image on the right.

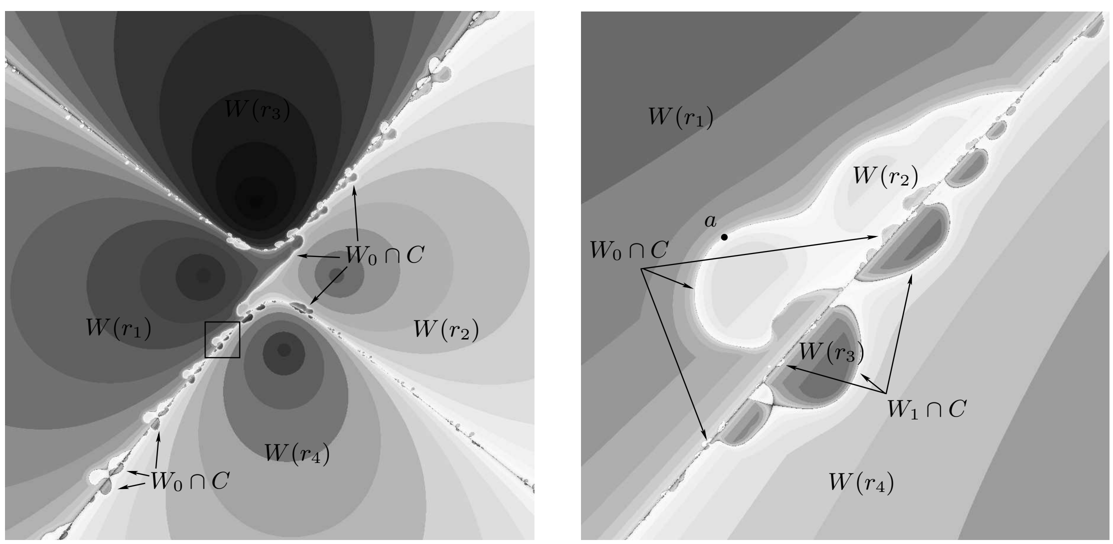

Figure 4: The critical value parabola C for $B = 0.7857 + 1.1161i$. The boundary between $W(r_1)$ and $W(r_2)$ is $W_0 \cap C$ and the boundary between the $W(r_3)$ and $W(r_4)$ is $W_1 \cap C$. The image on the right is a zoomed in view of the boxed region in the image on the left.

Figure 5 shows the vertical line $x = a$, where a is labeled in Figure 4, as well as the vertical lines through three inverse images of a. We have placed the center of the symmetry τ at the center of these images so that reflection across this point perfectly interchanges the basins.

Notice how the first inverse image of $x = a$ is divided into two regions that are in $W(r_1)$ and two regions in $W(r_2)$. This is because we chose a on the superstable separatrix W_0. The lines at

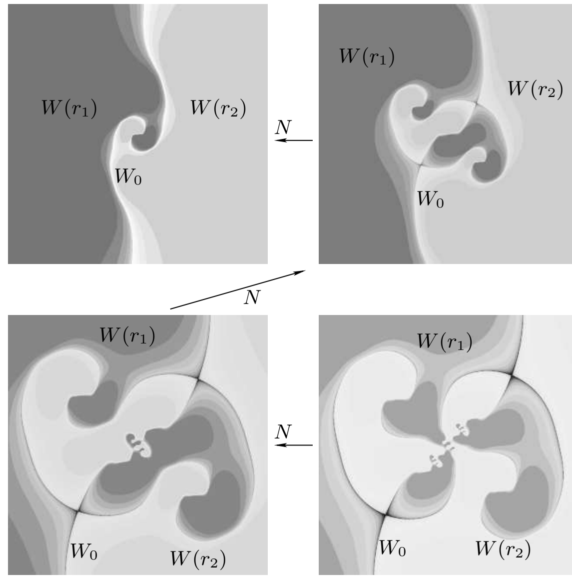

Figure 5: Vertical line through point a from Figure 4 and three inverse images of this line. The boundary between $W(r_1)$ and $W(r_2)$ is the intersection of W_0 with these vertical lines. Notice that there are many closed loops in W_0 within these vertical lines. The center of the symmetry τ is at the center of these images.

second and third inverse images of $x = a$ are divided into three regions in $W(r_1)$ and in $W(r_2)$ and five regions in $W(r_1)$ and in $W(r_2)$, respectively.

Case 2: $B = -0.7902 + 1.7232i$

Figure 6 shows the intersections of the basins of attraction for $W(r_1)$, $W(r_2)$, $W(r_3)$, and $W(r_4)$ with the critical value parabola C. Notice that there are clearly intersections of the superstable separatrix W_0 with C forming the visible boundary between $W(r_1)$ and $W(r_2)$. However, we see no boundaries between $W(r_3)$ and $W(r_4)$, indicating that W_1 might not intersect C. All of the further zoom-ins that we have done show no evidence of intersections between W_1 and C. We cannot prove that there are values of B for which $W_1 \cap C = \emptyset$, however it seems likely, based on computer experiments.

As for the previous value of B, the vertical lines above points of intersection of W_0 with C and the vertical lines mapped to them by N contain many interestingly loops that are in W_0.

3 Superstable separatrices W_0 and W_1.

The invariant circle S_0 is the set of points in the line $x = 0$ equidistant from r_1 and r_2 and the invariant circle S_1 is the set of points in the line $x = 1$ equidistant from r_3 and r_4.

Proposition 3.1. *The invariant circles S_0 and S_1 have multiplier 0 in the x-direction and they have multiplier 2 within the vertical line in the direction normal to the circle.*

Proof: The vertical lines $x = 0$ and $x = 1$ are superattracting in the x-direction, hence the circles S_0 and S_1 are as well. Within these vertical lines, N is the Newton's method for the quadratic polynomial with roots r_1 and r_2 (or r_3 and r_4), so the invariant circle is repelling in this line with multiplier 2. □

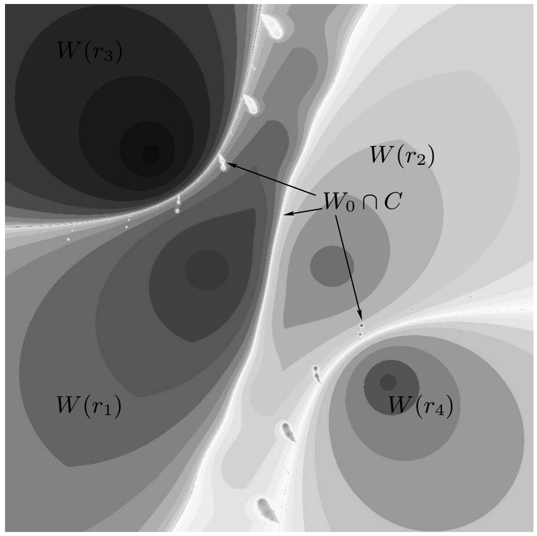

Figure 6: Critical value parabola C for $B = -0.7902 + 1.7232i$. The boundary between the $W(r_1)$ and $W(r_2)$ is $W_0 \cap C$. We see no boundaries between $W(r_3)$ and $W(r_4)$, indicating that W_1 might not intersect C.

Proposition 3.2. *The invariant circles S_0 and S_1 have local superstable manifolds W_0^{loc} and W_1^{loc}. More specifically, there are neighborhoods $U_0, U_1 \subset \mathbb{C}$ of $x = 0$ and $x = 1$ and subsets $W_0^{loc} \subset X_l$, $W_1^{loc} \subset X_r$ so that:*

- $N(W_0^{loc}) \subset W_0^{loc}$ *and* $N(W_1^{loc}) \subset W_1^{loc}$
- W_0^{loc} *is the image of some* $\Phi_0 : U_0 \times S_0 \to X_l$ *which is analytic in the first coordinate and quasiconformal in the second.*
- W_1^{loc} *is the image of some* $\Phi_1 : U_1 \times S_1 \to X_r$ *which is analytic in the first coordinate and quasiconformal in the second.*

We use a technique due to John Hubbard and Sebastien Krief which allows us to use the theory of holomorphic motions and the λ-Lemma of Mañe, Sad, and Sullivan [41], instead of the more standard graph transformation approach. A somewhat different stable manifold theorem for the invariant circles in the non-degenerate case ($A \neq 0$) is also proved using this technique in [37]. While points in the manifolds obtained in our proof are genuinely attracted to the circles S_0 and S_1, the situation in [37] is much more complicated, with dense sets of points that are not attracted to the invariant circles.

Proof: To simplify computations we will make the change of variables $z(x) = \frac{x}{x-1}$ and $w(y) = \frac{y}{y-1}$ which conjugates the first coordinate of N to $z \mapsto z^2$ and places the invariant circle S_0 at $\{z = 0, |w| = 1\}$. In the new coordinates (z, w), the Newton map becomes:

$$N\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} z^2 \\ \frac{w^2+(Bw-Bw^2)z-w^2z^2}{1+(B-Bw)z+(Bw^2+B-1-2Bw)z^2} \end{pmatrix} \tag{6}$$

and the critical value locus of N in these coordinates is the image of C under the change of variables, which we denote by C'.

Let

$$\Delta_{\epsilon,\delta} = \{(z,w) \in X_l : |z| < \epsilon \text{ and } 1-\delta < |w| < 1+\delta\}$$

so that $\Delta_{\epsilon,\delta}$ is an open neighborhood of S_0. The boundary of $\Delta_{\epsilon,\delta}$ consists of the vertical boundary $\partial^V \Delta_{\epsilon,\delta} = \{|z| = \epsilon\}$ and the horizontal boundary $\partial^H \Delta_{\epsilon,\delta} = \{|w| = 1 \pm \delta\}$.

We must choose ϵ and δ so that:

1. $\Delta_{\epsilon,\delta}$ is disjoint from the critical value locus C', and
2. $N(\partial^H \Delta_{\epsilon,\delta})$ is entirely outside of $\Delta_{\epsilon,\delta}$ and $N(\partial^V \Delta_{\epsilon,\delta})$ is entirely inside of $|z| < \epsilon$.

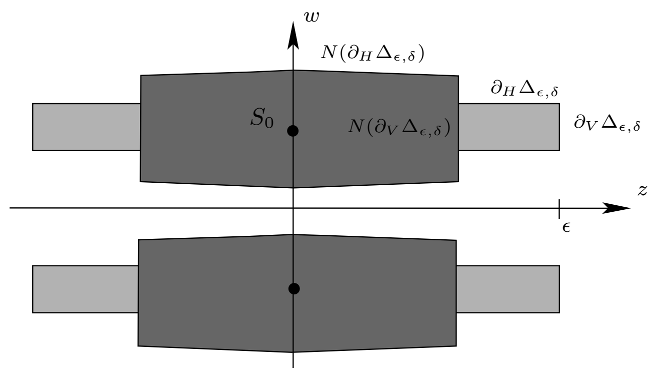

Figure 7: Here $\Delta_{\epsilon,\delta}$ is shown in light grey and $N(\Delta_{\epsilon,\delta})$ in dark grey.

The critical value locus C' intersects the vertical line $z = 0$ transversely at $w = 0$ and $w = \infty$, so we can choose ϵ sufficiently small so that C' intersects $\mathbb{D}_\epsilon \times \mathbb{P}$ outside of $\Delta_{\epsilon,\frac{1}{2}}$. Now, we reduce ϵ and δ so that the second condition holds. Because the first coordinate of N is $z \mapsto z^2$, $N(\partial^V \Delta_{\epsilon,\delta})$ is automatically inside of $|z| < \epsilon$. In the line $z = 0$, $N(z,w) = w^2$, so by continuity we can choose ϵ and δ small enough that $N(\partial^H \Delta_{\epsilon,\delta})$ is entirely outside of $\Delta_{\epsilon,\delta}$.

Let $\mathbb{D}_\epsilon$ be the open disc $|z| < \epsilon$ in $\mathbb{C}$ for this ϵ. Conditions 1 and 2 on ϵ and δ were chosen so that the following lemma is true:

Lemma 3.3. *Suppose that $D \subset \Delta_{\epsilon,\delta}$ is a complex disc that is the graph of an analytic function $\eta : \mathbb{D}_\epsilon \to \mathbb{P}$. Then $N^{-1}(D) \cap \Delta_{\epsilon,\delta}$ is the union of two disjoint complex discs, each given as the graph of analytic functions $\zeta_1, \zeta_2 : \mathbb{D}_\epsilon \to \mathbb{P}$.*

Proof of Lemma 3.3: The locus $N^{-1}(D) \cap \Delta_{\epsilon,\delta}$ satisfies the equation $N(z,w) \in D$, which is equivalent to $N_2(z,w) = \eta(z^2)$, because D is the graph of η. Since $D \subset \Delta_{\epsilon,\delta}$, D is disjoint from C', so $\frac{\partial}{\partial w} N_2(z,w)$ is non-zero in a neighborhood of $N^{-1}(D)$, and we can use the implicit function theorem to solve for $w = \zeta_1(z)$ and $w = \zeta_2(z)$. There are exactly two branches because $N_2(z,w)$ is degree 2 in w.

The graphs of ζ_1 and ζ_2 form the two complex discs $N^{-1}(D) \cap \Delta_{\epsilon,\delta}$. $\square$ Lemma 3.3.

The line $w = 1$ is invariant under N and attracted to the point $(0,1) \in S_0$. Let $D_0 = \{(z,w) : |z| < \epsilon, w = 1\}$. We will form W_0^{loc} by taking inverse images of D_0.

Since $D_0 \subset \Delta_{\epsilon,\delta}$ satisfies the conditions of Lemma 3.3, letting $D_1 = N^{-1}(D_0) \cap \Delta_{\epsilon,\delta}$ we obtain two complex discs in $\Delta_{\epsilon,\delta}$ each of which is given by the graph of an analytic function $\eta : \mathbb{D}_\epsilon \to \mathbb{P}$ and each of which is mapped within D_0 by N. These discs intersect S_0 at $w = 1$ and -1.

Because each of the discs in D_1 satisfies the hypotheses of Lemma 3.3 we can repeat this process, letting $D_2 = N^{-1}(D_1) \cap \Delta_{\epsilon,\delta}$, which this lemma guarantees is the union of four disjoint discs in

$\Delta_{\epsilon,\delta}$, each of which is the graph of some analytic function $\eta : \mathbb{D}_\epsilon \to \mathbb{P}$. These four discs intersect S_0 at the fourth roots of 1. Repeating this process, we obtain D_n consisting of 2^n disjoint complex discs in $\Delta_{\epsilon,\delta}$, each given by the graph of an analytic function intersecting S_0 at the 2^n-th roots of 1.

Let $D_\infty = \bigcup_{n=0}^{\infty} D_n$, which consists of a union of disjoint complex discs through each of the dyadic points $\mathcal{D}$ on S_0. Each of these discs is the graph of an analytic function from $\mathbb{D}_\epsilon$ to $\mathbb{P}$, and D_∞ is forward invariant to S_0 under N.

From a different perspective, D_∞ prescribes a *holomorphic motion*:

$$\phi : \mathbb{D}_\epsilon \times \mathcal{D} \to \mathbb{P}$$

where $\phi(z, \theta)$ is given by $\eta(z)$ where $\eta : \mathbb{D}_\epsilon \to \mathbb{P}$ is the analytic function whose graph is the disc in D_∞ containing $\theta \in \mathcal{D} \subset S_0$.

By the λ-lemma of Mañe-Sad-Sullivan [41], ϕ extends continuously to a holomorphic motion on S_0, the closure of $\mathcal{D}$.

$$\phi : \mathbb{D}_\epsilon \times S_0 \to \mathbb{P}$$

We define W_0^{loc} to be the image of $(z, w) \mapsto (z, \phi(z, w))$. Clearly $N(W_0^{loc}) \subset W_0^{loc}$ and every point in W_0^{loc} is forward invariant to S_0.

The construction of of W_1^{loc} is nearly identical and we omit it. □ Proposition 3.2.

Because the local superstable manifolds W_0^{loc} and W_1^{loc} are forward invariant under N, we can define global invariant sets W_0 and W_1 by:

$$W_0 = \bigcup_{n=0}^{\infty} N^{-n}(W_0^{loc}), \qquad W_1 = \bigcup_{n=0}^{\infty} N^{-n}(W_1^{loc}).$$

One might expect that W_0 and W_1 are manifolds, since the Inverse Function Theorem gives that the pull-back $N^{-k}(W_i^{loc})$ (or $N^{-k}(W_i^{loc})$) by N is "locally manifold" at points where $N^{-(k-1)}(W_i^{loc})$ is disjoint from or transverse to the critical value locus C. However, we do expect that there will be some values of the parameter B for which there is a tangency between $N^{-k}(W_i^{loc})$ and C. In fact, our computer images show that this must be the case, because we see the topology of $W_0 \cap C$ and of $W_1 \cap C$ change as we change B. For these parameter values W_i will not be a manifolds. To make this distinction, we will call W_0 and W_1 *separatrices* because they separate $W(r_1)$ from $W(r_2)$ and separate $W(r_3)$ from $W(r_4)$.

The following proposition requires that all iterates of N be defined for all points in a neighborhood of W_0 in X_l and in a neighborhood of W_1 in X_r. This will require a modification X_l^∞ of X_l that is obtained by blowing-up the points of indeterminacy p and q and all of their inverse images under N. We will prove Proposition 3.4, temporarily thinking that we are working in X_l and X_r, and then explain why it is necessary to blow-up the points of indeterminacy. The entire construction of X_l^∞ is given in the following section.

Proposition 3.4. *For every $B \in \Omega$, the separatrices W_0 and W_1 are real analytic subspaces of X_l^∞ and X_r, each defined as the zero set of a single non-constant real-analytic equation in an neighborhood of W_0 and in a neighborhood of W_1, respectively.*

The proof is similar of that of Böttcher's Theorem in one variable dynamics.

Proof: We express N in the variables $z = \frac{x}{x-1}$ and $w = \frac{y}{y-1}$ so that S_0 is given by $\{z = 0, |w| = 1\}$.

We will show that

$$\phi(z, w) = \lim_{n \to \infty} (N_2^n(z, w))^{1/2^n}$$

converges to a non-constant analytic function on a neighborhood of W_0. Then, for every $(z, w) \in W_0$, $|N_2^n(z, w)|$ converges to 1 because $S_0 = \{|w| = 1\}$, hence $\omega(z, w) := \log|\phi(z, w)| = \log|(N_2^n(z, w))^{1/2^n}|$ converges to 0 on W_0 and to non-zero values away from W_0.

If ϕ converges, then ϕ and ω transform nicely under the involution τ. For $|z|$ small and $|w|$ close to 1, τ is close to $(z,w) \mapsto (z, 1/w)$. Using this approximation, we have $\phi(\tau(z,w)) = \lim_{n\to\infty}(N_2^n(\tau(z,w)))^{1/2^n} = \lim(\tau(N^n(z,w))_2)^{1/2^n} \approx \lim(1/N_2^n(z,w))^{1/2^n} = 1/\phi(z,w)$. Consequently, $\omega(\tau(z,w)) = -\omega(z,w)$.

We can write $\phi(z,w) := \lim_{n\to\infty}(N_2^n(z,w))^{1/2^n}$ as a telescoping product:

$$\phi(z,w) = N_2(z,w)^{1/2} \cdot \frac{N_2^2(z,w)^{1/4}}{N_2(z,w)^{1/2}} \cdot \frac{N_2^3(z,w)^{1/8}}{N_2^2(z,w)^{1/4}} \cdots \tag{7}$$

We now check that we can restrict the neighborhood of W_0 where ϕ is defined so that we can use the binomial formula

$$(1+u)^\alpha = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} u^n, \text{ when } |u| < 1 \tag{8}$$

to define the $\frac{1}{2^n}$-th root in the n-th term of this product. We do this first for points in a neighborhood of W_0 in $\overline{W(r_1)}$ and a similar proof shows that the same works in a neighborhood of W_0 in $\overline{W(r_2)}$.

In the coordinates (z,w) the denominator of N_2 is of the form $1+r$ with $r = (B-Bw)z + (Bw^2+B-1-2Bw)z^2$, so for $|(B-Bw)z + (Bw^2+B-1-2Bw)z^2| < 1$ the second coordinate of N can me written as:

$$N_2(z,w) = w^2(1-Bz-z^2) + wzg(z,w) \tag{9}$$

with $g(z,w)$ analytic. We can write N_2 in form (9) in a neighborhood of $z=0$ (hence a neighborhood of S_0) since $|(B-Bw)z + (Bw^2+B-1-2Bw)z^2|$ vanishes when $z=0$. From this point on, we restrict our attention to this neighborhood of $z=0$.

The general term $\frac{N_2^{n+1}(z,w)^{1/2^{n+1}}}{N_2^n(z,w)^{1/2^n}}$ in the product (7) is of the form

$$\left(\frac{(N_2^n(z,w))^2(1-BN_1^n(z,w)-(N_1^n(z,w))^2) + N_2^n(z,w)N_1^n(z,w)\cdot g\left(N_1^n(z,w),N_2^n(z,w)\right)}{(N_2^n(z,w))^2}\right)^{1/2^{n+1}}$$
$$= \left(1-BN_1^n(z,w)-(N_1^n(z,w))^2 + \frac{N_1^n(z,w)}{N_2^n(z,w)}\cdot g\left(N_1^n(z,w),N_2^n(z,w)\right)\right)^{1/2^{n+1}}.$$

We need to check that we can restrict, if necessary, the neighborhood of definition for $\phi(z,w)$ so that

$$\left|-BN_1^n(z,w)-(N_1^n(z,w))^2 + \frac{N_1^n(z,w)}{N_2^n(z,w)}\cdot g\left(N_1^n(z,w),N_2^n(z,w)\right)\right| \tag{10}$$

$$\le \left|BN_1^n(z,w)+(N_1^n(z,w))^2\right| + \left|\frac{N_1^n(z,w)}{N_2^n(z,w)}\cdot g\left(N_1^n(z,w),N_2^n(z,w)\right)\right| \le \frac{1}{2}. \tag{11}$$

The first term is not a problem because $N_1^n(z,w) = z^{2n}$ and we are restricting to $|z|$ small. Since we are in $\overline{W(r_1)}$ the only difficulty can can occur if $N_2^n(z,w)$ goes to 0 fast enough to make (10) large. Detailed analysis of the behavior near r_1 resolves this concern:

In [37], the authors perform blow-ups at each of the four roots, and observe that the Newton map N induces rational functions of degree 2 on each of the exceptional divisors $E_{r_1}, E_{r_2}, E_{r_3}$, and E_{r_4}. Let's compute the rational function $s : E_{r_1} \to E_{r_1}$. In the coordinate chart $m = \frac{z}{w}$, the extension to E_{r_1} is obtained by:

$$s(m) = \lim_{w\to 0} \frac{m^2w^2(1+(B-Bw)mw+(Bw^2+B-1-2Bw)m^2w^2)}{w^2+(Bw-Bw^2)mw-w^2m^2w^2} = \frac{m^2}{1+Bm},$$

since $w = 0$ on E_{r_1}.

The rational function $s(m)$ has $m = 0$ as a superattracting fixed point, so there is a neighborhood of $m = 0 \in E_{r_1}$ within $\overline{W(r_1)}$ so that for any point (z, w) in this neighborhood, $\lim_{n\to\infty} \left|\frac{N_1^n(z,w)}{N_2^n(z,w)}\right| = 0$. Pulling back this neighborhood under N we find a neighborhood $V \subset \overline{W(r_1)}$ of the line $z = 0$ in which this limit holds.

Now we consider the case when $(z, w) \in \overline{W(r_2)}$. The concern is that $|w|$ may grow too fast for us to find a neighborhood of S_0 inequality (10) is true. Instead of analyzing the asymptotics of g, we can re-write N in the new coordinates (z, s) with $s = 1/w$ and a nearly identical construction to that of V gives the appropriate neighborhood V'.

Restricting the points $(z, w) \in V \cup V'$, the $\frac{1}{2^{n+1}}$-th root in the product (7) is well-defined. We check that the product converges on the neighborhood Λ of S_0. It is sufficient to show that the corresponding series of logarithms converges. The general term in this series is:

$$\log\left|\left(1 - BN_1^n(z,w) - (N_1^n(z,w))^2 + \frac{N_1^n(z,w)}{N_2^n(z,w)} \cdot g\left(N_1^n(z,w), N_2^n(z,w)\right)\right)^{1/2^{n+1}}\right| \leq \frac{\log 2}{2^{n+1}},$$

using Equation 10 and the triangle inequality so that

$$\left|1 - BN_1^n(z,w) - (N_1^n(z,w))^2 + \frac{N_1^n(z,w)}{N_2^n(z,w)} \cdot g\left(N_1^n(z,w), N_2^n(z,w)\right)\right| < 2.$$

This sequence of logarithms converges because it is dominated by a geometric series, and hence for the product (7) converges to the analytic function on $\phi(z, w)$ on Λ. This way $\omega(z, w) = \log|\phi(z, w)|$ is a real analytic function on Λ, and by the invariance properties of ϕ on $\omega(z, w)$ is an analytic function on a neighborhood of W_0.

The proof that W_1 is the zero locus of a non-constant analytic function is very similar. $\square$

It is important to notice that in this proof we assumed that all iterates of N are defined at every point in X_l, forgetting temporarily the points of indeterminacy p and q (and all of their inverse images in X_l.) This is a real problem because W_0 naturally goes through all of the points of indeterminacy: Under a high enough iterate of N, the line $x = \frac{1}{B(2-B)}$ is mapped by a ramified covering to a vertical line arbitrarily close to the line $x = 0$. Since these lines intersect W_0^{loc} in a topological circle, the line $x = \frac{1}{B(2-B)}$ intersects W_0 in a (possibly more complicated) curve. We will see that the resolution of the indeterminacy at p and q replaces p and q with exceptional divisors E_p and E_q that are mapped to $x = \frac{1}{B(2-B)}$ by isomorphisms. So, to make this proof correct, we will have to blow-up at p and q, and, in fact, at all inverse images of p and q.

An alternative approach would be to study W_0 on $X_l - \cup_{n=0}^{\infty} N^{-n}(\{p, q\})$, where we have already proven it is a real-analytic variety. However, we want to consider the topology of W_0, without all of these points removed, so we prefer to do the sequence of blow-ups.

4 Resolution of points of indeterminacy

By restricting to parameters $B \in \Omega$, the points of indeterminacy p and q are in X_l and there are no points of indeterminacy in X_r. In this section we will describe how to resolve the indeterminacy in N at p and q and in higher iterates of N at all of the inverse images of p and q in X_l, obtaining a new space X_l^∞ on which all iterates of N are defined at every point.

Writing W_0 as a real-analytic variety is not the only motivation for the construction of X_l^∞. We plan to study the detailed topology of the basins of attraction and of the separatrices W_0 and W_1. It is difficult to decide what is a reasonable alternative to the statement of Theorem 0.1 without blowing up points.

4.1 Construction of X_l^∞ and $N_\infty : X_l^\infty \to X_l^\infty$.

Most of the material in this section and in the following section closely follow the works of Hubbard and Papadopol [37] and Hubbard, Papadopol, and Veselov [33].

Substitution of the points p and q into $C(x,y)$ yields $\frac{1}{4}(B-1)$ and $\frac{B^2-7B+2}{4B-8}$, so for values of B at which these expressions are non-zero, neither p nor q is a critical value.

Let $S \subset \Omega$ be the subset of parameter space for which no inverse image of the point of indeterminacy p or of point of indeterminacy q is in the critical value locus C. We first describe the construction of X_l^∞ for parameter values $B \in S$, and then explain the necessary modifications for special circumstance when $B \notin S$.

Theorem 4.1. *The set S is generic in the sense of Baire's Theorem, i.e. uncountable and dense in Ω.*

Because of its computational nature, the proof of Theorem 4.1 is in Appendix B.

Construction of X_l^∞ when $B \in S$:

Proposition 4.2. *Let X_l^0 be the space X_l blown up at the points p and q and let $\pi_0 : X_l^0 \to X_l$ be the corresponding projection.*

- *The mapping N extends analytically to a mapping $N_0 : X_l^0 \to X_l$.*
- *N_0 maps the exceptional divisors E_p and E_q to the line $x = \frac{1}{B(2-B)}$ by isomorphisms.*

Proof: The definition of a blow-up at a point is available in Appendix A. Further details about blow-ups are available in books on Algebraic Geometry, including [28], and, in the context of complex dynamics, in the papers [33, 37].

We will work in the chart $(x,m) \mapsto (x, m(x-\frac{1}{B}), m) \in X_l \times \mathbb{P}^1$. We write $N(x,y) = (N_1(x,y), N_2(x,y))$ so that in the coordinates (x,m) we have $N_1(x,m) = \frac{1}{B(2-B)}$ and

$$N_2(x,m) = \frac{\frac{m}{B}(Bx^2 + 2xm(x-\frac{1}{B}) - Bx - m(x-\frac{1}{B}))}{(2x-1)(\frac{2m}{B}+1)}$$

When restricted to the exceptional divisor E_p (set $x = \frac{1}{B}$) the mapping becomes $m \mapsto \frac{m(1-B)}{(2-B)(2m+B)}$. If instead we had been working in the chart $(y,m') \mapsto (m'y + \frac{1}{B}, y, m')$, we would have obtained $m' \mapsto \frac{(1-B)}{(2-B)(2+m'B)}$. This is consistent with the extension in terms of m since one is obtained from the other by the change of variables $m = \frac{1}{m'}$. Both of the expressions for N restricted to E_p are linear-fractional transformations, hence N maps E_p to the line $y = \frac{1}{B(2-B)}$ by an isomorphism.

The exceptional divisor E_q can be treated similarly. $\square$

We will denote the vertical line $x = \frac{1}{B(2-B)}$ by V, since we use this line so frequently. This is the vertical line that is tangent to C at its "vertex".

By construction, the extension $N_0 : X_l^0 \to X_l$ has no points of indeterminacy. However, since we need to iterate we must consider N_0 as a rational map $N_0 : X_l^0 \to X_l^0$. Each of the inverse images of p and q become points of indeterminacy of N_0 because we have blown up at p and q. Because $B \in S$, neither p nor q are critical values and each has four inverse images under N_0. We can blow-up at each of these eight points obtaining the space X_l^1 and the projection $\pi_1 : X_l^1 \to X_l^0$. One can then extend N_0 to the exceptional divisors, obtaining $N_1 : X_l^1 \to X_l^0$.

To make iterates $N^{\circ k}$ of N well-defined for all k we must repeat this process for the k-th inverse images, obtaining successive blow-ups $\pi_k : X_l^k \to X_l^{k-1}$ for every k. The following proposition describes the extension of N to these spaces:

Proposition 4.3. *Denote by X_l^k the space X_l^{k-1} blown up at each of these $2 \cdot 4^k$ k-th inverse images of p and q.*

- *The mapping N_{k-1} extends analytically to a mapping $N_k : X_l^k \to X_l^{k-1}$.*
- *Suppose that z is one of the k-th inverse images of p or q and denote the exceptional divisor over z by E_z. Then, N_k maps E_z to $E_{N(z)}$ by isomorphism.*

Proof: As in Proposition 4.2, denote the first and second components of N by $N_1(x, y)$ and $N_2(x, y)$. Then, in the coordinates $(x, m) \mapsto (x, mx, m)$ in a neighborhood of E_z the mapping is given by: $m \mapsto \frac{\partial_x N_1 + \partial_y N_1 m}{\partial_x N_2 + \partial_y N_2 m}$. By the assumption that $B \in S$, DN is non-singular at z and this gives an isomorphism from E_z to $E_{N(z)}$. $\square$

Hence, by repeated blow-ups we obtain a sequence of spaces and projections:

$$X_l \xleftarrow{\pi_0} X_l^0 \xleftarrow{\pi_1} X_l^1 \xleftarrow{\pi_2} X_l^2 \xleftarrow{\pi_3} X_l^3 \xleftarrow{\pi_4} X_l^4 \xleftarrow{\pi_5} X_l^5 \xleftarrow{\pi_6} \cdots \tag{12}$$

The extensions of the Newton map N to these spaces that we calculated in Propositions 4.2 and 4.3 give another sequence of spaces and mappings:

$$X_l \xleftarrow{N_0} X_l^0 \xleftarrow{N_1} X_l^1 \xleftarrow{N_2} X_l^2 \xleftarrow{N_3} X_l^3 \xleftarrow{N_4} X_l^4 \xleftarrow{N_5} X_l^5 \xleftarrow{N_6} \cdots \tag{13}$$

However, we do not have a single space X_l^∞, nor a single mapping N_∞ from this space to itself. There is a standard procedure using *Inverse Limits* to create such a space and mapping from a sequence of spaces (12) and the sequence of mappings like (13). That is, we will let X_l^∞ be the inverse limit of the blown up spaces and projections in Sequence 12 and then use the sequence of extensions of the Newton maps 13 to define a mapping $N_\infty : X_l^\infty \to X_l^\infty$ which naturally corresponds to an extension of N.

Definition 4.4. *An* **Inverse system***, denoted (M_i, σ_i), is a family of objects M_i in a category C indexed by the natural numbers and for every i a morphism $\sigma_i : M_i \to M_{i-1}$.*

The **Inverse Limit** *of an inverse system (M_i, σ_i), denoted by $\varprojlim(M_i, \sigma_i)$, is the object X in C together with morphisms $\alpha_i : X \to M_i$ satisfying $\alpha_{i-1} = \sigma_i \circ \alpha_i$ for each i that is determined uniquely by the following universal property:*

For any other pair $Y, \beta_i : Y \to M_i$ such that $\beta_{i-1} = \sigma_i \circ \beta_i$, we have a unique morphism $u : Y \to X$ so that for each i we have $\beta_i = \alpha_i \circ u$.

For our uses, the category will always be analytic spaces and the morphisms holomorphic maps. While not needed here, inverse systems and inverse limits can be defined more generally, for objects M_i indexed by a filtering partially ordered set I. The following proposition gives a construction of $\varprojlim(M_i, \sigma_i)$ as a subset of the product space $\Pi_i M_i$.

Proposition 4.5. *Given an inverse system (M_i, σ_i) indexed by $\mathbb{N}$ (i.e. $\sigma_i : M_i \to M_{i-1}$), we can construct the inverse limit as follows:*

$$\varprojlim(M_i, \sigma_i) = \{(m_0, m_1, m_2, m_3, \cdots) | m_i \in M_i \text{ and } \sigma_i(m_i) = m_{i-1}\}.$$

We define $X_l^\infty = \varprojlim(X_l^k, \pi_k)$. Using Proposition 4.5 we can state more concretely that

$$X_l^\infty = \{(x_0, x_1, x_2, x_3, \cdots) | x_i \in X_l^i \text{ and } \pi_i(x_i) = x_{i-1}\}.$$

In this description of X_l^∞, the mappings $N_k : X_l^k \to X_l^{k-1}$ induce a mapping $N_\infty : X_l^\infty \to X_l^\infty$ given by $N_\infty((x_0, x_1, x_2, x_3, \cdots)) = (N_1(x_1), N_2(x_2), N_3(x_3), \cdots)$.

Construction of X_l^∞ when $B \notin S$:

For parameter values $B \notin S$, the blow-ups done at p and q in Proposition 4.2 are exactly the same, since N extends to these blow-ups for any value of B. (It is worth noticing that there is actually a

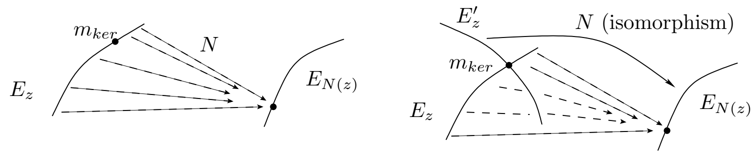

Figure 8: Blowing up a point on an exceptional divisor.

critical point of N on both E_p and on E_q since each is mapped to the line V, which contains a point of C.) Special care needs to be taken when a k-th inverse image of p and of q is a critical point of N. We describe the process here, although we leave some of the details for the appendix.

The goal is to produce a space X_l^k and a projection $\pi_k : X_l^k \to X_l^{k-1}$ in such a way that N extends to a map (without singularities) $N_k : X_l^k \to X_l^{k-1}$. If we can create the spaces X_l^k and extensions N_k at every "level" k, we can use the same inverse sequence process to make X_l^∞ and $N_\infty : X_l^\infty \to X_l^\infty$.

Suppose for the moment that z is a k-th inverse image of p and that none of the n-th inverse images of p for $n < k$ were in the critical locus $N^{-1}(C)$. In this case, there is a single exceptional divisor in X_l^{k-1} above $N(z)$. Because z is critical, the extension of N to E_z will map all of E_z (except for one point) to a single point in $E_{N(z)}$. However, at the slope $m_{ker} \in E_z$ corresponding to the kernel of DN, the extension to E_z has another point of indeterminacy! Consequently, one has to blow-up this point on E_z, obtaining a second exceptional divisor E'_z above m_{ker}. Figure 8 shows this situation. An easy check using Taylor series for N shows that N extends to E'_z giving isomorphism from E'_z to $E_{N(z)}$.

These two blow-ups above z are sufficient to extend N.

The two exceptional divisors above z result in a further complication at every point w that is mapped to z. Suppose that we have blown up at w. The extension of N to E_w has a point of indeterminacy at the point that is mapped to $m_{ker} \in E_z$. Because of this, one has to blow-up a second time above w to resolve this point of indeterminacy. In fact, at every repeated inverse image of z one will have to blow-up at least twice to resolve N.

There are further problems if an inverse image of z is again critical. At such a point, one will have to do even more blow-ups to resolve N. A detailed description of this process becomes rather tedious, and we will stop here.

4.2 The mappings from E_z to V

We saw in the previous section that N maps each exceptional divisor that was newly created in X_l^k to one of the exceptional divisors newly created in X_l^{k-1} by either an isomorphism, or a constant map. Since N maps each E_p and E_q isomorphically to the line V, the composition $N^{\circ k+1}$ maps each of the newly created exceptional divisors E_z in X_l^k to V either by an isomorphism, or a constant map. In summary:

Proposition 4.6. *Let E_z be one of the exceptional divisors newly created in X_l^k and let V be the line $x = 1/(B(2-B))$. Then $N^{\circ k+1}$ maps E_z to V by an isomorphism, or a constant map.*

4.3 Homology of X_r and of X_l^∞

Our goal in this paper is to relate the homology of the basins of attraction for the four roots of F to the homology of the spaces X_r and X_l^∞ and to the homology of the separatrices W_0 and W_1. In this section, we will compute the homology of X_r and X_l^∞.

Given a set T, we will denote by $\mathbb{Z}^{(T)}$ the submodule of the product $\mathbb{Z}^T$ for which each element has at most finitely many non-zero components.

We often find it necessary to encode information about the generators homology within our notation. For example, the module $\mathbb{Z}^{\{[K]\}}$ means the module $\mathbb{Z}$ that is generated by the fundamental class of $[K]$.

Proposition 4.7. *We have:* $H_0(X_r) = \mathbb{Z}$, $H_2(X_r) = \mathbb{Z}^{\{[\mathbb{P}^1]\}}$, *and* $H_i(X_r) = 0$, *for* $i \neq 0$ *or* 2.

Unfortunately homology does not behave nicely under inverse limits. So, instead of directly using the fact that X_l^∞ is an inverse limit to compute its homology, we will write X_l^∞ is a union of open subsets $U_0 \subset U_1 \subset U_2 \subset \cdots$ in such a way that $H_2(U_i) = \mathbb{Z}^{(L_i \cup \{[V]\})}$, where L_i is the set of fundamental classes of exceptional divisors contained in U_i and $[V]$ is the fundamental class of the vertical line V given by $x = \frac{1}{B(2-B)}$.

Recall that the projection $\pi : X_l^\infty \to X_l$ is continuous, we will create an exhaustion of X_l^∞ by open sets $U_0 \subset U_1 \subset U_2 \subset \cdots$ that are inverse images of open subsets forming an exhaustion of X_l. Let $V_k = X_l - \bigcup_{n=k}^\infty \{N^{-n}(p), N^{-n}(q)\}$. Clearly V_k is an open subset of X_l, so we will let $U_k = \pi^{-1}(V_k)$. It is also clear that $U_1 \subset U_2 \subset U_3 \subset \cdots$ and that $\bigcup_{k=1}^\infty U_k = X_l^\infty$.

Lemma 4.8. *For each* k, $H_2(U_k) \cong H_2(X_l^k)$

Proof: Notice that U_k canonically isomorphic to $X_l^k - \bigcup_{n=k}^\infty \{N^{-n}(p), N^{-n}(q)\}$. Removing a discrete set of points from a 4 (real) dimensional manifold does not affect the second homology. Hence, $H_2(U_k) \cong H_2(X_l^k)$. □

Lemma 4.9. $H_2(X_l^k) \cong \mathbb{Z}^{(L_k \cup \{[V]\})}$, *where* L_k *is the set of fundamental classes of exceptional divisors in* X_l^k.

Proposition 4.10. $H_2(X_l^\infty) \cong \mathbb{Z}^{(L \cup \{[V]\})}$, *where* L *is the set of fundamental classes of exceptional divisors in* X_l^∞ *and* $[V]$ *is the fundamental class of the vertical line* V.

Proof: Since $X_l^\infty = \bigcup_{k=1}^\infty U_k$ and $H_2(U_k) \cong H_2(X_l^k) \cong \mathbb{Z}^{(L \cup \{[V]\})}$, we have that $H_2(X_l^\infty) \cong \varinjlim \left(\mathbb{Z}^{(L_k \cup \{[V]\})}\right)$, which is $\mathbb{Z}^{(L \cup \{[V]\})}$. □

In the generic case $B \in S$ for which none of the inverse images of p or q under N are in the critical value parabola C we can describe $H_2(X_l^\infty)$ somewhat more explicitly:

Proposition 4.11. *For* $B \in S$, $H_2(X_l^\infty) = \mathbb{Z}^{\{[V]\}} \oplus \left(\bigoplus_{N^k(x)=p} \mathbb{Z}^{\{[E_x]\}}\right) \oplus \left(\bigoplus_{N^k(x)=q} \mathbb{Z}^{\{[E_x]\}}\right)$.

Proof: This is merely a restatement of Proposition 4.10 using that when $B \in S$, only a single blow-up is necessary at each k-th inverse image of p and of q for every k. □

We will need the following proposition about the intersection of classes in $H_2(X_l^\infty)$:

Proposition 4.12. *Let* $[V]$ *and* $[E_z]$ *be the fundamental classes of a vertical line* V *and an exceptional divisor* E_z *in* $H_2(X_l^\infty)$. *Then* $[V] \cdot [V] = 0$ *and* $[E_z] \cdot [E_z] \leq -1$.

Proof: We have chosen the vertical line V so that points on it are never blown up, hence within X_l^∞ it has self-intersection number 0, just as it did in X_l.

If no points on the exceptional divisor E_z have been blown up, then it is a classical result that $[E_z] \cdot [E_z] = -1$. Otherwise, if points in E_z have been blown up, it is a classical result that each blow-up reduces $[E_z] \cdot [E_z]$ by 1, hence $[E_z] \cdot [E_z] \leq -1$. (See [28].) □

4.4 Why we work in X_l

Suppose for a moment that we did this sequence of blow-ups in $X = \mathbb{P}^1 \times \mathbb{P}^1$, instead of just within the invariant subspace X_l. The repeated inverse images of the points of indeterminacy must accumulate because X is compact. The topology of the inverse limit X^∞ becomes very complicated near these points of accumulation. Hubbard and Papadopol develop elaborate techniques including *Farey Blow-ups* and *Real-oriented Blow-ups* to "tame" the topology at these points of accumulation. We avoid this problem caused by accumulation by working in the space X_l since the inverse images of p and q go to the "end" of $X_l = \{\mathrm{Re}(x) < 1/2\} \times \mathbb{P}^1$ instead of accumulating to a finite point.

5 Mayer-Vietoris sequences

We will study the topology of W_0 and W_1 in detail in order to prove Theorem 0.1. The following Mayer-Vietoris calculations will allow us to relate their homology to that of the basins of attraction for the four roots r_1, r_2, r_3, and r_4.

Let $\overline{W(r_1)}$ and $\overline{W(r_2)}$ be the closures of $W(r_1)$ and $W(r_2)$ in X_l^∞ and let $\overline{W(r_3)}$ and $\overline{W(r_4)}$ be the closures of $W(r_3)$ and $W(r_4)$ in X_r. Since W_0 and W_1 are real-analytic varieties in X_l^∞ and X_r, respectively, there are neighborhoods in X_l^∞ and X_r of W_0 and W_1 that deformation retract onto W_0 and W_1. Hence, we can consider the Mayer-Vietoris exact sequence (see [13, 31]) for the decompositions $\overline{W(r_1)} \cup \overline{W(r_2)} = X_l^\infty$, $\overline{W(r_1)} \cap \overline{W(r_2)} = W_0$ and $\overline{W(r_3)} \cup \overline{W(r_4)} = X_r$, $\overline{W(r_3)} \cap \overline{W(r_4)} = W_1$. We find that

$$\begin{aligned} 0 \to H_2(W_0) &\xrightarrow{i_{1*}\oplus i_{2*}} H_2\left(\overline{W(r_1)}\right) \oplus H_2\left(\overline{W(r_2)}\right) \\ &\xrightarrow{j_{1*}-j_{2*}} H_2(X_l^\infty) \xrightarrow{\partial} H_1(W_0) \xrightarrow{i_{1*}\oplus i_{2*}} H_1\left(\overline{W(r_1)}\right) \oplus H_1\left(\overline{W(r_2)}\right) \to 0 \end{aligned} \tag{14}$$

is exact, where i_1 and i_2 are the inclusions of W_0 into $\overline{W(r_1)}$ and $\overline{W(r_2)}$ and j_1 and j_2 are the inclusions of $\overline{W(r_1)}$ and $\overline{W(r_2)}$ into X_l^∞. Slightly more work shows that $\partial[V] = [S_0]$, where $[V]$ is the fundamental class of a vertical line in X_l^∞ and $[S_0]$ is the class of the invariant circle.

We have $H_2(X_r) = \mathbb{Z}^{\{[\mathbb{P}]\}}$ with $\partial([\mathbb{P}]) = [S_1]$. Using that $H_i(X_r) = 0$ for $i \neq 2, 0$, we find that the map

$$H_2(W_1) \quad \xrightarrow{i_{3*}\oplus i_{4*}} \quad H_2\left(\overline{W(r_3)}\right) \oplus H_2\left(\overline{W(r_4)}\right) \tag{15}$$

is an isomorphism and the sequence

$$0 \to \mathbb{Z}^{\{[\mathbb{P}]\}} \xrightarrow{\partial} H_1(W_1) \xrightarrow{i_{3*}\oplus i_{4*}} H_1\left(\overline{W(r_3)}\right) \oplus H_1\left(\overline{W(r_4)}\right) \to 0 \tag{16}$$

is exact, where where i_3 and i_4 are the inclusions of W_1 into $\overline{W(r_3)}$ and $\overline{W(r_4)}$.

6 Morse Theory for W_1 and W_0

In this section we prove that if there are parameter values B for which $W_1 \cap C = \emptyset$, then $H_1(\overline{W(r_3)})$ and $H_1(\overline{W(r_4)})$ are trivial. Our computer experiments suggest that such B exist, but we have not proven their existence. This proves the second half of the second part of Theorem 0.1, which we will finish proving in the following two sections.

Consider the function $h : \mathbb{C} \times \mathbb{P} \to \mathbb{R}$ given by $h\binom{x}{y} = \left|\frac{x}{x-1}\right|$ which is chosen so that

$$h\left(N\begin{pmatrix} x \\ y \end{pmatrix}\right) = \left|\frac{x^2}{x^2 - 2x + 1}\right| = h\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)^2. \tag{17}$$

We will consider the restriction of h to the super-stable separatrices W_0 and W_1 and use it as a Morse function to study their topology. Because W_0 and W_1 intersect the critical value parabola C in real-analytic sets, the following geometric description of the critical points of h makes sense:

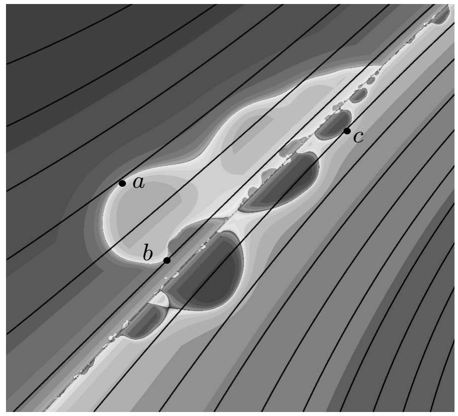

Figure 9: Level curves of the Morse function h in part of the critical value parabola C. The points labeled a and b on W_0 are in K and the point labeled c on W_1 is in K. Repeated inverse images of these points under N are critical points of h on W_0 and W_1. Clearly we have only labeled a few of the points in K that are visible.

Proposition 6.1. *Let K be the set of points in C where $W_0 \cap C$ (or $W_1 \cap C$) is parallel to the level curves of $h|C$. Then the set of critical points of h on W_0 and W_1 is $\bigcup_{k=1}^{\infty} N^{-k}(K)$.*

Proof: Applying the chain rule to Equation 17 we find:

$$Dh\left(N\begin{pmatrix}x\\y\end{pmatrix}\right)\cdot DN\begin{pmatrix}x\\y\end{pmatrix} = 2h\begin{pmatrix}x\\y\end{pmatrix}\cdot Dh\begin{pmatrix}x\\y\end{pmatrix}. \tag{18}$$

Because $\left|h\binom{x}{y}\right| = 0$ only when $x = 0$, Equation 18 gives that if $Dh\binom{x}{y} = 0$ for a point $\binom{x}{y}$ on W_0 then either:

1. $Dh\left(N\binom{x}{y}\right) = 0$ giving that $\binom{x}{y}$ is an inverse image (possibly an n-th inverse image) of another critical point of h. Or,

2. $DN\binom{x}{y}$ is singular and $Dh\left(N\binom{x}{y}\right)$ is killed by $DN\binom{x}{y}$.

The condition in the second case says that (x, y) is on the critical points locus of N and that the curve $W_0 \cap C$ is tangent to the level curves of $h|C$ at $N(x, y)$. $\square$

Notice that if $h : W_0 \to \mathbb{R}$ or if $h : W_1 \to \mathbb{R}$ has no critical points aside from at $x = 0$ or 1, then the negative gradient flow $-\nabla h$ gives a deformation retraction of W_0 to S_0 or the gradient flow ∇h gives a deformation retraction of W_1 to S_1.

Proposition 6.2. *If there are no points of intersection between W_1 and the critical value parabola C, then W_1 is homotopy equivalent to S_1.*

Proof: By Proposition 6.1 if W_1 and C are disjoint, there are no critical points of h.$\square$

Corollary 6.3. *If there are no points of intersection between W_1 and the critical value parabola C, then the basins of attraction $W(r_3)$ and $W(r_4)$ for the roots $r_3 = (1, 0)$ and $r_4 = (1, 1 - B)$ have trivial first and second homology groups.*

Proof: This follows for the second homology from the isomorphism between $H_2(W_1)$ and $H_2(\overline{W(r_3)})\oplus H_2(\overline{W(r_4)})$. For the first homology, $H_1(W_1) \cong \mathbb{Z}^{\{[S_1]\}} = \mathrm{Image}(\partial)$ and exactness of (16) gives that

$H_1(\overline{W(r_3)}) = 0 = H_1(\overline{W(r_4)})$. Because W_1 is disjoint from C, B is not in the bifurcation locus, we can replace $\overline{W(r_3)}$ and $\overline{W(r_4)}$ with the basins themselves. □

Proposition 6.4. *There are always critical points of $h : W_0 \to \mathbb{R}$.*

Proof: Using implicit differentiation of $C(x, y) = 0$, one can check that there is a unique critical point of $h|C$ at the intersection of C with the line $Bx + 2y - 1 = 0$. Since this line is the axis of symmetry for τ, it is in W_0. □

Instead of studying the Morse function h when W_0 or W_1 intersects C, in the next two sections we will use linking numbers to show that such intersections result in infinitely many loops corresponding to distinct generators of homology.

7 Many loops in W_0 and W_1

Denote the vertical line in X_l at a fixed value of x by $\mathbb{P}_x$. Such vertical lines in X_l correspond naturally to lines in X_l^∞ by means of the "proper transform" that is induced by the blow-up operation.

The Newton Map N maps $\mathbb{P}_x$ to $\mathbb{P}_{x^2/(2x-1)}$ by the rational map:

$$R_x(y) = \frac{y(Bx^2 + 2xy - Bx - y)}{(2x-1)(Bx + 2y - 1)}.$$

Notice that when $x = \frac{1}{B}$ and when $x = \frac{1}{2-B}$, a common term cancels from the numerator and denominator of R_x, giving $R_x(y) = \frac{y}{2} + \frac{1-B}{2(2-B)}$ and $R_x(y) = \frac{y}{2}$, respectively. The critical values of R_x are the intersections of the critical value parabola C with the line $\mathbb{P}_{x^2/(2x-1)}$. There are two distinct critical values, except when $x = \frac{1}{B}$ or $\frac{1}{2-B}$.

Lemma 7.1. *There are $\epsilon_0 > 0$ and $\epsilon_1 > 0$ so that if $|x - 0| < \epsilon_0$, then $W_0 \cap \mathbb{P}_x$ forms a simple closed curve and so that if $|x - 1| < \epsilon_1$, then $W_1 \cap \mathbb{P}_x$ forms a simple closed curve.*

Proof: This is a direct consequence of the existence of W_0^{loc} and W_1^{loc} and the fact that there is no possible recurrence in the dynamics for x within X_l^∞ or X_r. □

Most vertical lines $\mathbb{P}_x$ will be divided by W_i ($i = 0$ or 1) into exactly two simply connected domains. However, if $W_i \cap C$ is non-empty in any forward image of $\mathbb{P}_x$, then W_i will divide $\mathbb{P}_x$ into many more simply connected domains. These are counted in the following proposition.

Proposition 7.2. *Let $\mathbb{P}_x$ be a vertical line whose k-th forward image $\mathbb{P}_{\hat{x}}$ is divided by W_i into exactly two simply connected domains. If $W_i \cap C \neq \emptyset$ in $\mathbb{P}_{\hat{x}}$, then $\mathbb{P}_x$ is divided by W_i into between $2^k + 2$ and 2^{k+1} simply connected domains.*

We prove Proposition 7.2 for W_0 in X_l^∞ because the proof is identical in X_r. The following lemma and corollary are direct consequences of the Riemann-Hurwitz Theorem.

Lemma 7.3. *Let $R : \mathbb{P} \to \mathbb{P}$ be a ramified covering map of degree d and let $U \subset \mathbb{P}$ be a simply connected open subset of $\mathbb{P}$ containing the image of at most one point of ramification of R. Then, $R^{-1}(U)$ consists of a finite number of disjoint simply connected domains.*

The symmetry (1.6) guarantees that there is at most one of the two critical values of R_x is in each simply connected domain, hence the inverse image of each domain is a finite number of simply connected domains. An easy check shows that if U contains one of the critical values of R_x, then $R_x^{-1}(U)$ is a single simply connected domain, while if U does not contain a critical value of R_x it is two simply connected domains.

Corollary 7.4. *Let $\mathbb{P}_x$ be a vertical line whose image $\mathbb{P}_{\hat{x}}$ is divided by W_i into m simply connected domains. If $W_i \cap C \neq \emptyset$ in $\mathbb{P}_{\hat{x}}$, then W_i divides $\mathbb{P}_x$ into $2m$ simply connected domains. Otherwise it divides $\mathbb{P}_x$ into $2m-2$ simply connected domains.*

The proof of Proposition 7.2 follows from this Corollary and the fact that there is a sufficiently high k so that $|x_k| < \epsilon$ so that W_0 divides $\mathbb{P}_{x_k}$ into exactly two domains.

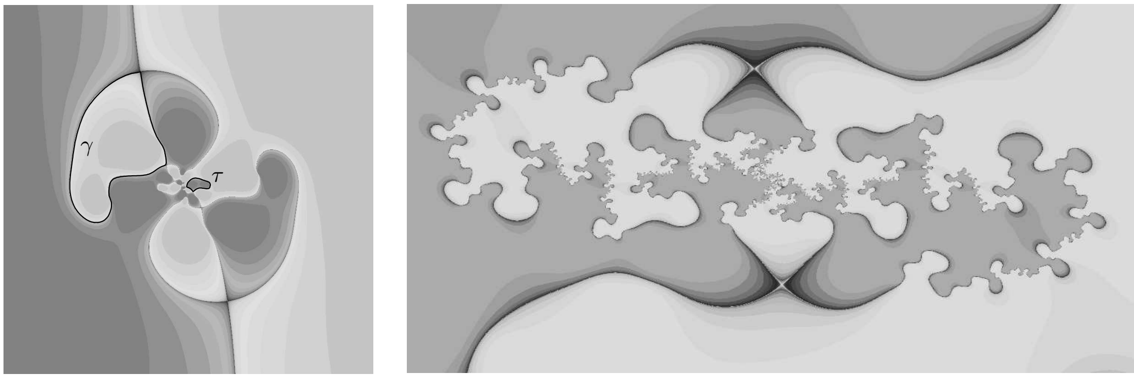

Figure 10: Left: a vertical line divided by W_0 into 10 regions. Loop γ has size$(\gamma) = 1/10$ and loop τ has size$(\tau) = 1/10$. Right: a different vertical line that is divided by W_0 into 10 regions. This time, the loops bounding the regions are much more ornate.

Suppose that W_0 divides $\mathbb{P}_x$ into $2m$ simply connected domains $U_1, \cdots, U_m$ in $W(r_1)$ and $V_1, \cdots, V_m$ in $W(r_2)$. Let k be chosen so that W_0 forms a simple closed curve in $\mathbb{P}_{\tilde{x}}$ (where $\tilde{x}$ is the k-th iterate of x under $x \mapsto \frac{x^2}{2x-1}$.) Denote by U the domain in $\mathbb{P}_{\tilde{x}}$ within $W(r_1)$ and by V the domain in $\mathbb{P}_{\tilde{x}}$ within $W(r_2)$.

Under the mapping N^k, each domain U_i covers U with some degree l_i and each domain V_j covers V with degree p_j. Then: $\sum_{i=1}^m l_i = 2^k$, $\quad \sum_{i=1}^m p_i = 2^k$ because U is covered by $\cup_{i=1}^m U_i \subset \mathbb{P}_x$ with degree 2^k.

For such a U_i we will assign size$(U_i) = -\frac{l_i}{2^k}$ and such a V_i we can assign size$(U_i) = \frac{p_i}{2^k}$. This is well defined because given k_1 and k_2 as above, the l_i corresponding to k_1 and the l_i corresponding to k_2 will differ by $2^{k_1-k_2}$.

$$\sum_{i=1}^m \text{size}(U_i) = -1, \qquad \sum_{i=1}^m \text{size}(V_i) = 1$$

In the next section we will see that size(U_i) for such a region equals the linking number between $\gamma_i = \partial U_i$ and an appropriate geometric object in X_l^∞.

8 Linking numbers

Classically one considers the linking number of two oriented loops c and d in $\mathbb{S}^3$. The linking number $lk(c,d) \in \mathbb{Z}$ is found by taking any oriented surface Γ with oriented boundary c and defining $lk(c,d)$ to be the signed intersection number of Γ with d as in Figure 8. For this and many equivalent definitions of linking number in $\mathbb{S}^3$ see [45, pp. 132-133], [12, pp. 229-239], and [43, Problems 13 and 14].

To see that this linking number is well-defined notice that assigning $lk(c,d) = [\Gamma] \cdot [d]$, where $\cdot$ indicates the intersection product on $H_*(\mathbb{S}^3, c)$, coincides with the classical definition. If Γ' is any

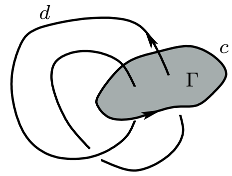

Figure 11: Here $lk(c, d) = +2$.

other 2-chain with $\partial\Gamma' = c$ then $\partial(\Gamma - \Gamma') = [c] - [c] = 0$ and $(\Gamma - \Gamma')$ represents a homology class in $H_2(\mathbb{S}^3)$. Since $H_2(\mathbb{S}^3) = 0$, $[\Gamma - \Gamma'] = 0$ forcing $[\Gamma - \Gamma'] \cdot [d] = 0$. Therefore: $[\Gamma] \cdot [d] = [\Gamma'] \cdot [d]$, so that $lk(c, d)$ is well defined.

Linking kernel: $\mathcal{L}Z_p(M)$

Suppose that M is a 3-dimensional manifold with $H_2(M) \neq 0$. We can define a linking number $lk(c, d)$ so long as the second argument d has $[d] \cdot \sigma = 0$ for every $\sigma \in H_2(M)$. We define $\mathcal{L}Z_1(M) \subset Z_1(M)$ to be the sub-module of one chains having this property. As before, given $d \in \mathcal{L}Z_1(M)$ the linking number $lk(c, d)$ is well-defined for c disjoint from d with $[c] = 0$.

Linking numbers work in a similar way if a manifold M has dimension m: one requires that c and d have dimensions summing to $m - 1$ and one must restrict to c disjoint from d with $[c] = 0$ and restrict to $d \in \mathcal{L}Z_p(M)$, where p is the dimension of d.

8.1 Linking kernel for X_l^∞

Recall from Section 4 that

$$H_2(X_l^\infty) = \mathbb{Z}^{(L \cup \{[V]\})}$$

where L is the set of exceptional divisors E_i introduced in the sequence of blow-ups and V is the vertical line $x = \frac{1}{B(2-B)}$.

Recall from Proposition 4.12 that each exceptional divisor $[E_i]$ has $[E_i] \cdot [E_i] \leq -1$ and that $[V] \cdot [V] = 0$ so that if $\omega = a_0[V] + a_1[E_1] + \cdots + a_n[E_n]$ satisfies $\omega \cdot \sigma = 0$ for every $\sigma \in H_2(X_l^\infty)$ then $a_i = 0$ for all $i \neq 0$.

In summary, $\mathcal{L}Z_2(X_l^\infty) = \mathbb{Z}^{\{V\}}$. The curves γ_i considered in the previous section each have linking number 0 with V, since each γ_i is entirely within some (other) vertical line. To show that any of the curves γ_i are non-trivial we will need to look for a different kind of object to link with. We do this by extending the definition of linking to closed currents.

8.2 Generalities on currents

Just as distributions are defined as the topological dual of smooth functions with compact support, currents are the topological dual of smooth differential forms with compact support.

Let $A_c^{n-q}(M)$ denote the $(n - q)$-forms with compact support on a smooth manifold M. The linear maps $T : A_c^{n-q}(M) \to \mathbb{C}$ that are continuous are the **currents of degree** q (or, as some say, the currents of **dimension** $n - q$) and are denoted by $\mathcal{D}^q(M)$. If M has a complex structure, one defines the currents of bi-degree (p, q), denoted $\mathcal{D}^{p,q}(M)$ as the topological dual of the $(n-p, n-q)$-forms with compact support $A_c^{n-p,n-q}(M)$. For more background on currents, consult [28, section 3.1 and 3.2] or the articles on complex dynamics [38, 49, 48].

An exterior derivative $d : \mathcal{D}^q(M) \to \mathcal{D}^{q+1}(M)$ is defined as the adjoint to the classical exterior derivative on smooth forms with compact support: $dT(\eta) = -T(d\eta)$.

On a complex manifold, one has two derivatives $\partial : \mathcal{D}^{p,q}(M) \to \mathcal{D}^{p+1,q}(M)$ and $\overline{\partial} : \mathcal{D}^{p,q}(M) \to \mathcal{D}^{p,q+1}(M)$, defined in the analogous way. However, the real operators $d = \partial + \overline{\partial}$ and $d^c = \frac{i}{2\pi}(\overline{\partial} - \partial)$

are more often used in dynamics. Currents that satisfy $dT = 0$ are referred to as *d-closed*. We will denote the d-closed currents of degree q by $Z^q(M)$, and when desiring to emphasize bi-degree, we will denote the d-closed currents of bi-degree p, q by $Z^{p,q}(M)$.

Given a smooth form $\psi \in A^q(M)$, there is a current $T_\psi \in \mathcal{D}^q(M)$ defined by $T_\psi(\eta) = \int_M \psi \wedge \eta$ for any $\eta \in A_c^{n-q}(M)$. Currents of this form are referred to as *smooth currents*. Using Stokes Theorem, one can check that $dT_\psi = T_{d\psi}$ so that the inclusion $A_c^{n-q}(M) \to \mathcal{D}^q(M)$ given by $\psi \mapsto T_\psi$ is a cochain map.

A piecewise smooth, oriented $(n-q)$ chain Γ in M also defines a current $T_\Gamma \in \mathcal{D}^q(M)$ given by $T_\Gamma(\eta) = \int_\Gamma \eta$ for any $\eta \in A_c^{n-q}(M)$. We will refer to currents that can be represented this way as *currents of integration*.

We will often use work with **closed, positive (1,1) currents**. These are $(1,1)$ currents that are locally expressed as $T = dd^c\phi$ for a plurisubharmonic function ϕ. (See the dd^c-Poincaré Lemma.) We denote the closed-positive $(1,1)$ currents on M by $Z_+^{1,1}(M)$.

Typically one cannot pull back a current under a ramified mapping F. One very special property of positive closed currents is that they can be pulled-back, using the potential function: $F^*(\lambda) := dd^c(\phi \circ F)$, where ϕ is a potential function for λ. When $\lambda = T_\eta$ is a smooth current, this pull-back coincides with the classical pull-back of smooth forms: $F^*(T_\eta) = T_{F^*(\eta)}$.

8.3 Linking with currents

The operator $d : \mathcal{D}^q(M) \to \mathcal{D}^{q+1}(M)$ satisfies $d \circ d = 0$ and we denote the corresponding cohomology theory by $H^*(\mathcal{D}^*(M), d)$. There is a natural map from the DeRham cohomology $H^*_{DR}(M)$ into $H^*(\mathcal{D}^*(M), d)$ induced by the inclusion of smooth forms into the currents.

Theorem 8.1. (Approximation by smooth currents) *The map $H^*_{DR}(M) \to H^*(\mathcal{D}^*(M), d)$ is an isomorphism. Furthermore, the cohomology class of any closed current L can be represented by a closed smooth form η_L with support in an arbitrarily small neighborhood of the support of L.*

See [28, pages 382-385] for a proof.

Given $T \in Z^2(M)$, and a piecewise smooth 2-chain σ having $\partial\sigma$ disjoint from the support of T, there is a pairing:

$$C_2(M) \times Z^2(M) \to \mathbb{R}$$

defined by $\langle \sigma, T \rangle = \int_\sigma \eta_T$ were η_T is a smooth form within the same cohomology class as T with support is bounded away from $\partial\sigma$. The existence of η_T is garunteed by the approximation by smooth currents, and the pairing is well defined since the integral depends only on the cohomology class of η_T.

When T is a current of integration integration over a piecewise smooth chain this pairing coincides with the usual intersection number of piecewise smooth chains and when T is given by a smooth form, it coincides, by definition, with the standard pairing $\int_\sigma T$. (In fact, our pairing is a special case of the general intersection number for closed currents of complimentary degrees [28, p. 392] and [17].)

Proposition 8.2. *If λ is a positive closed current and F is a ramified mapping, we have $\langle F_*\sigma, \lambda \rangle = \langle \sigma, F^*\lambda \rangle$.*

Proof: Let η_λ be a smooth approximation of λ in the same cohomology class. Then, $\langle F_*\sigma, \lambda \rangle = \int_{F_*\sigma} \eta_\lambda = \int_\sigma F^*\eta_\lambda = \langle \sigma, F^*\lambda \rangle$, since $F^*\eta_\lambda$ is a smooth approximation of $F^*\lambda$. $\square$

We define the linking kernel $\mathcal{L}Z^2(M)$ to be the space of closed currents T having $\langle \sigma, T \rangle = 0$ for every $\sigma \in H_2(M)$. Given $T \in \mathcal{L}Z^2(M)$, let $B_1^T(M)$ be the 1-boundaries in M that are disjoint from the support of T. We can define a linking number with respect to T

$$lk(\cdot, T) : B_1^T(M) \to \mathbb{R}$$

by $lk(c, T) = \langle \Gamma, T \rangle$, where Γ is any 2-chain with $\partial\Gamma = c$. Since $T \in \mathcal{L}Z^2(M)$, we have that $\langle \Gamma, T \rangle = \langle \Gamma', T \rangle$ for any other Γ' with $\partial\Gamma' = c$.

8.4 Finding an element of $\mathcal{L}Z^2(X_l^\infty)$

In this subsection, we will find an element of $\mathcal{L}Z^2(X_l^\infty)$ by successively determining elements of $\mathcal{L}Z^2(X_l)$, $\mathcal{L}Z^2(X_l^0)$, $\mathcal{L}Z^2(X_l^1)$, $\mathcal{L}Z^2(X_l^2), \cdots$ where X_k^j is the space X_k after having completed the blow-ups at level j. In the limit, we will find an element of $\mathcal{L}Z^2(X_l^\infty)$, which in the next subsection will be useful for linking.

Let L_1 be the invariant line that goes through $(0, 0)$ and $(1, 0)$, i.e. $y = 0$ and L_2 be the invariant line that goes through $(0, 1)$ and $(1, 1 - B)$, i.e. $y + Bx - 1 = 0$. (To remember the indexing, think that L_1 contains r_1 and L_2 contains r_2.) We can use the Poincaré-Lelong formula ([28, p. 388] or [49]) to express the fundamental classes of these lines as positive-closed currents:

$$[L_1] = \frac{1}{2\pi} dd^c \log |y|, \qquad [L_2] = \frac{1}{2\pi} dd^c \log |y + Bx - 1|.$$

Both L_1 and L_2 intersect any given vertical line $\mathbb{P}$ with intersection number 1. Because $[V]$ is the sole generator of $H_2(X_l)$ we have that $[L_2] - [L_1] \in \mathcal{L}Z^2(X_l)$.

Now, suppose that we want to find an element of $\mathcal{L}Z^2(X_l^0)$, that is, a closed 2 current that evaluates to 0 on every element of $H_2(X_l^0) \cong \mathbb{Z}^{\{[V],[E_p],[E_q]\}}$. In fact:

$$\langle E_p, [L_1] \rangle = 1 = \langle E_p, [L_2] \rangle \text{ and } \langle E_q, [L_1] \rangle = 0 = \langle E_q, [L_2] \rangle,$$

using standard intersection numbers for piecewise smooth chains, so that $[L_2] - [L_1] \in \mathcal{L}Z^2(X_l^0)$.

This luck will not continue. Let z be one of the two preimages of p that is in the invariant line L_1. Since L_1 and L_2 intersect at the single point p, this forces that $z \notin L_2$. Consequently: $\langle E_z, [L_1] \rangle = 1 \neq 0 = \langle E_z, [L_2] \rangle$ so that $[L_2] - [L_1] \notin \mathcal{L}Z^2(X_l^1)$.

We consider the k-th inverse images $N^{-k}(L_1)$ and $N^{-k}(L_2)$. If we denote by $N_1^k(x, y)$ and $N_2^k(x, y)$ the first and second coordinates of N^k, then the Poincaré-Lelong formula gives

$$\begin{aligned} [N^{-k}(L_1)] &= \frac{1}{2\pi} dd^c \log |N_2^k(x, y)|, \\ [N^{-k}(L_2)] &= \frac{1}{2\pi} dd^c \log |N_1^k(x, y) + B \cdot N_2^k(x, y) - 1|. \end{aligned}$$

Lemma 8.3. *For every $k \geq 0$ we have*

$$\left\langle V, [N^{-k}(L_1)] \right\rangle = \left\langle V, [N^{-k}(L_2)] \right\rangle$$

Proof: The k-th inverse images $N^{-k}(L_1)$ and $N^{-k}(L_2)$ both have degree 2^k in y, so they each intersect a generic vertical line transversely exactly 2^k times. These intersection numbers coincide with the pairings. □

Proposition 8.4. $[N^{-(k+1)}(L_2)] - [N^{-(k+1)}(L_1)] \in \mathcal{L}Z^2(X_l^k)$

Proof Let E_z be any one of the exceptional divisors in X_l^k. Using Proposition 4.6, there is some d and some $l \leq k + 1$ so that N^l maps E_z to V by a ramified cover of degree d (possibly with $d = 0$.) Just as in the discussion above:

$$\left\langle E_z, [N^{-(k+1)}(L_1)] \right\rangle = \left\langle N^l(E_z), [N^{-(k+1)+l} L_1] \right\rangle = d \left\langle V, [N^{-(k+1)+l} L_1] \right\rangle$$
$$\left\langle E_z, [N^{-(k+1)}(L_2)] \right\rangle = \left\langle N^l(E_z), [N^{-(k+1)+l} L_2] \right\rangle = d \left\langle V, [N^{-(k+1)+l} L_2] \right\rangle$$

Here we are using Proposition 8.2 to obtain the first equality in each equation. (One must check that the Poincaré-Lelong equation gives that $[N^{-(k+1)}(L_i)] = (N^l)^*[N^{-(k+1)+l}L_i]$ for $l \leq k+1$.) Then, Lemma 8.3 gives that the two terms on the right hand side of each equation are equal.

Since $H_2(X_l^k)$ is generated by the fundamental classes of V and the fundamental classes of each of the exceptional divisors E_z we conclude that $[N^{-(k+1)}(L_2)] - [N^{-(k+1)}(L_1)] \in \mathcal{L}Z^2(X_l^k)$. $\Box$

Since $X_l^\infty = \varprojlim(X_l^k, \pi)$ and $[N^{-(k+1)}(L_2)] - [N^{-(k+1)}(L_1)] \in \mathcal{L}Z^2(X_l^k)$ we expect that a limit as $k \to \infty$ of $[N^{-(k+1)}(L_2)] - [N^{-(k+1)}(L_1)]$ will be an element of $\mathcal{L}Z^2(X_l^\infty)$. For such a limit to converge we must normalize $[N^{-(k+1)}(L_2)]$ and $[N^{-(k+1)}(L_1)]$. Dividing by the degrees, we define:

$$\begin{aligned}
\lambda_1^k &= \frac{1}{2^k}[N^{-k}(L_1)] = \frac{1}{2\pi}dd^c\frac{1}{2^k}\log|N_2^k(x,y)|, \\
\lambda_2^k &= \frac{1}{2^k}[N^{-k}(L_2)] = \frac{1}{2\pi}dd^c\frac{1}{2^k}\log|N_1^k(x,y) + B\cdot N_2^k(x,y) - 1|
\end{aligned}$$

Let

$$\begin{aligned}
\lambda_1 &= \lim_{k\to\infty}\lambda_1^k = \frac{1}{2\pi}dd^c\lim_{k\to\infty}\frac{1}{2^k}\log|N_2^k(x,y)|, \\
\lambda_2 &= \lim_{k\to\infty}\lambda_2^k = \frac{1}{2\pi}dd^c\lim_{k\to\infty}\frac{1}{2^k}\log|N_1^k(x,y) + B\cdot N_2^k(x,y) - 1|.
\end{aligned}$$

We will first check that these limits exist and define positive-closed 1-1 currents, and then we will show that $\lambda_2 - \lambda_1 \in \mathcal{L}Z^2(X_l^\infty)$.

Proposition 8.5. *The limits*

$$\begin{aligned}
G_1(x,y) &= \lim_{k\to\infty}\frac{1}{2^k}\log|N_2^k(x,y)| \\
G_2(x,y) &= \lim_{k\to\infty}\frac{1}{2^k}\log|N_1^k(x,y) + B\cdot N_2^k(x,y) - 1|
\end{aligned}$$

converge and are plurisubharmonic functions in the basins of attraction $W(r_1)$ and $W(r_2)$, respectively. Hence, $\lambda_1 = \frac{1}{2\pi}dd^cG_1(x,y)$ and $\lambda_2 = \frac{1}{2\pi}dd^cG_2(x,y)$ are positive closed 1-1 currents on X_l^∞: $\lambda_1, \lambda_2 \in Z_+^{1,1}(X_l^\infty)$.

Proof: To see that $G_1(x,y)$ and $G_2(x,y)$ are well-defined and plurisubharmonic, we will show that $G_1(x,y)$ and $G_2(x,y)$ coincide with the potential functions that were described by Hubbard and Papadopol in [37, p. 21] and [38]. We will do this for $G_1(x,y)$, and leave necessary modifications for $G_2(x,y)$ to the reader.

Supposing that $(0,0)$ is a root, Hubbard and Papadopol [37] consider the limit

$$G_{HP}(x,y) = \lim_{k\to\infty}\frac{1}{2^k}\log||N^k(x,y)||$$

which they show converges to a plurisubharmonic function on the basin of $(0,0)$. The reader should notice that G_{HP} does not depend on the choice of the norm $||\cdot||$ used in the definition because any two different norms on a finite dimensional vector space are equivalent by a finite multiplicative constant, which is eliminated by the multiplicative factor of $\frac{1}{2^k}$. Therefore, we can use the supremum norm.

We will show that $G_1 = G_{HP}$ on $W(r_1)$, to see that G_1 is plurisubharmonic.

If $|N_2^k(x,y)| \geq |N_1^k(x,y)|$ for all (x,y) as $k\to\infty$, then the supremum norm coincides with $|N_2^k(x,y)|$ giving $G_1(x,y) = G_{HP}(x,y)$. This condition is equivalent to the condition:

$$\lim_{k\to\infty}\frac{1}{2^k}\log\left|\frac{N_2^k(x,y)}{N_1^k(x,y)}\right| \geq 0. \tag{19}$$

which will now show is a consequence of a standard result from the dynamics of one complex variable.

In [37], the authors perform blow-ups at each of the four roots, and observe that the Newton map N induces rational functions of degree 2 on each of the exceptional divisors $E_{r_1}, E_{r_2}, E_{r_3}$, and E_{r_4}. Let's compute the rational function $s : E_{r_1} \to E_{r_1}$. In the coordinate chart $m = \frac{y}{x}$, the extension to E_{r_1} is obtained by:

$$s(m) = \lim_{x\to 0} \frac{mx(Bx^2 + 2mx^2 - Bx - mx)}{x^2(Bx + 2mx - 1)} = m(B + m)$$

since $x = 0$ on E_{r_1}.

Since condition (19) is a limit, it suffices to check it in an arbitrarily small neighborhood of the origin. In a small enough neighborhood, we can replace $\frac{N_2^k(x,y)}{N_1^k(x,y)}$ with $s\left(\frac{y}{x}\right)$ obtaining

$$\lim_{k\to\infty} \frac{1}{2^k} \log \left| \frac{N_2^k(x,y)}{N_1^k(x,y)} \right| = \lim_{k\to\infty} \frac{1}{2^k} \log |s^k(m)| = G_s(m). \tag{20}$$

where $G_s(m)$ is the standard Green's function from one variable complex dynamics associated to the polynomial $s(m)$. This last equality is actually a delicate but well-known result that was proved by Brolin [16]. A more friendly proof is available in [49, Section 9].

Having the last equality, it is a standard result, for example see Milnor [42] pages 95 and 96, that $G_s(m) = 0$ on the filled Julia set $K(s)$ and that $G_s(m) > 0$ outside of $K(s)$.

This justifies the replacement of the supremum norm from G_{HP} by $|N_2^k(x,y)|$, and hence gives that $G_1(x,y) = G_{HP}(x,y)$. □

Corollary 8.6. *Let $s : E_{r_1} \to E_{r_1}$ be the polynomial induced by the Newton map N and let $G_s : E_{r_1} \to \mathbb{R}$ be its Green's function. In a sufficiently small neighborhood of r_1,*

$$G_1(x,y) = G_s\left(\frac{y}{x}\right) - \log\left|\frac{1}{x}\right|.$$

Proof: This comes directly from the algebra:

$$\begin{aligned} G_1(x,y) &= \lim_{k\to\infty} \frac{1}{2^k} \log |N_2^k(x,y)| = \lim_{k\to\infty} \frac{1}{2^k} \left(\log \left| \frac{N_2^k(x,y)}{N_1^k(x,y)} \right| + \log |N_1^k(x,y)| \right) \\ &= G_s\left(\frac{y}{x}\right) + \lim_{k\to\infty} \frac{1}{2^k} \log |N_1^k(x,y)| = G_s\left(\frac{y}{x}\right) + \log|x| = G_s\left(\frac{y}{x}\right) - \log\left|\frac{1}{x}\right| \end{aligned}$$

because $\frac{N_2^k(x,y)}{N_1^k(x,y)} \approx s\left(\frac{y}{x}\right)$ and $N_1^k(x,y) = \frac{x^2}{2x-1} \approx -x^2$ near r_1. □

8.5 Nice properties of λ_2 and λ_1:

In this subsection, we will prove some of the useful properties if λ_2 and λ_1. We will finish the subsection by showing that $\lambda_2 - \lambda_1 \in \mathcal{L}Z^2(X_l^\infty)$.

Lemma 8.7. (**Normalization**) *Suppose that $\mathbb{P}_x$ is a vertical line that is divided into exactly two simply connected domains $U \subset W(r_1)$ and $V \subset W(r_2)$ by W_0. Then:*

$$\langle V, \lambda_2 \rangle = 1 = \langle U, \lambda_1 \rangle \;\; \textit{and} \;\; \langle U, \lambda_2 \rangle = 0 = \langle V, \lambda_1 \rangle$$

Proof: Because $N_2^k(x,y)$ and $BN_1^k(x,y) + N_2^k(x,y) - 1$ are of degree 2^k in y, both λ_1^k and λ_2^k are normalized to that $\langle V, \lambda_1^k \rangle = 1$ and $\langle V, \lambda_2^k \rangle = 1$. Since the potentials for λ_1^k and λ_2^k converge to the potentials for λ_1 and λ_2, we have

$$\langle U, \lambda_1 \rangle = \left\langle U, \lim_{k\to\infty} \lambda_1^k \right\rangle = \lim_{k\to\infty} \langle U, \lambda_1^k \rangle = \lim_{k\to\infty} 1 = 1.$$

and similarly for λ_2. The proof that $\langle U, \lambda_2 \rangle = 0 = \langle V, \lambda_1 \rangle$ is identical. $\square$

Corollary 8.8. *Suppose that $\mathbb{P}_x$ is vertical line, then $\langle \mathbb{P}_x, \lambda_2 \rangle = 1 = \langle \mathbb{P}_x, \lambda_1 \rangle$.*

It follows directly from the definitions of λ_1 and λ_2 that $N^*(\lambda_1) = 2 \cdot \lambda_1$ and $N^*(\lambda_2) = 2 \cdot \lambda_2$. (For example $N^*(\lambda_1) = \frac{1}{2\pi} dd^c \lim_{k\to\infty} \frac{1}{2^k} \log |N_1^k \circ N| = \frac{1}{2\pi} dd^c \lim_{k\to\infty} \frac{1}{2^k} \log |N_1^{k+1}| = 2 \cdot \lambda_1$.) In combination with Proposition 8.2, this gives:

Lemma 8.9. (**Invariance**) *Suppose that Γ is a piecewise smooth 2-chain, then*

$$\langle N(\Gamma), \lambda_1 \rangle = 2 \cdot \langle \Gamma, \lambda_1 \rangle \qquad \langle N(\Gamma), \lambda_2 \rangle = 2 \cdot \langle \Gamma, \lambda_2 \rangle$$

Proposition 8.10. (**Support disjoint from** W_0) *There is a neighborhood Θ of W_0 in X_l^∞ which is disjoint from the support of λ_1 and λ_2.*

Proof: By construction, λ_1 has support in $\overline{W(r_1)}$ and λ_2 has support in $\overline{W(r_2)}$. We will find a neighborhood, which we also call Θ, of W_0 in $\overline{W(r_1)}$ that is disjoint from the support of λ_1. Clearly similar methods will work in $\overline{W(r_2)}$ and the desired neighborhood is the union of the two.

Recall from Corollary 8.6 that $G_1(x, y) = G_s\left(\frac{y}{x}\right) - \log\left|\frac{1}{x}\right|$, where G_s is the Green's function associated to the polynomial $s : E_{r_1} \to E_{r_1}$ induced by N at r_1. Recall that $s(m) = m(B + m)$ in the coordinates $m = \frac{x}{y}$ on E_{r_1}, so that $m = \infty$ is a superattracting fixed point. (This is the standard situation for a quadratic polynomial.)

It is a standard result from one-variable dynamics, for example see [42] p. 96, that G_s is harmonic outside of the Julia set $J(s)$. In particular, G_s is harmonic in a neighborhood of ∞ (not including ∞). A related standard result that G_s has the singularity

$$G(m) = \log |m| + O(1) \text{ as } m \to \infty$$

We check that this singularity exactly cancels with $-\log\left|\frac{1}{x}\right|$ coming from $G_1(x, y) = G_s\left(\frac{y}{x}\right) - \log\left|\frac{1}{x}\right|$:

$$\begin{aligned} G_1(x, y) &= \log\left|\frac{y}{x}\right| - \log\left|\frac{1}{x}\right| + O(1) \text{ as } \left|\frac{y}{x}\right| \to \infty \\ &= \log|y| + O(1) \text{ as } \left|\frac{y}{x}\right| \to \infty \end{aligned}$$

Therefore, $G_1(x, mx)$ is harmonic on a neighborhood U of $m = \infty$, including the point ∞. Choose $\theta > 0$ so that if $|m| > \theta$, then $G_1(x, mx)$ is harmonic.

Let $\Theta_0 = \{(x, y) \in \overline{W(r_1)}$ such that $|\frac{y}{x}| > \theta\}$. This is the open cone of points in $W(r_1)$ with slope to the origin greater than θ. Since the invariant circle S_0 is above $m = \infty$, Θ_0 is a neighborhood of S_0 (within $\overline{W(r_1)}$.)

By construction, $\Theta = \bigcup_{n=0}^{\infty} N^{-n}(\Theta_0)$ will be invariant under N and open. Because Θ_0 is disjoint from the support of λ_1, the invariance properties for λ_1 from Lemma 8.9 give that all of Θ must be disjoint from the support of λ_1.

Finally, since Θ_0 contains a neighborhood of S_0, and both W_0 and Θ are invariant under N, Θ forms an open neighborhood of W_0. $\square$

In fact, using the smooth approximation theorem, one can also choose the smooth approximations of λ_1 and λ_2 to have support bounded away from W_0.

Corollary 8.11. *Given any piecewise smooth chain $\sigma \in W_0$, we have that $\langle \sigma, \lambda_1 \rangle = 0$ and $\langle \sigma, \lambda_2 \rangle = 0$.*

Proposition 8.12. $\lambda_1 - \lambda_2 \in \mathcal{L}Z^2(X_l^\infty)$

Proof: This proof will be even simpler than the proof of Proposition 8.4 because we directly use the invariance of λ_1 and λ_2 shown in Lemma 8.9.

By Corollary 8.8, we have $\langle V, \lambda_1\rangle = \langle V, \lambda_2\rangle$. Any exceptional divisor E_z was created during the blow-ups at some level k, and using Proposition 4.6 there is some l so that $N^{\circ(k+1)}$ maps E_z to $V = \mathbb{P}_{1/(B(2-B))}$ by a ramified covering mapping of degree l, (possibly $l = 0$). Then:

$$\langle E_z, \lambda_1\rangle = \frac{l}{2^{k+1}} \langle V, \lambda_1\rangle = \frac{l}{2^{k+1}} \langle V, \lambda_2\rangle = \langle E_z, \lambda_2\rangle .$$

Hence $\langle E_z, \lambda_2 - \lambda_1\rangle = 0$ for any exceptional divisor E_z.

Since an element of $H_2(X_l^\infty)$ is a linear combination of the fundamental class $[V]$ with a finite number of fundamental classes of exceptional divisors E_z, we have shown that $\lambda_2 - \lambda_1 \in \mathcal{L}Z^2(X_l^\infty)$. $\square$

8.6 $H_1(W_0)$ is infinitely generated.

From Section 7 we have infinitely many cycles γ_i in W_0 of arbitrarily small "size," and we now have $(\lambda_2 - \lambda_1) \in \mathcal{L}Z^2(X_l^\infty)$ with which we can try to link them.

Since $H_1(X_l^\infty) = 0$, every 1-cycle in X_l^∞ is a 1-boundary in X_l^∞. In particular, $Z_1(W_0) \subset B_1(X_l^\infty)$. By Lemma 8.10, the support of $\lambda_2 - \lambda_1$ is disjoint from W_0, giving that $Z_1(W_0) \subset B_1^{\lambda_2-\lambda_1}(X_l^\infty)$. Hence, we can restrict $lk(\cdot, \lambda_2 - \lambda_1)$ to 1-cycles in W_0:

$$lk(\cdot, \lambda_2 - \lambda_1) : Z_1(W_0) \to \mathbb{R}$$

Proposition 8.13. *For every $\gamma \in Z_1(W_0)$, $lk(\gamma, \lambda_2 - \lambda_1)$ depends only on $[\gamma] \in H_1(W_0)$.*

Proof: Suppose that $\gamma_1 - \gamma_2 = \partial\sigma$, with $\sigma \in C_2(W_0)$. Then, Corollary 8.11 gives that $\langle \sigma, \lambda_2 - \lambda_1\rangle = 0$, hence $lk(\gamma_1, \lambda_2 - \lambda_1) = lk(\gamma_1, \lambda_2 - \lambda_1)$. $\square$

Proposition 8.14. *The image of $lk(\cdot, \lambda_2 - \lambda_1) : H_1(W_0) \to \mathbb{R}$ is contained in the rationals $\mathbb{Q}$.*

Proof: Recall from Section 3 that there is an $\epsilon > 0$ for which W_0 restricted to $|x| < \epsilon$ is homeomorphic to the product $S_0 \times \mathbb{D}_\epsilon$. Because any $\gamma \in Z_1(W_0)$ is compact, there exists a sufficiently high iterate N^k so that $N^k(\gamma)$ lies within $|x| < \epsilon$. Then $[N^k(\gamma)] = n \cdot [S_0]$ for some appropriate n. Using the invariance property, this gives $lk(\gamma, \lambda_2 - \lambda_1) = \frac{n}{2^k} \in \mathbb{Q}$. $\square$

From here on we will write $lk(\cdot, \lambda_2 - \lambda_1) : H_1(W_0) \to \mathbb{Q}$.

Proposition 8.15. *Suppose that γ_i is a curve in a vertical line bounded by a simply connected domain U_i. Then: $lk(\gamma_i, \lambda_2 - \lambda_1) = \mathrm{size}(U_i)$, where $\mathrm{size}(U_i)$ was defined in Section 7.*

Proof of Proposition 8.15: Recall that $\mathrm{size}(U_i)$ is defined as $\pm\frac{l_i}{2^k}$ where k is such that N^k maps to a vertical line $\mathbb{P}_x$ that is divided by W_0 into only two domains $U \subset W(r_1)$ and $V \subset W(r_2)$ and where l_i is the degree of this mapping to U or V. The sign is $-$ if U_i is mapped to U and $+$ if U_i is mapped to V. Without loss in generality, suppose that U_i is mapped to U, and hence $\mathrm{size}(U_i) < 0$. Using Lemma 8.9 we have that:

$$\langle U_i, \lambda_2 - \lambda_1\rangle = \frac{1}{2^k} \left\langle N^k(U_i), \lambda_2 - \lambda_1\right\rangle = \frac{1}{2^k} \langle l_i U, -\lambda_1\rangle \quad = \quad -\frac{l_i}{2^k} \langle U_i, \lambda_1\rangle = -\frac{l_i}{2^k} = \mathrm{size}(U_i)$$

where we are using that $\langle U, \lambda_2\rangle = 0$ and $\langle U, \lambda_1\rangle = 1$. $\square$

Corollary 8.16. *The image of the homomorphism $lk(\cdot, \lambda_2 - \lambda_1) : H_1(W_0) \to \mathbb{Q}$ contains elements of arbitrarily small, but non-zero, absolute value.*

This gives us our desired result:

Corollary 8.17. *The homology group $H_1(W_0)$ is infinitely generated.*

Notice that an additive subgroup of $\mathbb{Q}$ that is dense must be infinitely generated, but a dense additive subgroup of $\mathbb{R}$ typically is not infinitely generated because the generators can be incommensurable.

Recall the Mayer-Vietoris exact sequence (14):

$$H_2\left(\overline{W(r_1)}\right) \oplus H_2\left(\overline{W(r_2)}\right) \to H_2(X_l^\infty) \xrightarrow{\partial} H_1(W_0) \to H_1\left(\overline{W(r_1)}\right) \oplus H_1\left(\overline{W(r_2)}\right) \to 0$$

If Image$(\partial) = 0$, or even if we knew that $|\text{size}(\partial(\sigma))|$ were bounded away from 0 for every $\sigma \in H_2(X_l^\infty)$, we would be able to conclude that $H_1\left(\overline{W(r_1)}\right)$ and $H_1\left(\overline{W(r_2)}\right)$ are infinitely generated. However, this is not the case.

Proposition 8.18. *There are $\sigma \in H_2(X_l^\infty)$ with $|lk(\partial(\sigma), \lambda_2 - \lambda_1)| > 0$ arbitrarily small.*

Proof: For every k, there exists some exceptional divisor E having $N^k : E \to V$ an isomorphism. For generic parameter values $B \in S$, any exceptional divisor at a $(k-1)$-st inverse image of p will have this property, since, for generic B there is a single exceptional divisor above each point that we have blown up, and $N : E_z \to E_{N(z)}$ is always an isomorphism.

For the values of $B \notin S$, there may be many blow-ups done at each $(k-1)$-st inverse image of p. We take a detailed look at the sequence of blow-ups from Section 4.1 that was used to create X_l^{k-1} from X_l^{k-2}. One must check that for each exceptional divisor $E^i_{N(z)}$ that occurs in the sequence of blow-ups at $N(z)$, there is an exceptional divisor in the sequence of blow-ups at z that maps isomorphically to $E^i_{N(z)}$. Therefore, for any k, one can find an exceptional divisor E so that $N^{k-1} : E \to E_p$ is an isomorphism. Since $N : E_p \to V$ is always an isomorphism, E is the desired exceptional divisor.

Because N^k maps E isomorphically to V, it maps $\partial([E])$ to $\partial([V])$. The invariance property from Lemma 8.9 gives that

$$lk(\partial([E]), \lambda_2 - \lambda_1) = \frac{1}{2^k} lk(\partial([V]), \lambda_2 - \lambda_1) = \frac{1}{2^k}.$$

$\square$ Proposition 8.18.

8.7 $H_1\left(\overline{W(r_1)}\right)$ and $H_1\left(\overline{W(r_2)}\right)$ are infinitely generated.

The following idea will allow us to show that $H_1\left(\overline{W(r_1)}\right)$ and $H_1\left(\overline{W(r_2)}\right)$ are infinitely generated, despite the fact that $|lk(\partial(\sigma), \lambda_2 - \lambda_1)|$ can be arbitrarily small, but non-zero, for $\sigma \in H_2(X_l^\infty)$.

Even and odd parts of Homology:

Recall from Proposition 1.6 that N has a symmetry of reflection τ about the line $Bx + 2y - 1 = 0$ which exchanges the basins of attraction. This τ induces an involution τ_* on $H_*(X_l^\infty), H_*(W_0)$, and $H_*(W(r_1)) \oplus H_*(W(r_2))$. Every homology class σ will have $\tau_*^2(\sigma) = \sigma$ and consequently the eigenvalues of τ are ± 1.

We say that a homology class σ *is even* if it is in the eigenspace of τ_* corresponding to eigenvalue $+1$, and we say that σ *is odd* if it is in the eigenspace of τ_* corresponding to eigenvalue -1.

Because the Mayer-Vietoris exact sequence commutes naturally with induced maps, we have a decomposition of the sequence (14) into even and odd parts. We will only need the odd part:

$$(H_2(\overline{W(r_1)}) \oplus H_2(\overline{W(r_2)}))^{\text{odd}} \to H_2^{\text{odd}}(X_l^\infty) \xrightarrow{\partial} H_1^{\text{odd}}(W_0) \to (H_1(\overline{W(r_1)}) \oplus H_1(\overline{W(r_2)}))^{\text{odd}} \to 0$$

Lemma 8.19. *If σ is some piecewise smooth chain, then: $\langle \sigma, \lambda_2 \rangle = \langle \tau(\sigma), \lambda_1 \rangle$ and $\langle \sigma, \lambda_1 \rangle = \langle \tau(\sigma), \lambda_2 \rangle$.*

Proof:
Recall the definition of λ_2 and λ_1:

$$\begin{aligned}\lambda_1 &= \frac{1}{2\pi} dd^c \lim_{k\to\infty} \frac{1}{2^k} \log |N_2^k(x,y)|,\\ \lambda_2 &= \frac{1}{2\pi} dd^c \lim_{k\to\infty} \frac{1}{2^k} \log |N_1^k(x,y) + B\cdot N_2^k(x,y) - 1|.\end{aligned}$$

Since precomposition with τ exchanges the line $Bx + y - 1 = 0$ with the line $y = 0$, Equation 8.19 holds. □

Corollary 8.20. *For every $[\gamma] \in H_1(W_0)$ we have: $lk(\gamma, \lambda_2 - \lambda_1) = -lk(\tau(\gamma), \lambda_2 - \lambda_1)$.*

Proof:
Suppose that σ is a piecewise smooth 2-chain with $\partial\sigma = \gamma$. Then we certainly have $\partial(\tau(\sigma)) = \tau(\gamma)$. Lemma 8.19 gives:

$$lk(\gamma, \lambda_2 - \lambda_1) = \langle \sigma, \lambda_2 - \lambda_1\rangle = \langle \tau(\sigma), \lambda_1 - \lambda_2\rangle = -\langle \tau(\sigma), \lambda_2 - \lambda_1\rangle = -lk(\tau(\gamma), \lambda_2 - \lambda_1)$$

□

Proposition 8.21. *If $\gamma \in H_1^{\mathrm{odd}}(W_0)$ is in the image of the boundary map $\partial : H_2^{\mathrm{odd}}(X_l^\infty) \to H_1^{\mathrm{odd}}(W_0)$, then $lk(\gamma, \lambda_2 - \lambda_1) = 0$.*

We first need the following lemma:

Lemma 8.22. *For any exceptional divisor E_z we have*

$$\partial(\tau_*([E_z])) = -\tau_*(\partial([E_z])) \tag{21}$$

Proof: This proof will depend *essentially* on the explicit interpretation of the boundary map ∂ from the Mayer-Vietoris sequence. In the following paragraph we closely paraphrase Hatcher [31], p. 150:

The boundary map $\partial : H_n(X) \to H_{n-1}(A \cap B)$ can be made explicit. A class $\alpha \in H_n(X)$ is represented by a cycle z. By appropriate subdivision, we can write z as a sum $x + y$ of chains in A and B, respectively. While it need not be true that x and y are cycles individually, we do have $\partial x = -\partial y$ since $z = x + y$ is a cycle. The element $\partial\alpha$ is represented by the cycle $\partial x = -\partial y$.

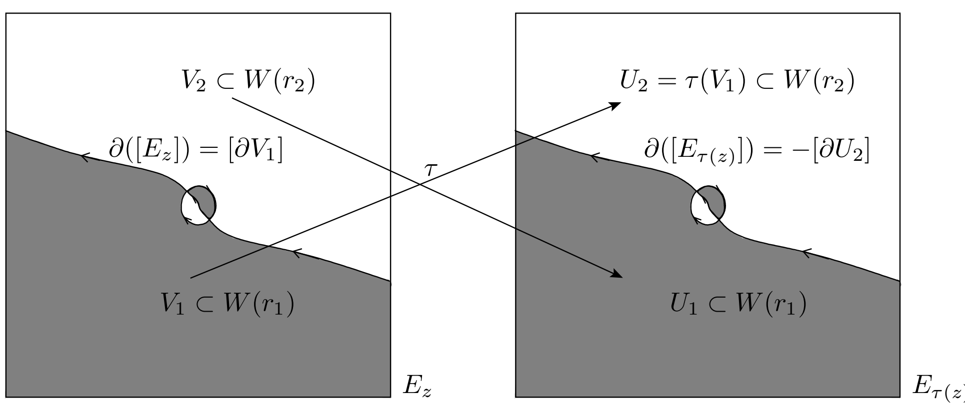

Figure 12: Showing that $\partial(\tau_*([E_z])) = -\tau_*(\partial([E_z]))$.

We use this explicit interpretation of ∂ to check Equation 21. Notice that $\tau_*([E_z]) = [E_{\tau(z)}]$ consistent with the orientation that E_z and $E_{\tau(z)}$ have as Riemann surfaces. Therefore we have that $\partial(\tau_*([E_z])) = \partial([E_{\tau(z)}]) = [\partial U_1] = -[\partial U_2]$, where U_1 is the oriented region of $E_{\tau(z)}$ that is in $\overline{W(r_1)}$ and U_2 is the oriented region of $E_{\tau(z)}$ that is in $\overline{W(r_2)}$.

Similarly $\partial([E_z]) = [\partial V_1] = -[\partial V_2]$, where V_1 and V_2 are $E_z \cap \overline{W(r_1)}$ and $E_z \cap \overline{W(r_2)}$. Because τ maps E_z to $E_{\tau(z)}$ swapping $\overline{W(r_1)}$ with $\overline{W(r_2)}$ we have:

$$\tau_*(\partial([E_z])) = [\partial U_2] = -\partial(\tau_*([E_z]))$$

□

Proof of Proposition 8.21:
Since elements of the form $[E_z] - [\tau(E_z)]$ span $H_2^{\text{odd}}(X_l^\infty)$, we need only check that the images of differences like this under ∂ have 0 linking number:

$$\begin{aligned} lk(\partial([E_z] - [\tau(E_z)]), \lambda_2 - \lambda_1) &= lk(\partial([E_z]) - \partial(\tau_*([E_z])), \lambda_2 - \lambda_1) \\ &= lk(\partial([E_z]) + \tau_*(\partial([E_z])), \lambda_2 - \lambda_1) = 0 \end{aligned}$$

The last term is 0 by Lemma 8.22. □

Proposition 8.23. *The image of $lk(\cdot, \lambda_2 - \lambda_1) : H_1^{\text{odd}}(W_0) \to \mathbb{Q}$ contains elements of arbitrarily small, but non-zero absolute value.*

Proof of Proposition 8.23:
Recall from Proposition 8.16 that we can find 1-cycles γ that have $lk(\gamma, \lambda_2 - \lambda_1)$ arbitrarily small, but non-zero. Notice that $[\gamma - \tau(\gamma)]$ is obviously odd, and using Lemma 8.22:

$$\begin{aligned} lk(\gamma - \tau(\gamma), \lambda_2 - \lambda_1) &= lk(\gamma, \lambda_2 - \lambda_1) - lk(\tau(\gamma), \lambda_2 - \lambda_1) \\ &= lk(\gamma, \lambda_2 - \lambda_1) + lk(\gamma, \lambda_2 - \lambda_1) = 2lk(\gamma, \lambda_2 - \lambda_1). \end{aligned}$$

Hence, by choosing γ so that $lk(\gamma, \lambda_2 - \lambda_1)$ is arbitrarily small, but non-zero, we can make $lk(\gamma - \tau(\gamma), \lambda_2 - \lambda_1)$ arbitrarily small, but non-zero with $[\gamma - \tau(\gamma)] \in H_1^{\text{odd}}(W_0)$. □

Recall the last part of the exact sequence on the odd parts of homology:

$$\to H_2^{\text{odd}}(X_l^\infty) \xrightarrow{\partial} H_1^{\text{odd}}(W_0) \xrightarrow{i_{1*} \oplus i_{2*}} \left(H_1\left(\overline{W(r_1)}\right) \oplus H_1\left(\overline{W(r_2)}\right)\right)^{\text{odd}} \to 0$$

where i_1 and i_2 are the inclusions $W_0 \hookrightarrow \overline{W(r_1)}$and $W_0 \hookrightarrow \overline{W(r_2)}$ respectively.

As a consequence of Proposition 8.21, given any $\eta \in \left(H_1\left(\overline{W(r_1)}\right) \oplus H_1\left(\overline{W(r_2)}\right)\right)^{\text{odd}}$ we can define $lk(\eta, \lambda_2 - \lambda_1) = lk(\gamma, \lambda_2 - \lambda_1)$ for any $\gamma \in H_1^{\text{odd}}(W_0)$ whose image under $i_{1*} \oplus i_{2*}$ is η. As a consequence of Proposition 8.23 we know that there are $\eta \in \left(H_1\left(\overline{W(r_1)}\right) \oplus H_1\left(\overline{W(r_2)}\right)\right)^{\text{odd}}$ with arbitrarily small $|lk(\eta, \lambda_2 - \lambda_1)|$. This proves the desired result:

Theorem 8.24. *Let $\overline{W(r_1)}$ and $\overline{W(r_2)}$ be the closures in X_l^∞ of the basins of attraction of the roots $r_1 = (0,0)$ and $r_2 = (0,1)$ under the Newton Map N. Then $H_1\left(\overline{W(r_1)}\right)$ and $H_1\left(\overline{W(r_2)}\right)$ are infinitely generated.*

Recall also:

Corollary 8.25. *For parameter values $B \in \Omega_r$, we can replace $\overline{W(r_1)}$ and $\overline{W(r_2)}$ with $W(r_1)$ and $W(r_2)$ finding that $H_1(W(r_1))$ and $H_1(W(r_2))$ are also infinitely generated.*

8.8 Linking with currents in X_r

Much of the work in the previous few subsections was to make linking numbers well defined in X_l^∞, overcoming the indeterminacy from the fact that $H_2(X_l^\infty)$ is infinitely generated. Because $H_2(X_r) \cong \mathbb{Z}^{\{[\mathbb{P}]\}}$ it is relatively easy to find elements in $\mathcal{L}Z_2(X_r)$. However, we can just mimic the work from the previous sub-sections in an appropriate way.

The major difference is that in X_l^∞ there is always an intersection of W_0 with C resulting in loops in W_0 of arbitrarily small size. In X_r, we must stipulate that an intersection of W_1 with C exists before proving that the homology is infinitely generated, because there appear to be parameter values for which there is no intersection.

If we define λ_3 and λ_4 in a similar way as we defined λ_1 and λ_2, then the following are proven in an easy way:

Proposition 8.26. *If W_1 intersects the critical value parabola C, then $H_1(W_1)$ is infinitely generated.*

Since there is only one generator of $H_2(X_r)$ this directly gives:

Theorem 8.27. *If W_1 intersects the critical value parabola C, then $H_1\left(\overline{W(r_3)}\right)$ and $H_1\left(\overline{W(r_4)}\right)$ are infinitely generated.*

where $\overline{W(r_3)}$ and $\overline{W(r_4)}$ are the closures in X_r of the basins of attraction of roots $r_3 = (1,0)$ and $r_4 = (1, 1-B)$ under N.

Corollary 8.28. *For parameter values $B \in \Omega_{reg}$, we can replace $\overline{W(r_1)}$ and $\overline{W(r_2)}$ with $W(r_1)$ and $W(r_2)$ finding that $H_1(W(r_1))$ and $H_1(W(r_2))$ are also infinitely generated.*

This is the last part of the proof of Theorem 0.1. $\square$

A Blow-ups of complex surfaces at a point.

Further material is available in [28, pp. 182-189 and 473-478] and the introduction of [33].

Suppose that $R : \mathbb{C}^2 \to \mathbb{C}^2$ has a point of indeterminacy at $(0,0)$. Blowing-up at $(0,0)$ produces a new space:

$$\widetilde{\mathbb{C}}^2_{(0,0)} = \left\{(z,l) \in \mathbb{C}^2 \times \mathbb{P}^1 : z \in l\right\} \tag{22}$$

to which we can often find an extension $R : \widetilde{\mathbb{C}}^2_{(0,0)} \to \mathbb{C}^2$ with no indeterminacy. Here $\mathbb{P}^1$ is identified with the space of directions through $(0,0)$ in $\mathbb{C}^2$. There is a natural projection $\rho : \widetilde{\mathbb{C}}^2_{(0,0)} \to \mathbb{C}^2$ given by $\rho(z,l) = z$ and the set $E_{(0,0)} = \rho^{-1}((0,0))$ is referred to as the *exceptional divisor.* A standard check shows that the blow-up is independent of the choice of coordinates hence well defined on a complex surface M at a point z.

A rational map $R : \mathbb{C}^2 \to \mathbb{C}^2$ can be lifted to a new rational mapping $\widetilde{R} : \widetilde{\mathbb{C}}^2_{(0,0)} - E_{(0,0)} \to \mathbb{C}^2$ be defining $R(z,l) = R(z)$ for $z \neq 0$. If the indeterminacy in R at $(0,0)$ was reasonably tame, $\widetilde{R}$ extends by continuity to all of $E_{(0,0)}$. Otherwise, there will be points of indeterminacy of $\widetilde{R}$ on $E_{(0,0)}$ to which $\widetilde{R}$ cannot be extended. One can try further blow-ups at these points to resolve these new points of indeterminacy. The extension of $\widetilde{R}$ to $E_{(0,0)}$ is analytic except at any new points of indeterminacy because $E_{(0,0)}$ is a space of complex co-dimension 1.

Proposition A.1. *If M is a complex surface and z is any point in M, then the blow-up $\widetilde{M}_z$ has $H_2(\widetilde{M}_z) \cong H_2(M) \oplus \mathbb{Z}^{(\{[E_z]\})}$ and $H_i(\widetilde{M}_z) \cong H_i(M)$ for $i \neq 2$.*

The proof is an application of the Mayer-Vietoris sequence on homology and the fact that $\widetilde{\mathbb{C}}^2_{(0,0)}$ has the homotopy type of $\mathbb{P}$.

Further analysis shows that the fundamental class $[E_z]$ of an exceptional divisor has self-intersection number -1. Meanwhile, blowing up a smooth point on any complex curve C decreases the self-intersection of its homology class $[C]$ by one. (See [28], for proof.)

B Proof of Theorem 4.1

Let $S \subset \Omega$ be the set of parameter values B for which no inverse image of the point of indeterminacy p or the point of indeterminacy q is in the critical value locus C. We are especially interested in $B \in S$ because the sequence of blow-ups from Section 4.1 is especially easy to describe for these B.

Theorem 4.1 states that S is generic in the sense of Baire's Theorem, i.e. uncountable and dense in Ω. The proof will follow as a corollary to:

Theorem. *(***Baire***) Let X be either a complete metric space, or a locally compact Hausdorf space. Then, the intersection of any countable family of dense open sets in X is dense.*

See Bredon [13] for a proof of Baire's Theorem.

Proof of Theorem 4.1: Let $S_n \subset \mathbb{C}$ be the subset of parameter values B for which none of the n-th inverse images of p or q under N are in the critical value locus C.

Lemma B.1. *S_n is a dense open set in $\mathbb{C}$*

Proof: Let R_n be the set of B for which an n-th inverse image of p is in C and let T_n be the set of B for which an n-th inverse image of q in C. We will show that R_n and T_n are finite, so that $S_n = \Omega - (R_n \cup T_n)$ is a dense open set.

Lemma B.2. *T_n is a finite set.*

Proof: We have $B \in T_n$ if:

$$y^2 + Bxy + \frac{B^2}{4}x^2 - \frac{B^2}{4}x - y = 0, \qquad N_1^n(x,y) = \frac{1}{2-B}, \qquad N_2^n(x,y) = \frac{1-B}{2-B} \tag{23}$$

has a solution. As always, N_1^n and N_2^n denote the first and second coordinates of N^n. By clearing the denominators in the second and third equations, condition 23 can be expressed as the common zeros of 3 polynomials $P_1(x,y,B)$, $P_2(x,y,B)$, and $P_3(x,y,B)$ in the three variables x, y, and B. We will check there is no common divisor of $P_1(x,y,B), P_2(x,y,B)$, and $P_3(x,y,B)$ so that the solutions to 23 form a finite set.

First, notice that $P_1(x,y,B) = y^2+Bxy+\frac{B^2}{4}x^2-\frac{B^2}{4}x-y$ is irreducible. It is sufficient to write an explicit biholomorphic map from $\mathbb{C}^2$ to $\{P_1 = 0\} \subset \mathbb{C}^3$. At a given B, the line $Bx+2y=t$ intersects $\{P_1 = 0\}$ at a single point which we denote by $f_B(t)$. It is easy to check that $(t,B) \mapsto (f_B(t), B)$ provides the desired isomorphism.

Hence P_1 has a factor in common with P_2 or P_3 if and only if P_1 divides P_2 or P_3. We will show that this is impossible by examining the lowest degree terms of P_2 and P_3. If P_1 divides P_2 or P_3, then the lowest degree term, $-y$, of P_1 must divide the lowest degree term of P_2 or the lowest degree term of P_3.

We check by induction that the lowest degree term of P_2 is ± 1 for every n. To simplify notation, let $a_k(x,y,B)$ be the polynomial obtained by clearing the denominators from $N_1^k(x,y) = \frac{1}{2-B}$.

By clearing denominators of $N_1(x,y) = \frac{1}{2-B}$, we find $a_1(x,y,B) = x^2(2-B)-1(2x-1) = 2x^2 - Bx^2 - 2x + 1$, so $a_1(x,y,B)$ has constant term ± 1. Now suppose that $a_n(x,y,B)$ has constant term ± 1. By definition, $a_{n+1}(x,y,B)$ is obtained by clearing the denominators of $a_n(N_1(x,y), N_2(x,y), B) =$

0. Because the denominators of both $N_1(x,y)$ and $N_2(x,y)$ have constant term ± 1 and because $a_n(x,y,B)$ has constant term 1 we find that $a_{n+1}(x,y,B)$ has constant term ± 1.

Because P_2 has constant term ± 1 for every n P_1 cannot divide P_2, and we conclude that there are no common factors between P_1 and P_2.

A nearly identical proof by induction shows that lowest degree term of P_3 is also ± 1 for each n. Hence P_1 does not divide P_3, and we conclude that P_1 and P_3 have no common divisors.

To see that P_2 and P_3 have no common divisors, notice that $P_2(x,y,B)=0$ is an equation for many disjoint vertical lines, while $P_3(x,y,B)=0$ stipulates that the n-th image of this locus has constant $y=0$. Since vertical lines are mapped to vertical lines by N, P_2 and P_3 can have no common factors.

Hence, P_1, P_2, and P_3 are algebraically independent, so they have a finite number of common zeros, giving that T_n is a finite set. □ Lemma B.2.

Lemma B.3. *R_n is a finite set.*

Proof: Now we show that R_n, the set of B so that an n-th inverse image of p under N is in C, is finite. In terms of equations, R_n is the set of B so that:

$$y^2 + Bxy + \frac{B^2}{4}x^2 - \frac{B^2}{4}x - y = 0, \qquad N_1^n(x,y) = \frac{1}{B}, \qquad N_2^n(x,y) = 0 \tag{24}$$

has a solution. Let Q_1, Q_2, and Q_3 be the polynomials equations resulting from clearing the denominators in Equation 24.

The proof is the same as for T_n except that a different proof is needed to see that Q_1 does not divide Q_3. An adaptation of the proof that P_1 does not divide P_3 fails because the lowest degree term of Q_3 has positive degree in y. We will check that Q_1 does not divide Q_3 and leave the remainder of the proof to the reader.

The x-axis, $y=0$, is one of the invariant lines of N and it intersects the basins $W(r_1)$, $W(r_3)$ and the separator $\mathrm{Re}(x)=1/2$. Therefore it is disjoint from the two basins $W(r_2)$ and $W(r_4)$. By definition, $Q_3(x,y,B)$ is the equation for the n-the inverse image of the x-axis. So, for a given B, the locus $Q_3(x,y,B)=0$ is also disjoint from the two basins $W(r_2)$ and $W(r_4)$.

For every B, the critical value parabola C goes through the four roots r_1, r_2, r_3, and r_4, so it intersects all four basins of attraction. By definition, C is the zero locus $Q_1(x,y,B)=0$. Therefore, if Q_1 divides Q_3, there is a component of the zero locus $Q_3(x,y,B)=0$ intersecting all four basins $W(r_1)$, $W(r_2)$, $W(r_3)$ and $W(r_4)$ for every B. This is impossible, so Q_1 cannot divide Q_3. □ Lemma B.3 and □ Lemma B.1.

Since S_n is a dense open set in Ω for each n and $S = \cap_{n=0}^{\infty} S_n$, so it follows from Baire's Theorem that S is uncountable and dense in the parameter space Ω. □ Theorem 4.1.

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