TWO-DIMENSIONAL BLASCHKE PRODUCTS: DEGREE GROWTH AND ERGODIC CONSEQUENCES

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Abstract. We study the dynamics of Blaschke products in two dimensions, particularly the rates of growth for the degrees of iterates and the corresponding implications for the ergodic properties of the map.

1. Introduction

For dominant rational maps of compact, complex, Kahler manifolds there is a conjecture specifying the expected ergodic properties of the map depending on the relationship between the rates of growth for certain degrees under iteration of the map. (See Conjecture 1.1, as presented in [16], as well as the results towards this conjecture [17, 6, 7, 8].) We observe that the two-dimensional Blaschke products fit naturally within this conjecture, having examples from each of the three cases that the conjecture gives for maps of a surface. We then consider the dynamics of Blaschke products from these dramatically distinct cases, relating it to the behavior predicted by this conjecture.

Furthermore, generic (in an appropriate sense) Blaschke products do not have the Julia set contained within $\mathbb{T}^2$. Rather, "the majority of it" is away from $\mathbb{T}^2$ within the support of the measure of maximal entropy. This is very different from the case of 1-dimensional Blaschke products for which the Julia set is the unit circle (see below).

A (finite) Blaschke product is a map of the form

$$E(z) = \theta_0 \prod_{i=1}^{n} \frac{z - e_i}{1 - z\overline{e_i}}, \tag{1}$$

where $n \geq 2$, $e_i \in \mathbb{C}$ for each $i = 1, \ldots, n$, and $\theta_0 \in \mathbb{C}$ with $|\theta_0| = 1$. The simplest dynamical situation occurs if one restricts that $|e_i| < 1$ for $i = 1 \ldots n$. It implies that the Julia set J_E is contained within the unit circle $\mathbb{T}^1$.

In this paper we study Blaschke products in two variables generalizing this situation. Let

$$f(z,w) = \left(\theta_1 \prod_{i=1}^{m} \frac{z - a_i}{1 - \bar{a}_i z} \prod_{i=1}^{n} \frac{w - b_i}{1 - \bar{b}_i w}, \theta_2 \prod_{i=1}^{p} \frac{z - c_i}{1 - \bar{c}_i z} \prod_{i=1}^{q} \frac{w - d_i}{1 - \bar{d}_i w}\right), \tag{2}$$

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with $|\theta_1| = |\theta_2| = 1$ and all of the zeros $a_1, \ldots, d_q$ of modulus less than one. We will often denote the corresponding 1-variable Blaschke products by $A(z), B(w), C(z)$, and $D(w)$. Such maps were introduced in [23].

Note that if one allows some of the zeros e_i of a one-variable Blaschke product (1) to have modulus greater than 1, a much more complicated structure for the Julia set can occur [24]. We do not consider the 2-variable analog of that situation in this paper, but we expect that it may be interesting for further study.

We describe the degrees of a given Blaschke product f by a matrix

$$N = \begin{bmatrix} m & n \\ p & q \end{bmatrix}.$$

(As in [23], we assume that m, n, p, and q are greater than or equal to 1).

Given any matrix of degrees N, any choice of rotations θ_1, θ_2, and any zeros $a_1, \ldots, a_m$, $b_1, \ldots, b_n$, $c_1, \ldots, c_p$, and $d_1, \ldots, d_q$ (all of modulus less than 1) there is a Blaschke product. We denote the space of all such Blaschke products by $\mathcal{B}_N$ and we will call any $f \in \mathcal{B}_N$ a *Blaschke Product associated to* N. We typically will use the notation $\sigma \in \mathbb{D}^{m+n+p+q}$ to represent the collection of zeros $a_1, \ldots, d_q$. Notice that $\mathcal{B}_N$ can be identified with $\mathbb{D}^{m+n+p+q} \times \mathbb{T}^2$, an identification that we use when discussing sets of full measure on $\mathcal{B}_N$.

In the case that all of the zeros are equal to 0, a 2-dimensional Blaschke product becomes a *monomial map*

$$f(z, w) = (z^m w^n, z^p w^q), \tag{3}$$

whose dynamics was studied extensively in [13, 18]. For any N we will call this map the *monomial map associated to* N. (It is also interesting to note that monomial maps occur frequently "outside of dynamical systems", for example in the description of cusps for Inoue-Hirzebruch surfaces [12]).

One nice reason to study Blaschke products is that they preserve the unit torus $\mathbb{T}^2 := \{(z, w) \;:\; |z| = |w| = 1\}$. The monomial map associated to N induces a linear map on $\mathbb{T}^2$. If $\det N > 0$, this is an orientation preserving local diffeomorphism of topological degree $\det N$. (The topological degree is the number of preimages of a generic point for $f_{|\mathbb{T}^2}$). *Throughout the paper we will assume* $\det N > 0$. Furthermore, the action on $\pi_1(\mathbb{T}^2)$ is described by N, in terms of the obvious choice of generators.

Any $f \in \mathcal{B}_N$ is homotopic on $\mathbb{T}^2$ to this monomial map and therefore has the same action on $\pi_1(\mathbb{T}^2)$ and the same topological degree. However, it may fail to be a local diffeomorphism.

We will often consider the special case in which $f_{|\mathbb{T}^2}$ is an orientation preserving diffeomorphism of $\mathbb{T}^2$. We call such an f as a *Blaschke product diffeomorphism* (although generally it is only a diffeomorphism on $\mathbb{T}^2$, not globally on $\mathbb{P}^2$). Blaschke product diffeomorphisms were studied extensively in [23] and they can only occur if $\det N = 1$. The corresponding monomial map induces a linear Anosov map on $\mathbb{T}^2$ and a Blaschke product whose zeros are sufficiently small will also be a Blaschke product diffeomorphism, inducing an Anosov map on $\mathbb{T}^2$. For any Blaschke product diffeomorphism, the restriction $f|_{\mathbb{T}^2}$ has topological entropy $\log(c_+(N))$, where $c_+(N)$ is the largest eigenvalue of N. There is also a unique invariant measure μ_{tor} of maximal entropy for $f_{|\mathbb{T}^2}$. (See Appendix A.)

A rational map $g : \mathbb{P}^2 \to \mathbb{P}^2$ can be lifted to a system of three homogeneous equations on $\mathbb{C}^3$ having no common factors. The *algebraic degree* $d_{\mathrm{alg}}(g)$ is the common degree of these homogeneous equations. In some cases, the degree of iterates drops, $d_{\mathrm{alg}}(g^n) < (d_{\mathrm{alg}}(g))^n$, because a common factor appears in the homogeneous equations for g^n. (See [14]). However, a limiting degree called *the first dynamical degree*

$$\lambda_1(g) = \lim_{n\to\infty} (d_{\mathrm{alg}}(g^n))^{1/n} \tag{4}$$

always exists, describing the asymptotic rate of growth in the sequence $\{d_{\mathrm{alg}}(g^n)\}$, [25]. Note that $\lambda_1(g) \le d_{\mathrm{alg}}(g)$. In §2 we briefly describe a common technique for computing $\lambda_1(f)$.

The ergodic properties of g are believed, see [16, Conj. 3.2], to depend heavily on the relationship between $\lambda_1(g)$ and the topological degree $d_{\mathrm{top}}(g)$. (Here, $d_{\mathrm{top}}(g)$ is defined as the number of preimages under g of a generic point from $\mathbb{P}^2$.) Actually, the conjecture stated in [16] is far more general, pertaining to dominant maps of Kahler manifolds X of arbitrary dimension. We provide a brief summary in the case that X is a surface.

Conjecture 1.1. *The ergodic properties of a rational map $g : X \to X$ are believed to fall into three cases:*

- **Case I: Large topological degree:** $d_{\mathrm{top}}(g) > \lambda_1(g)$. *This case has been solved by* [17] *(see also* [25]*) where it was shown that there is an ergodic invariant measure μ of maximal entropy* $\log(d_{\mathrm{top}}(f))$. *The measure μ is not supported on hypersurfaces, it does not charge the points of indeterminacy, and the repelling points of f are equidistributed according to this measure. It is the unique measure of maximal entropy.*
- **Case II: Small topological degree:** $d_{\mathrm{top}}(g) < \lambda_1(g)$. *It is believed that there is an ergodic invariant measure μ of maximal entropy* $\log(\lambda_1(g))$ *that is not supported on hypersurfaces and does not charge the points of indeterminacy. Saddle-type points are believed to be equidistributed according to this measure. It is the unique measure of maximal entropy.*

  A recent series of preprints [6, 7, 8] *has appeared where it is proven that these expected properties (except for uniqueness of μ) hold, provided that certain technical hypotheses are met.*
- **Case III: Equal degrees:** $d_{\mathrm{top}}(g) = \lambda_1(g)$. *Little is known or conjectured in this case.*

Remark 1. *Suppose that f is the monomial map associated to N. According to* [13], $\lambda_1(f) = c_+(N)$ *and* $d_{\mathrm{top}}(f) = \det N$. *Therefore, by choosing N appropriately we can find monomial maps in each of the three cases from Conjecture 1.1.*

We now summarize the main results of this paper:

In §3 we prove

Theorem 1.2. *Any $f \in \mathcal{B}_N$ has the same dynamical degree as the monomial map associated to N. That is:*

$$\lambda_1(f) = c_+(N) = \frac{m+q+\sqrt{(m-q)^2+4np}}{2},$$

where $c_+(N)$ is the leading eigenvalue of N.

In §4 we consider Blaschke products falling into Case I of Conjecture 1.1.

Notice that $d_{\rm top}(f) \geq d_{\rm top}(f_{|\mathbb{T}^2}) = \det N$, so that $d_{\rm top}(f)$ is greater than or equal to the topological degree of the monomial map associated to N. In particular, if the monomial map associated to N falls into Case I of the conjecture (i.e. $\det N > c_+(N)$) then so does every other $f \in \mathcal{B}_N$.

On the other hand, for any N, most Blaschke products fall into Case I:

Theorem 1.3. *For any matrix of degrees N there is an open dense set of full measure $\hat{\mathcal{B}}_N \subset \mathcal{B}_N$ so that if $f \in \hat{\mathcal{B}}_N$ then $d_{\rm top}(f) = mq+np > \lambda_1(f)$.*

The results from [17] apply, giving the existence of a unique measure of maximal entropy μ having entropy $\log(d_{\rm top}(f))$. For particular choices of f we can have $\mathrm{supp}(\mu) \subset \mathbb{T}^2$. However, if $f \in \hat{\mathcal{B}}_N$, this measure does not charge the invariant torus $\mathbb{T}^2$. Furthermore, in certain situations, an analysis of the dynamics near $\mathbb{T}^2$ allows one to see that $\mathrm{supp}(\mu)$ is isolated away from $\mathbb{T}^2$.

In §5 we consider Blaschke products falling into Case II of Conjecture 1.1. If $\det N < c_+(N)$, this occurs for the monomial maps associated to N, as well as certain non-generic $f \in \mathcal{B}_N$.

Many of the examples in §5 induce a diffeomorphism of $\mathbb{T}^2$ in which case there is an invariant measure $\mu_{\rm tor}$ supported on $\mathbb{T}^2$ of entropy $\log c_+(N) = \log \lambda_1(f)$. As a consequence of the bound on entropy provided in [11], we find

Proposition 1.4. *Let f be a Blaschke product diffeomorphism of small topological degree $d_{\rm top}(f) < \lambda_1(f)$. Then, $f : \mathbb{P}^2 \to \mathbb{P}^2$ has a measure of maximal entropy $\mu_{\rm tor}$ supported within $\mathbb{T}^2$.*

We do not know if $\mu_{\rm tor}$ is the unique measure of maximal entropy in all of $\mathbb{P}^2$ for these Blaschke product diffeomorphisms. Furthermore, it would also be interesting to see how these maps fit within the framework presented in [6, 7, 8].

In §6 we briefly consider the case of Blaschke products falling into Case III of Conjecture 1.1.

We conclude with Appendix A by proving basic facts about the entropy of Blaschke product diffeomorphisms in $\mathbb{T}^2$.

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2. A standard technique for computing dynamical degree

This section describes the work of many other authors (see the references within) and none of it is original to this paper. We provide it as a brief summary of the technique that we will use for computing the dynamical degree of Blaschke products.

One can recasts the dynamical degree (4) as:

$$\lambda_1(f) = \limsup (r_1((f^n)^*))^{1/n}, \tag{5}$$

where $r_1((f^n)^*)$ is the spectral radius of the linear action of $(f^n)^*$ on $H^{1,1}_a(X,\mathbb{R})$. Here, $H^{1,1}_a(X,\mathbb{R})$ is the part of the $(1,1)$ cohomology that is generated by algebraic curves in X, see [17, Prop 1.2(iii)]. (The cohomology class $[D]$ of an algebraic curve D is taken in the sense of closed-positive $(1,1)$ currents.) When $X = \mathbb{P}^2$ this definition agrees with (4) and this new definition is invariant under birational conjugacy (see [17, Prop 1.5]).

Definition 2.1. *A rational mapping $f : X \to X$ of a Kahler surface X is called* algebraically stable *if there is no integer n and no hypersurface V so that each component of $f^n(V)$ is contained within the indeterminacy set I_f.*

For the case $X = \mathbb{P}^2$, see [27, p. 109] and more generally, see [9].

If $f : X \to X$ is algebraically stable then, according to [9, Thm 1.14], one has that the action of $f^* : H^{1,1}_a(X,\mathbb{R}) \to H^{1,1}_a(X,\mathbb{R})$ is well-behaved: $(f^n)^* = (f^*)^n$. In this case, (5) simplifies to

$$\lambda_1(f) = r_1(f^*). \tag{6}$$

Therefore, *computation of dynamical degree for an algebraically stable mapping reduces to the study of a single iterate.*

If $f : \mathbb{P}^2 \to \mathbb{P}^2$ that is not algebraically stable, a typical way to compute $\lambda_1(f)$ is as follows. One tries to find an appropriate finite sequence of blow-ups at certain points in $\mathbb{P}^2$ in an attempt to obtain a new surface X on which the extension $\tilde{f}$ of f is algebraically stable. Note that in this approach $\tilde{f}$ and f are birationally conjugate using the canonical projection $\pi : X \to \mathbb{P}^2$, and hence $\lambda_1(f) = \lambda_1(\tilde{f})$.

A surface X that is birationally equivalent to $\mathbb{P}^2$ is called a *rational surface.* In this paper we will always construct X using the strategy described in the previous paragraph, so it will be an ongoing assumption that any surface X is rational (unless otherwise explicitly stated). In this case, $H^{1,1}_a(X)$ coincides with the full cohomology $H^{1,1}(X)$, allowing us a further simplification.

Suppose that one has created such a new surface X so that $\tilde{f} : X \to X$ is algebraically stable. Then, $\lambda_1(f) = \lambda_1(\tilde{f}) = r_1(\tilde{f}^*)$, where $r_1(\tilde{f}^*)$ is the spectral radius of the action $\tilde{f}^* : H^{1,1}(X,\mathbb{R}) \to H^{1,1}(X,\mathbb{R})$. This latter number can be computed by considering the pull-backs f^* of an appropriate finite set of curves that form a basis for $H^{1,1}(X,\mathbb{R})$. Nice descriptions of this procedure and explicit examples are demonstrated in [2, 1, 3] and the references therein. (The latter two of these references work in terms of $Pic(X)$, rather than $H^{1,1}(X)$, but the technique is essentially the same.)

In fact, such a modification does not exist for all rational maps. In [13] it was shown that for certain monomial maps (with some negative powers) there is no finite sequence of blow-ups that one can do, starting with $\mathbb{P}^2$, in order to obtain a surface X on which the map is algebraically stable. However, in the case that f is a monomial map with all positive powers (as assumed in this paper) it was shown in [13] that one can always find a toric surface $\check{X}$ on which f becomes algebraically stable. In this case, $\check{X}$ is obtained first by blowing-up $\mathbb{P}^2$ *and then extending to a ramified cover* (so that it is typically no longer a rational surface). See Question 1 at the end of §3.2.

3. Computation of dynamical degree for Blaschke products

In this section we prove Theorem 1.2, which states that for any Blaschke product $f \in \mathcal{B}_N$ we have $\lambda_1(f) = c_+(N)$.

We employ the following strategy: In §3.1 we obtain a lower bound $\lambda_1(f) \geq c_+(N)$ for all $f \in \mathcal{B}_N$. It will be a consequence of the dynamics of $f|_{\mathbb{T}^2}$. Then, in §3.2 we use the strategy described in §2 to find a dense set of full measure $\mathcal{B}'_N \subset \mathcal{B}_N$ on which $\lambda_1(f) = c_+(N)$. In §3.3 we combine the results of §3.1 and §3.2 to show $\lambda_1(f) = c_+(N)$ everywhere.

3.1. **Lower bound.**

Proposition 3.1. *For any Blaschke product $f \in \mathcal{B}_N$ we have that $\lambda_1(f) \geq c_+(N)$.*

Proof. We use the definition given in Equation (4) for $\lambda_1(f)$.

Consider the basis $\{[\gamma_1],[\gamma_2]\}$ for $H_1(\mathbb{T}^2)$ generated by the unit circle γ_1 in the plane $w = 0$ and the unit circle γ_2 in the plane $z = 0$. As noted earlier, the action $f_* : H_1(\mathbb{T}^2) \to H_1(\mathbb{T}^2)$ with respect to this basis is given by multiplication by the matrix N.

We will show that $d_{\mathrm{alg}}(f^n) \geq ||N^n||_\infty$, i.e. that $d_{\mathrm{alg}}(f^n)$ grows at least as fast as the largest element of N^n. This suffices to prove the assertion since $||N^n||_\infty \geq a \cdot c_+(N)^n$ for some positive constant a.

Notice that f_* acts "stably" on $H_1(\mathbb{T}^2)$ in the sense that the action of f_*^n is given by N^n with respect to the previously mentioned basis. Consider now the largest element of N^n, which we suppose (for the moment) is the $(1,1)$ element. Then $f_*^n([\gamma_1]) = k[\gamma_1]$ where $k \geq a \cdot c_+(N)^n$.

Write f in affine coordinates (z,w). We will show that the first coordinate of f^n is a rational function of degree at least k in z. This is sufficient to give that any homogeneous expression for f^n has degree at least k, as well.

Let π be the projection $\pi(z,w) = z$ so that the first coordinate of f^n is given by $\pi \circ f^n$. Also let $\iota(z) = (z,1)$. The iterate f^n is holomorphic on the open bidisc $\mathbb{D} \times \mathbb{D}$ because f^n forms a normal family there. Then $\pi \circ f^n \circ \iota : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ is a holomorphic function preserving the unit circle. By the previous homological considerations this map has degree k on the circle, so by a standard theorem it must be a (one variable)

Blaschke product of degree k with no poles inside of $\mathbb{D}$. This gives a lower bound for the degree in z of the first coordinate of f by $k \geq a \cdot c_+(N)^n$.

In the case that some other element than the $(1,1)$ element of N^n were largest, an identical proof works by choosing ι to be the appropriate inclusion and π to be the appropriate projection.

For each n the same argument can be applied to show that one of the affine coordinates of f^n is a rational function of degree at least $a \cdot c_+(N)^n$. The same holds for the homogeneous expression for f^n, giving $d_{\mathrm{alg}}(f^n) \geq a \cdot c_+(N)^n$, which is sufficient for the desired bound on $\lambda_1(f)$. $\square$

3.2. **Equality on a dense set of full measure.**

Proposition 3.2. *There is a dense set of full measure $\mathcal{B}'_N \subset \mathcal{B}_N$ so that $\lambda_1(f) = c_+(N)$ for $f \in \mathcal{B}'_N$.*

Before proving Proposition 3.2, we make some comments about the action of Blaschke products on $\mathbb{P}^2$. It simplifies the discussion to consider only Blaschke products f for which the zeros are distinct and non-zero. This will be standing assumption in this subsection.

We begin by writing f in homogeneous coordinates $[Z : W : T]$, with $T = 0$ corresponding to the line at infinity with respect to the usual affine coordinates (z, w). We write

$$f([Z : W : T]) = [f_1(Z,W,T) : f_2(Z,W,T) : f_3(Z,W,T)]$$

with

$$\begin{aligned} f_1(Z,W,T) &= \theta_1 \prod_{i=1}^{m}(Z - a_iT) \prod_{i=1}^{n}(W - b_iT) \prod_{i=1}^{p}(T - Z\overline{c_i}) \prod_{i=1}^{q}(T - W\overline{d_i}) \\ f_2(Z,W,T) &= \theta_2 \prod_{i=1}^{p}(Z - c_iT) \prod_{i=1}^{q}(W - d_iT) \prod_{i=1}^{m}(T - Z\overline{a_i}) \prod_{i=1}^{n}(T - W\overline{b_i}) \\ f_3(Z,W,T) &= \prod_{i=1}^{m}(T - Z\overline{a_i}) \prod_{i=1}^{n}(T - W\overline{b_i}) \prod_{i=1}^{p}(T - Z\overline{c_i}) \prod_{i=1}^{q}(T - W\overline{d_i}). \end{aligned} \tag{7}$$

Since the zeros of f distinct and non-zero, no common factors occur in Equation (7). Therefore, $d_{\mathrm{alg}}(f) = m + n + p + q$.

We will call the lines $Z - a_iT = 0$ the *zeros* of A and denote the union of such lines by $Z(A)$. Similarly, we will call the lines $T - Z\overline{a_i} = 0$ the *poles* of A, denoting the union of such lines by $P(A)$. The collections of lines $Z(B), P(B), Z(C), P(C), Z(D)$, and $P(D)$ are all defined similarly. Because of the standing assumption on the zeros of f, none of these lines coincide with either the z or w-axes.

From (7) we see that the "vertical" lines from $P(A)$ and the "horizontal" lines from $P(B)$ are collapsed to the point at infinity $[1 : 0 : 0]$. In a similar way $P(C)$ and $P(D)$ are collapsed to $[0 : 1 : 0]$.

For the monomial map associated to N, the lines of zeros collapse to $[0:0:1]$ and (often) the line at infinity $T=0$ collapses to either $[1:0:0]$ or $[0:1:0]$. If the zeros of f are distinct and non-zero, these lines no longer collapse.

The points of indeterminacy for f are precisely the points for which all three coordinates of (7) vanish. In particular:

Lemma 3.3. *If the zeros of f are distinct and non-zero, then f has 2 points of indeterminacy on the line at infinity:* $[1:0:0]$ *and* $[0:1:0]$.

Remark 2. *Such f are never algebraically stable on $\mathbb{P}^2$: As mentioned previously the lines in $P(A)\cup P(B)$ collapse under f to* $[1:0:0]$ *and the lines in $P(C)\cup P(D)$ collapse to* $[0:1:0]$.

Each of the intersection points from

$$\begin{aligned}&Z(A)\cap P(B),\ P(A)\cap Z(B),\\ &Z(C)\cap P(D),\ P(C)\cap Z(D),\ \text{and}\\ &(P(A)\cup P(B))\cap(P(C)\cup P(D))\end{aligned}$$

is a point of indeterminacy. There are $2(mn+pq)+(mq+np)$ such points of indeterminacy in $\mathbb{C}^2$ and none of them lie on the z or w-axes.

We write I_f to denote the indeterminacy points of f and C_f to denote the critical set of f (within which are all of the collapsing curves of f). Let $P_f = P(A)\cup P(B)\cup P(C)\cup P(D)$ be the union of all lines of poles for f.

Lemma 3.4. *Given any f and g differing by rotations (but with the same zeros: $\sigma_1=\sigma_2$), we have the following:*

- $I_f = I_g$,
- $C_f = C_g$, *and*
- $P_f = P_g$.

Proof. For each f and g, the indeterminacy points are given by the points where the corresponding lift F or G, respectively, to $\mathbb{C}^3$ has all three coordinates vanishing. The rotation multiplies the first two coordinates of each map by non-zero constants θ_1 and θ_2, hence has no affect on the indeterminacy points.

Similarly, any rotation of f by factors θ_1,θ_2 (non-zero) changes $\det(DF)$ by the non-zero factor $\theta_1\theta_2$, so the critical curves are unaffected.

The third item follows similarly. □

We now prove Proposition 3.2, defining $\mathcal{B}'_N$ within the proof.

Proof of Proposition 3.2: The strategy of proof is as follows. We begin by restricting that for any $f\in\mathcal{B}'_N$, the zeros of f are distinct and non-zero, so that the action of f on $\mathbb{P}^2$ is as described above.

We fix the zeros σ (satisfying the above restriction) and let $f_{(\theta_1,\theta_2)}$ be the mapping with zeros σ and rotations (θ_1,θ_2). According to Lemma 3.4, each of these mappings will have the same indeterminacy set, critical set, and collection of poles. Let $I_f^0\subset\mathbb{C}^2$ be the collection of finite indeterminate points.

We will select a full-measure subset $\Omega \equiv \Omega_\sigma \subset \mathbb{T}^2$ so that if $(\theta_1, \theta_2) \in \Omega$, then any collapsing curve (other than the lines of poles) does not have orbit landing in I_f^0 or on one of the lines of poles. (Landing on the lines of poles is dangerous since they are mapped to the indeterminate points $[1:0:0]$ and $[0:1:0]$). *It is not clear that any such curves exist, but we are unable to rule them out in the general case.*

We will then blow up $\mathbb{P}^2$ at $[1:0:0]$ and $[0:1:0]$ obtaining $\widetilde{\mathbb{P}^2}$ and show that every such $f_{(\theta_1,\theta_2)}$ extends to algebraically stable map on $\widetilde{\mathbb{P}^2}$, allowing us to use the technique described in §2.

Therefore, we let

$$\mathcal{B}'_N := \{f \in \mathcal{B}_N \,:\, \text{the zeros}\, \sigma \,\text{of}\, f \,\text{are distinct and non-zero, and}\, (\theta_1, \theta_2) \in \Omega_\sigma\}.$$

We fix σ (as in the definition of $\mathcal{B}'_N$) and construct $\Omega \equiv \Omega_\sigma$. For simplicity of exposition, we suppose that there is only collapsing curve C (other than the poles). It can be generalized to a finite number of them in the obvious way. (Note that since C is not a pole we have that $f(C) \in \mathbb{C}^2$.)

It is convenient to allow the rotations θ_1, θ_2 to be any complex numbers, and later restrict that they each have modulus equal to one. So long as neither θ_1 or θ_2 is zero, the proof of Lemma 3.4 gives that the indeterminacy set and poles of f remain unchanged, allowing to denote them by I_f^0 and P_f, independent of $\theta_1, \theta_2 \neq 0$.

If either θ_1 or θ_2 is zero, then $f_{(\theta_1,\theta_2)}$ degenerates, and some poles and indeterminacy points may disappear. However, this degenerate map remains a holomorphic map away from I_f^0 and P_f.

Let

$$\Psi_1 := \{(\theta_1, \theta_2) \in \mathbb{C}^2 \,:\, f_{(\theta_1,\theta_2)}(C) \not\in (I_f^0 \cup P_f) = \emptyset\}$$

and, inductively, let

$$\Psi_{n+1} := \{(\theta_1, \theta_2) \in \Psi_n \,:\, f^{n+1}_{(\theta_1,\theta_2)}(C) \not\in (I_f^0 \cup P_f) = \emptyset\}.$$

We will show that each Ψ_n is the complement of a (proper) analytic subset of $\mathbb{C}^2$ and that $(0,0) \in \Psi_n$. The proof is by induction on n.

Notice that $\rho(\theta_1, \theta_2) = f_{(\theta_1,\theta_2)}(C)$ is a holomorphic function defined on $\mathbb{C}^2$ with $\rho(0,0) = (0,0) \not\in (I_f^0 \cup P_f)$. Therefore, the set of (θ_1, θ_2) with $\rho(\theta_1, \theta_2) \in (I_f^0 \cup P_f)$ is a proper analytic subset of $\mathbb{C}^2$. We let Ψ_1 be its complement.

Suppose that Ψ_n is the complement of a proper analytic set in $\mathbb{C}^2$ and that $(0,0) \in \Psi_n$. Let $\varrho(\theta_1, \theta_2) = f^{n+1}_{(\theta_1,\theta_2)}(C)$, which is holomorphic on Ψ_n. Suppose that $\varrho(\Psi_n) \cap (I_f^0 \cup P_f) \neq \emptyset$. (Otherwise, $\Psi_{n+1} = \Psi_n$ and we are done.)

Since Ψ_n is connected and $\varrho(0,0) = (0,0) \not\in (I_f^0 \cup P_f)$ we see that ϱ is a non-constant holomorphic function on Ψ_n. In particular, the set of $(\theta_1, \theta_2) \in \Psi_n$ having $\varrho(\theta_1, \theta_2) \in (I_f^0 \cup P_f)$ is a proper analytic subset of Ψ_n. We let Ψ_{n+1} be its complement.

We now let $\Omega_n = \Psi_n \cap \mathbb{T}^2$, which is the complement of a proper real-analytic subset. Thus, the set $\Omega = \cap \Omega_n$ is the complement of a countable union of sets of measure zero, and hence a set of total measure.

The set of zeros σ that are distinct and non-zero is a dense set of full measure in $\mathbb{D}^{m+n+p+q}$ and for each such σ, Ω_σ is of full measure in $\mathbb{T}^2$. It follows that $\mathcal{B}'_N \subset \mathcal{B}_N$ is also a dense subset and, by Fubini's Theorem, of full measure.

We now blow up $[1:0:0]$ and $[0:1:0]$ obtaining $\widetilde{\mathbb{P}^2}$. Recall that the blow up of $\mathbb{C}^2$ at $(0,0)$ is

$$\widetilde{\mathbb{C}}^2_{(0,0)} = \left\{((w,t),l) \in \mathbb{C}^2 \times \mathbb{P}^1 : (w,t) \in l\right\}$$

There is a canonical projection $\pi : \widetilde{\mathbb{C}}^2_{(0,0)} \to \mathbb{C}^2$ and the fiber $E_{(0,0)} = \pi^{-1}((0,0))$ is referred to as the *exceptional divisor*. See [15]. In fact, this definition is coordinate independent so that the notion of blowing up a complex surface X at a point $p \in X$ is well-defined. The exceptional divisor above p will be denoted by E_p.

For $f \in \mathcal{B}'_N$ we check that f extends continuously (and hence holomorphically) to the blow-up at $[1:0:0]$. The calculation at $[0:1:0]$ is identical, and we omit it. We write f in the affine coordinates $w = W/Z$ and $t = T/Z$ so that the point of indeterminacy $[1:0:0]$ is at the origin with respect to these coordinates.

We work in the chart $(t,\lambda) \mapsto (\lambda t, t, \lambda) \in \widetilde{\mathbb{C}}^2_{(0,0)}$. With domain in this chart and codomain in the typical chart $(z,w) = (Z/T, W/T)$ we find that f induces:

$$(t,\lambda) \mapsto \left(\prod_{i=1}^{m} \frac{1-a_i t}{t-\overline{a_i}} \prod_{i=1}^{n} \frac{\lambda t - b_i t}{t - \overline{b_i}\lambda t} \;,\; \prod_{i=1}^{m} \frac{1-c_i t}{t-\overline{c_i}} \prod_{i=1}^{n} \frac{\lambda t - d_i t}{t - \overline{d_i}\lambda t}\right)$$

so that the extension to $E_{[1:0:0]}$ is given by taking the limit $t \to 0$:

$$\lambda \mapsto \left(\prod_{i=1}^{m} \frac{-1}{\overline{a_i}} \prod_{i=1}^{n} \frac{\lambda - b_i}{1 - \overline{b_i}\lambda} \;,\; \prod_{i=1}^{p} \frac{-1}{\overline{c_i}} \prod_{i=1}^{q} \frac{\lambda - d_i}{1 - \overline{d_i}\lambda}\right).$$

The calculation can also be done in the coordinates $\lambda' = \frac{1}{\lambda}$, where one sees that the extension is continuous to all of $E_{[1:0:0]}$, hence holomorphic. (We are essentially using that none of the zeros a_i or c_i are equal to 0.) Since the extension is non-constant with respect to λ, the extension of f sends $E_{[1:0:0]}$ to a non-trivial rational curve.

The blow-up at $[0:1:0]$ follows similarly and the extension of f also sends $E_{[0:1:0]}$ to a non-trivial rational curve. We denote by $\widetilde{f} : \widetilde{\mathbb{P}^2} \to \widetilde{\mathbb{P}^2}$ this extension of f to the space $\widetilde{\mathbb{P}^2}$ that is obtained by doing both blow-ups.

Having blown up $[1:0:0]$ and $[0:1:0]$ we will now observe that each of the lines of poles from $P(A) \cup P(B)$ covers $E_{[1:0:0]}$ with non-zero degree and each of the lines of poles from $P(C) \cup P(D)$ covers $E_{[0:1:0]}$ with non-zero degree. *In particular $\widetilde{f}$ does not collapse any of these lines to points.*

The calculation is the same for each line, so we show it for $z = \frac{1}{a_1}$. If we parameterize this line by $w = W/T$, then the image in coordinate $\rho = \frac{w'}{t}$ (here $w' = W/Z$) can be found by substituting $W = w$, $Z = \frac{1}{a_1}$, and $T = 1$ into the quotient $f_2(Z,W,T)/f_3(Z,W,T)$. We obtain

$$\rho(w) = \frac{\prod_{i=1}^{p}(1/\overline{a_1} - c_i)\prod_{i=1}^{q}(w - d_i)}{\prod_{i=1}^{p}(1 - \overline{c_i}/\overline{a_1})\prod_{i=1}^{q}(1 - w\overline{d_i})}.$$

This is a rational map of degree q if $a_1 \neq c_i$ for all $i = 1, \ldots, p$, which holds by hypothesis that $f \in \mathcal{B}'_N$. (In fact one can do the same calculation in the other coordinate charts on the line $z = \frac{1}{\overline{a_1}}$ and on $E_{[1:0:0]}$, but the result will also be a rational map of degree q in those coordinates, as well.)

Similar calculations show that under $\widetilde{f}$, each of the lines of poles from $P(A)$ covers $E_{[1:0:0]}$ with degree q and each of the lines the poles from $P(B)$ cover $E_{[1:0:0]}$ with degree p. The poles from $P(C)$ cover $E_{[0:1:0]}$ with degree n and the poles from $P(D)$ with degree m.

Let $\widetilde{X}$ be the blow-up of complex surface X at point p and $\pi : \widetilde{X} \to X$ be the corresponding projection. Given an algebraic curve $D \subset X$ there are two natural ways to "lift" D to $\widetilde{X}$: the *total transform* and the *proper transform*. The total transform is just $\pi^{-1}(D)$ while the proper transform is obtained by the closure $\overline{\pi^{-1}(D \setminus \{p\})}$. Clearly when $p \notin D$ there is no difference, however when $p \in D$ they differ by the exceptional divisor $E_p \subset \widetilde{X}$. In the case the many points have been blown-up the analogous definitions hold, see [15].

We now check that the only collapsing curves for $\widetilde{f} : \widetilde{\mathbb{P}^2} \to \widetilde{\mathbb{P}^2}$ are the proper transforms of the curves $C_1, \ldots, C_k$ that are collapsed under f to points in $\mathbb{C}^2$. In fact any collapsing curve must be either the proper transform of a collapsing curve for $f : \mathbb{P}^2 \to \mathbb{P}^2$ or be one of the exceptional divisors $E_{[1:0:0]}$ or $E_{[0:1:0]}$. Since $\widetilde{f}$ maps each of $E_{[1:0:0]}$ or $E_{[0:1:0]}$ to a non-trivial rational curve, neither is a collapsing curve. Furthermore, we have just checked that the lines from $P(A) \cup P(B) \cup P(C) \cup P(D)$ are no longer collapsed by $\widetilde{f}$. All that remains are the proper transforms of $C_1, \ldots, C_k$.

By the choice of $f \in \mathcal{B}'_N$ we have that the orbits of these collapsing curves avoid the indeterminate points as well as all of the lines of poles. Therefore, under the extension $\widetilde{f}$, their orbits cannot land on $E_{[1:0:0]}$, $E_{[0:1:0]}$, or the line at infinity. Thus, the orbits under $\widetilde{f}$ coincide with those under f, and they do not hit points in I_f^0, which are the only indeterminate points for $\tilde{f}$.

We can now compute $\lambda_1(f) = \lambda_1(\widetilde{f})$ as the spectral radius of the action of $\widetilde{f}^*$ on $H^{1,1}\left(\widetilde{\mathbb{P}^2}, \mathbb{R}\right)$.

Let $\widetilde{L}_v \subset \widetilde{\mathbb{P}^2}$ be the proper transform of the vertical line $L_v := \{Z = 0\}$ and let $\widetilde{L}_h$ be the proper transform of the horizontal line $L_h := \{W = 0\}$. We choose the fundamental classes $[\widetilde{L}_v]$, $[E_{[0:1:0]}]$ and $[E_{[1:0:0]}]$ as our basis of $H^{1,1}\left(\widetilde{\mathbb{P}^2}, \mathbb{R}\right)$. It will be useful in our calculation to express $[\widetilde{L}_h]$ in terms of this basis.

Lemma 3.5. *We have that:*

$$[\widetilde{L}_h] \sim [\widetilde{L}_v] + [E_{[0:1:0]}] - [E_{[1:0:0]}].$$

Proof. Both $[L_v]$ and $[L_h]$ are cohomologous in $\mathbb{P}^2$ so that their total transforms $\pi^*([L_v]) = [\widetilde{L}_v] + [E_{[0:1:0]}]$ and $\pi^*([L_h]) = [\widetilde{L}_h] + [E_{[1:0:0]}]$ are cohomologous, as well. □

We have that $f : \mathbb{P}^2 \to \mathbb{P}^2$ maps the lines of zeros $Z(A), Z(B)$ and the lines of poles $P(C), P(D)$ to L_v. However, after blowing up $[0:1:0]$ the lines of poles cover $E_{[0:1:0]}$ so that they should be considered as part of $\widetilde{f}^*[E_{[0:1:0]}]$ and not part of $\widetilde{f}^*[\widetilde{L}_v]$.

To see this more formally we write $\widetilde{f}$ with domain in the affine coordinates $z = Z/T, w = W/T$ and image in the coordinates $t = T/W$, $\lambda = Z/T$. These image coordinates are chosen so that when $t = 0$, λ parameterizes $E_{[0:1:0]}$ (except for one point). The second coordinate of the the image $(t, \lambda) = \widetilde{f}(z, w)$ is given by:

$$\lambda \;=\; \frac{\Pi_{i=1}^m (z - a_i) \Pi_{i=1}^n (w - b_i)}{\Pi_{i=1}^m (1 - z\bar{a}_i) \Pi_{i=1}^n (1 - w\bar{b}_i)} \tag{8}$$

In the (t, λ) coordinates $\widetilde{L_v}$ is given by $\lambda = 0$. Therefore,

$$\widetilde{f}^*(\widetilde{L_v}) \sim m[\widetilde{L_v}] + n[\widetilde{L_h}] \tag{9}$$

because of the m factors of $z - a_i$ and n factors of $w - b_i$ in the numerator of Equation (8). Using Lemma 3.5 we find:

$$\widetilde{f}^*[\widetilde{L}_v] \sim (m+n)[\widetilde{L}_v] + n[E_{[0:1:0]}] - n[E_{[1:0:0]}].$$

Suppose that we had parameterized the image of $\widetilde{f}$ in the other natural set of coordinates in a neighborhood of $E_{[0:1:0]}$, given by $\hat{z} = Z/W$ and $\eta = T/Z$. Then, the total transform $\pi^{-1}(L_v) = \widetilde{L_v} \cup E_{[0:1:0]}$ is given by $\hat{z} = 0$. The first coordinate of the image $(\hat{z}, \eta) = \widetilde{f}(z, t)$ is given by

$$\hat{z} = \frac{\theta_1 \prod_{i=1}^m (z - a_i) \prod_{i=1}^n (w - b_i) \prod_{i=1}^p (1 - z\overline{c_i}) \prod_{i=1}^q (1 - w\overline{d_i})}{\Pi_{i=1}^p (z - c_i) \Pi_{i=1}^q (w - d_i) \Pi_{i=1}^m (1 - z\bar{a}_i) \Pi_{i=1}^n (1 - w\bar{b}_i)}$$

so that

$$\widetilde{f}^* \left([L_v] + [E_{[0:1:0]}]\right) \sim m[L_v] + n[L_h] + p[L_v] + q[L_h].$$

By substracting (9) and using Lemma 3.5 we find:

$$\widetilde{f}^*([E_{[0:1:0]}]) \sim (p+q)[\widetilde{L}_v] + q[E_{[0:1:0]}] - q[E_{[1:0:0]}].$$

A similar calculation gives that

$$\widetilde{f}^*([E_{[1:0:0]}]) \sim (m+n)[\widetilde{L}_v] + n[E_{[0:1:0]}] - n[E_{[1:0:0]}].$$

Therefore, in terms of the basis $\left\{[\widetilde{L}_v], [E_{[0:1:0]}], [E_{[1:0:0]}]\right\}$ we have $\widetilde{f}^*$ given by:

$$\begin{bmatrix} (m+n) & (p+q) & (m+n) \\ n & q & n \\ -n & -q & -n \end{bmatrix}. \tag{10}$$

Therefore, $\lambda_1\left(\widetilde{f}\right) = r_1\left(\widetilde{f}^*\right)$ is the largest eigenvalue of this matrix, which one can see coincides with $c_+(N)$. Since dynamical degrees are invariant under birational conjugacy with $\widetilde{f}$ and f conjugate under the projection π we find $\lambda_1(f) = \lambda_1\left(\widetilde{f}\right) = c_+(N)$, as well.

This concludes the proof of Proposition 3.2. $\square$

Question 1. *As mentioned earlier, in* [13] *it is shown that for any matrix of degrees N with positive coefficients there is some toric surface $\check{X}_N$ on which monomial map corresponding to N becomes algebraically stable. Do all $f \in \mathcal{B}_N$ extend to algebraically stable maps on $\check{X}_N$, as well? This would be particularly helpful for studying bifurcations within the family.*

In the case that f is birational, [9, Thm. 0.1] *gives the existence of a modification by blow-ups X of $\mathbb{P}^2$ so that $f : X \to X$ is algebraically stable. Does $X = \widetilde{\mathbb{P}^2}$ for birational Blaschke products?*

3.3. **Equality everywhere.**

Lemma 3.6. *Given $\epsilon > 0$ there exist K_ϵ so that for all $f \in \mathcal{B}'_N$ we have*

$$d_{\mathrm{alg}}(f^n) \leq K_\epsilon(c_+(N) + \epsilon)^n. \tag{11}$$

Proof. The action $\tilde{f}^* : H^{1,1}\left(\widetilde{\mathbb{P}^2}, \mathbb{R}\right) \to H^{1,1}\left(\widetilde{\mathbb{P}^2}, \mathbb{R}\right)$ is given by the matrix (10) and hence independent of $f \in \mathcal{B}'_N$. Therefore, the sequence of degrees $\{d_{\mathrm{alg}}(f^n)\}$ is independent of $f \in \mathcal{B}'_N$ and satisfies $\lim(d_{\mathrm{alg}}(f^n))^{1/n} = c_+(N)$. This is sufficient to give (11). $\square$

Proof of Theorem 1.2. Given $\epsilon > 0$, let K_ϵ be given according to Lemma 3.6. We now show that (11) actually holds for every $f \in B_N$. This will give $\lambda_1(f) \leq c_+(N)$. Combined with the lower bound from Proposition 3.1, it will complete the proof of Theorem 1.2.

Consider the family of n-th iterates f^n, where f ranges over all of $\mathcal{B}_N$. If we write f^n in homogeneous coordinates as $[h_1 : h_2 : h_3]$, certain common factors will exist independent of the choice of $f \in \mathcal{B}_N$. After eliminating all such common factors, we obtain a homogeneous representation $[\hat{h}_1 : \hat{h}_2 : \hat{h}_3]$ for f^n that has no common factor for at least one particular $f_0 \in \mathcal{B}_N$.

It is a consequence of elimination theory (see, e.g., [5, §3.5]) that $\hat{h}_1, \hat{h}_2$, and $\hat{h}_3$ have a common factor if and only if their coefficients satisfy an algebraic condition. Since the coefficients of the $\hat{h}_i$ are polynomial in the coefficients of $f = [f_1 : f_2 : f_3]$, the $\hat{h}_i$ have a common factor if and only if the coefficients of f satisfy an algebraic condition.

Since f depends polynomially on $a_1, \ldots, d_q$, $\bar{a}_1, \ldots \bar{d}_q, \theta_1$, and θ_2, such a common factor occurs if and only if $a_1, \ldots, d_q, \theta_1$, and θ_2 satisfy a real-algebraic equation. Because a common factor does not exist when representing f_0^n, this is a proper real-algebraic subset of $\mathcal{B}_N$.

By Proposition 3.2, $\mathcal{B}'_N$ is a dense in $\mathcal{B}_N$. So, we find some $f_1 \in \mathcal{B}'_N$, so that $[\hat{h}_1 : \hat{h}_2 : \hat{h}_3]$ represents of f_1^n and has no common factor. It follows from Lemma 3.6 that by $\deg(\hat{h}_i) = d_{\rm alg}(f_1^n) \leq K_\epsilon (c_+(N) + \epsilon)^n$.

However, any $f \in \mathcal{B}_N$, has $[\hat{h}_1 : \hat{h}_2 : \hat{h}_3]$ as a homogeneous representation of f^n (possibly with common factors). Therefore, $d_{\rm alg}(f^n) \leq \deg(\hat{h}_i) \leq K_\epsilon(c_+(N)+\epsilon)^n$. □

4. Case I: Large topological degree

As noted in the introduction, if $\det N > c_+(N)$, then every $f \in \mathcal{B}_N$ falls into Case I, having $d_{\rm top}(f) > \lambda_1(f)$. We now check that for every N, generically chosen Blaschke products are also from Case I.

Let $\hat{\mathcal{B}}_N$ be the set of Blaschke products for which all of the zeros from σ are distinct and none of the zeros are critical for their corresponding one-variable Blaschke factor. I.e. $A'(a_i) \neq 0$ for all i, and similarly for B, C, and D. It is straightforward that $\hat{\mathcal{B}}_N$ is an open dense subset of $\mathcal{B}_N$ having total measure. Furthermore it is invariant under rotations by θ_1, θ_2.

Recall:

Theorem. 1.3 *For any matrix of degrees N there is an open dense set of full measure $\hat{\mathcal{B}}_N \subset \mathcal{B}_N$ so that if $f \in \hat{\mathcal{B}}_N$ then $d_{\rm top}(f) = mq + np > \lambda_1(f)$.*

Proof. It suffices to count the preimages of any point that is not a critical value of f. For $f \in \hat{\mathcal{B}}_N$, the origin $(0,0)$ is not a critical value. This follows because the preimages of $(0,0)$ in $\mathbb{C}^2$ are precisely the collection of points from $Z(A) \cap Z(D)$ and $Z(B) \cap Z(C)$. Substituting into $Jf = A'(z)B(w)C(z)D'(w) - A(z)B'(w)C'(z)D(w)$ we see that the definition of $\hat{\mathcal{B}}_N$ prevents the Jf from vanishing on these points.

Since the zeros are distinct for $f \in \hat{\mathcal{B}}_N$ we have $mq + np$ such points. In $\mathbb{P}^2$ the line at infinity $T = 0$ is forward invariant, so that there are no additional preimages of $(0,0)$ that are not in $\mathbb{C}^2$. Then, this total number of preimages of the non-critical value $(0,0)$ is $d_{\rm top}(f)$.

Notice that

$$\begin{aligned}\lambda_1(f) &= \frac{m+q+\sqrt{(m-q)^2+4np}}{2} < \frac{m+q+\sqrt{(m-q)^2}+2\sqrt{np}}{2} \\ &= \frac{m+q+|m-q|}{2} + \sqrt{np} < mq + np = d_{\rm top}(f).\end{aligned}$$

□

For mappings in Case I of Conjecture 1.1, [17, Thm 2.1] gives a unique ergodic invariant measure μ of maximal entropy $\log(d_{\rm top}(f))$. This measure is also backwards invariant, satisfying $f^*\mu = d_{\rm top}(f)\ \mu$. (The notion of pulling back a measure is not canonical; see [25, p. 899] for the precise definition). In particular, $\text{supp}(\mu)$ is totally invariant.

Proposition 4.1. *There are points $x \in \mathbb{T}^2$ for which the weighted sequence of measures*

$$\frac{1}{(d_{\text{top}}(f))^n} f^{n*}\delta_x \tag{12}$$

converges weakly to μ. (Here δ_x indicates the Dirac mass.)

Proof. It is a consequence of [17, Thm 3.1] that the sequence of measures (12) converges weakly to μ, so long as x is not in a pluripolar exceptional set $\mathcal{E}_f$. Since $\mathbb{T}^2$ is *generating* (i.e. the complexification of each tangent space to $\mathbb{T}^2$ spans the full tangent space in $\mathbb{P}^2$), it follows from a theorem by Sadullaev [26, Theorem 4] that $\mathcal{E}_f \cap \mathbb{T}^2$ has zero Haar measure in $\mathbb{T}^2$. □

If $\det N > c_+(N)$ then mappings $f \in \mathcal{B}_N$ having $d_{\text{top}}(f) = \det N$ can be constructed by making an appropriate non-generic choice of zeros σ.

Corollary 4.2. *If $f \in \mathcal{B}_N$ satisfies $d_{\text{top}}(f) = \det N > c_+(N)$, then* $\text{supp}(\mu) \subset \mathbb{T}^2$.

Proof. According to Proposition 4.1 the sequence of measures (12) converges weakly to μ when starting with a generic point $x \in T^2$ (chosen with respect to the Haar measure). Since $d_{\text{top}}(f) = \det N = d_{\text{top}}(f_{|\mathbb{T}^2})$, all preimages of x remain in $\mathbb{T}^2$ so that each of the measures (12) is supported in $\mathbb{T}^2$ and, therefore, μ is also. □

The hypothesis of Corollary 4.2 are not satisfied for generic Blaschke products. Rather, for any $f \in \hat{\mathcal{B}}_N$ we have $d_{\text{top}}(f) > \lambda_1(f)$ and $d_{\text{top}}(f) > \det N$. In this case we have:

Proposition 4.3. *Suppose that $f \in \mathcal{B}_N$ satisfies $d_{\text{top}}(f) > \lambda_1(f)$ and $d_{\text{top}}(f) > \det N$, then μ does not charge $\mathbb{T}^2$. In particular, this holds for any $f \in \hat{\mathcal{B}}_N$.*

Proof. Since $d_{\text{top}}(f) > \det N = d_{\text{top}}(f|\mathbb{T}^2)$ we have that $\mathbb{T}^2$ is not totally invariant. Since $\text{supp}(\mu)$ is totally invariant, we cannot have $\text{supp}(\mu) \subset \mathbb{T}^2$. However, since $\mathbb{T}^2$ is forward invariant and μ is ergodic, we must have $\mu(\mathbb{T}^2) = 0$. □

Notice that even though μ does not charge $\mathbb{T}^2$, its support may accumulate to $\mathbb{T}^2$. In certain cases, we can rule out this possibility, using hyperbolic theory in a complex neighborhood of $\mathbb{T}^2$.

Suppose that the eigenvalues of N satisfy $c_-(N) < 1 < c_+(N)$ so that monomial map associated to N induces a linear Anosov map with one-dimensional stable direction and one dimensional unstable directions. Then, any $f \in \mathcal{B}_N$ with sufficiently small choice of zeros will also be an Anosov map of $\mathbb{T}^2$ again with one dimensional stable and unstable directions. Furthermore, since $\mathbb{T}^2$ is hyperbolic for $f_{|\mathbb{T}^2}$ and $\mathbb{T}^2$ is generating, $\mathbb{T}^2$ is also a hyperbolic set for $f : \mathbb{P}^2 \to \mathbb{P}^2$ with one-complex dimensional stable and unstable directions. For details on the hyperbolic theory of endomorphisms, see [19].

Proposition 4.4. *Suppose that the eigenvalues of N satisfy $c_-(N) < 1 < c_+(N)$ and that $f \in \mathcal{B}_N$. If the zeros of f are chosen sufficiently small, then $\mathbb{T}^2$ is isolated in the recurrent set of f.*

Proof. We can assume that $\mathbb{T}^2$ is a hyperbolic set for $f : \mathbb{P}^2 \to \mathbb{P}^2$. Since $f_{|\mathbb{T}^2}$ is typically just an endomorphism, we go to the natural extension $\hat{\mathbb{T}}^2 := \{(x_i)_{i\leq 0} : x_i \in \mathbb{T}^2 \text{ and } f(x_i) = x_{i+1}\}$. We denote such histories by $\hat{x} = (x_i)_{i\leq 0} \in \hat{\mathbb{T}}^2$. See [19].

Let us check that $\mathbb{T}^2$ is maximally invariant, i.e. for a sufficiently small complex neighborhood U of $\mathbb{T}^2$ we have:

$$\bigcap_{n\in\mathbb{Z}} f^n(U) = \mathbb{T}^2, \ \bigcap_{n>0} f^n(U) = W^u_{\text{loc}}(\mathbb{T}^2), \text{ and } \bigcap_{n<0} f^n(U) = W^s_{\text{loc}}(\mathbb{T}^2).$$

where $W^s_{\text{loc}}(\mathbb{T}^2) = \cup_{x\in\mathbb{T}^2} W^s_{\text{loc}}(x)$ and $W^u_{\text{loc}}(\mathbb{T}^2) = \cup_{\hat{x}\in\hat{\mathbb{T}}^2} W^u_{\text{loc}}(\hat{x})$.

This is equivalent to the existence of a local product structure for the natural extension $\hat{\mathbb{T}}^2$; see Definition 2.2 and Corollary 2.6 from [19]. Since $f_{|\mathbb{T}^2}$ is Anosov, there is a local product structure (within $\mathbb{T}^2$) for $\hat{\mathbb{T}}^2$. That is: the unique point of intersection between $W^s_{\text{loc}}(x)$ and $W^u_{\text{loc}}(\hat{y})$ occurs at a point $z \in \mathbb{T}^2$ having some appropriate preorbit z_j (for $j < 0$) with $z_j \in W^u_{\text{loc}}(\hat{f}^{-j}(\hat{q}))$. This local product structure naturally carries over when we consider $\mathbb{T}^2 \subset \mathbb{P}^2$: the unique point of intersection between the complex manifolds $W^s_{\text{loc},\mathbb{C}}(x)$ and $W^u_{\text{loc},\mathbb{C}}(\hat{y})$ must be the same point $z \in \mathbb{T}^2$ and the previously chosen preorbit z_j $(j < 0)$ satisfies $z_j \in W^u_{\text{loc},\mathbb{C}}(\hat{f}^{-j}(\hat{q}))$.

We suppose that there are recurrent points r_i for $f : \mathbb{P}^2 \to \mathbb{P}^2$ that accumulate arbitrarily close to $\mathbb{T}^2$ from outside of $\mathbb{T}^2$.

In [23, Prop. 3.8] it was shown there are unique (semi) attracting points $e \in \overline{\mathbb{D}}^2$ and $e' \in \overline{(\mathbb{C} \setminus \mathbb{D})^2}$ so that $\mathbb{D}^2 \subset W^s(e)$ and $(\mathbb{C} \setminus \bar{\mathbb{D}})^2 \subset W^s(e')$. In particular, these bidiscs contain no recurrent points other than e and e', which are isolated from $\mathbb{T}^2$, since $f_{|\mathbb{T}^2}$ is Anosov. Therefore, the r_i must accumulate to $\mathbb{P}^2$ outside of $\mathbb{D}^2 \cup (\mathbb{C} \setminus \bar{\mathbb{D}})^2$ and their orbits cannot enter these bidiscs. We will show that this is impossible.

We can find some recurrent point r_I in an arbitrarily small neighborhood $V \subset U$ or $\mathbb{T}^2$. Since $\mathbb{T}^2$ is maximally invariant and $r_I \notin \mathbb{T}^2$, we must have $f^n(r_I) \notin U$ for some $n > 0$. Let n_0 be the first such n. If we choose V sufficiently small, we can make n_0 arbitrarily large. Therefore, we can assume that $f^{n_0-1}(r_I)$ is arbitrarily close to $W^u_{\text{loc},\mathbb{C}}(\hat{x})$ for some $\hat{x} \in \hat{\mathbb{T}}^2$.

It is sufficient to check that $W^u_{\text{loc}}(\hat{x}) \subset \mathbb{T}^2 \cup \mathbb{D}^2 \cup (\mathbb{C} \setminus \bar{\mathbb{D}})^2$. Notice that $f^{n_0-1}(r_I) \notin f^{-1}(U)$, which is some neighborhood of $\mathbb{T}^2$. Thus, $f^{n_0-1}(r_I)$ is arbitrarily close to $W^u_{\text{loc}}(\hat{x}) \setminus f^{-1}(U) \subset \mathbb{D}^2 \cup (\mathbb{C} \setminus \bar{\mathbb{D}})^2$, which will contradict the assumption that r_I is recurrent.

For all $x \in \mathbb{T}^2$, consider the complex conefield:

$$\mathcal{K}(x) = \{v \in T_x\mathbb{P}^2 : |\text{Im}(d\phi + d\psi)(v)| > |\text{Im}(d\phi - d\psi)(v)|\}.$$

which states precisely that v is pointing into the pair of invariant bidiscs $\mathbb{D}^2 \cup (\mathbb{C} \setminus \mathbb{D})^2$. Therefore, forward invariance of $\mathcal{K}$ follows from forward invariance of these bidiscs. Following general principles, since $\mathcal{K}$ is invariant on the compact invariant set $\mathbb{T}^2$, we can extend it to an invariant conefield in some small complex neighborhood of $\mathbb{T}^2$.

Because $\mathcal{K}$ is forward invariant, the complex unstable manifold $W^u_{\text{loc}}(\hat{x})$ of each $\hat{x} \in \hat{\mathbb{T}}^2$ is constrained within the cones. This gives $W^u_{\text{loc}}(\hat{x}) \subset \mathbb{T}^2 \cup \mathbb{D}^2 \cup (\mathbb{C} \setminus \mathbb{D})^2$, as needed. □

Remark 3. *If $f_{|\mathbb{T}^2}$ is Anosov with two unstable directions (i.e. $1 < c_-(N) \leq c_+(N)$) then $\mathbb{T}^2$ is repelling. In this case, points arbitrarily near to $\mathbb{T}^2$ can escape a neighborhood of $\mathbb{T}^2$ outside of $\mathbb{D}^2 \cup (\mathbb{C} \setminus \mathbb{D})^2$. If $d_{\text{top}}(f) > d_{\text{top}}(f_{\mathbb{T}^2})$, such an orbit could possibly approach a preimage of $\mathbb{T}^2$, allowing for recurrence. However, if $d_{\text{top}}(f) = d_{\text{top}}(f_{|\mathbb{T}^2})$, then elementary considerations give that $\mathbb{T}^2$ is isolated in the recurrent set.*

Corollary 4.5. *Suppose that $f \in \mathcal{B}_N$ satisfies the hypothesis of Proposition 4.4 and that $d_{\text{top}}(f) > \lambda_1(f)$, then $\text{supp}(\mu)$ is isolated from $\mathbb{T}^2$.*

In particular, if $c_-(N) < 1 < c_+(N)$ and $f \in \hat{\mathcal{B}}_N$ with sufficiently small choice of zeros σ, then $\text{supp}(\mu)$ is isolated from $\mathbb{T}^2$.

Proof. Since $d_{\text{top}}(f) > \lambda_1(f) = c_+(N) > \det N$, Proposition 4.3 gives that $\text{supp}(\mu)$ is not contained in $\mathbb{T}^2$. Therefore Proposition 4.4 implies that they are isolated. $\square$

Corollary 4.6. *Suppose that the eigenvalues of N satisfy $c_-(N) < 1 < c_+(N)$ and $f \in \hat{\mathcal{B}}_N$ has sufficiently small zeros σ. Then $\text{supp}(\mu)$, $\mathbb{T}^2$, and $\{e, e'\}$ are isolated pieces of the recurrent set with $W^u(\text{supp}(\mu)) \cap W^s(\mathbb{T}^2) \neq \emptyset$, $W^u(\mathbb{T}^2) \cap W^s(e) \neq \emptyset$, and $W^u(\mathbb{T}^2) \cap W^s(e') \neq \emptyset$.*

Proof. By Proposition 4.4, $\text{supp}(\mu)$ is isolated from $\mathbb{T}^2$ and in the proof we saw that e and e' are not in $\mathbb{T}^2$. By Proposition 4.1, preimages of $\mathbb{T}^2$ accumulate to $\text{supp}(\mu)$ so that $W^u(\text{supp}(\mu)) \cap W^s(\mathbb{T}^2) \neq \emptyset$. In the proof of Proposition 4.4 we saw that $W^u(\mathbb{T}^2)$ enters both $W^s(e)$ and $W^s(e')$. $\square$

Remark 4. *Saddle sets for globally holomorphic maps were studied by [10], where it was shown that the topological entropy of the saddle set is bounded above by $\log(d_{\text{alg}}(f))$, with equality holding if and only if the saddle set is terminal. The saddle set given by $\mathbb{T}^2$ that is discussed above conforms with a possible generalization to meromorphic maps of the result of [10], since $\mathbb{T}^2$ is terminal and has topological entropy $\log(\lambda_1(f))$ (because it is conjugate to the linear Anosov map).*

Question 2. *Suppose N satisfy $c_-(N) < 1 < c_+(N)$, is there some open $U \subset \hat{\mathcal{B}}_N$ so that every $f \in U$ is Axiom-A, with non-wandering set $\Omega(f) = \text{supp}(\mu) \cup \mathbb{T}^2 \cup \{e, e'\}$?*

5. Case II: Small topological degree

Given a choice of degrees N with $c_+(N) > \det N$, the associated monomial map has small topological degree (see Remark 1). Non-monomial Blaschke products f with $\lambda_1(f) > d_{\text{top}}(f)$ can also be constructed by choosing the zeros σ to have many repeated values. We present one specific family, for concreteness:

Example 1. *For $a \neq b \neq c$, consider the family*

$$f_{a,b,c}(z, w) = \left(\theta_1 \left(\frac{z-a}{1-\bar{a}z} \right)^5 \left(\frac{w-b}{1-\bar{b}w} \right)^2, \ \theta_2 \frac{z-a}{1-\bar{a}z} \cdot \frac{z-c}{1-\bar{c}z} \cdot \frac{w-b}{1-\bar{b}w} \right).$$

Members of this family are not in $\hat{\mathcal{B}}_N$ because $A(z)$ and $C(z)$ have a common zero, however one can directly check that $d_{\text{top}}(f_{a,b,c}) = 5$. Meanwhile, Theorem 1.2 gives $\lambda_1(f) = c_+(N) = \frac{6+\sqrt{32}}{2} > d_{\text{top}}(f_{a,b,c})$.

If $|a|, |b|$, and $|c|$ are sufficiently small then $f_{a,b,c}$ is a diffeomorphism on $\mathbb{T}^2$. Therefore, Lemma A.2 gives an invariant measure $\mu_{\rm tor}$ supported on $\mathbb{T}^2$ with entropy $\log c_+(N) = \log \lambda_1(f)$. *Since $\lambda_1(f) > d_{\rm top}(f)$,* [11] *gives that the topological entropy of f is* $\log \lambda_1(f)$. *Therefore, $\mu_{\rm tor}$ is a measure of maximal entropy for $f : \mathbb{P}^2 \to \mathbb{P}^2$.*

Because $d_{\rm top}(f) = 5$ there are many preimages of $\mathbb{T}^2$ in $\mathbb{P}^2$ that are away from $\mathbb{T}^2$. This raises the question of whether there exists a dynamically non-trivial invariant set outside of $\mathbb{T}^2$.

Recall

Proposition. 1.4 *Let f be a Blaschke product diffeomorphism of small topological degree $d_{\rm top}(f) < \lambda_1(f)$. Then, $f : \mathbb{P}^2 \to \mathbb{P}^2$ has a measure of maximal entropy $\mu_{\rm tor}$ supported within $\mathbb{T}^2$.*

Proof. The general statement follows precisely as in Example 1. □

Question 3. *Is $\mu_{\rm tor}$ is the* unique *invariant measure of maximal entropy for $f : \mathbb{P}^2 \to \mathbb{P}^2$?*

Remark 5. *It would be interesting to determine if the Blaschke products for which $\lambda_1(f) > d_{\rm top}(f)$ satisfy the hypothesis of* [6, 7, 8].

Remark 6. *Suppose that* $\det N < c_1(N)$. *Within $\mathcal{B}_N$, a change in the choice of zeros σ can result in the change between Case I and Case II, and therefore a big change in the global dynamics on $\mathbb{P}^2$. For example, consider a perturbation of the monomial map associated to N by introducing arbitrarily small generically chosen zeros. Aside from degree considerations, it would be interesting to know the mechanism(s) for this bifurcation.*

This bifurcation may be similar to that obtained when applying a perturbation of a monomial map, in a way that makes the resulting in a map that is globally holomorphic on $\mathbb{P}^2$. Because the monomial map f_0 is hyperbolic on $\mathbb{T}^2$, for small enough ϵ the perturbation to f_ϵ produces a continuation of $\mathbb{T}^2$ to an invariant hyperbolic set for f_ϵ. However, the topological degree of f_ϵ jumps to d^2, so the perturbation creates another invariant set of larger entropy $\log(d^2)$ *(using either* [17]*, or earlier work as referenced in* [27, §3]*).*

6. Case III: Equal degrees

The monomial map associated to N will have $\lambda_1(f) = d_{\rm top}(f)$ if and only if $\det N = c_+(N)$. In this case we will have that the smaller eigenvalue of N satisfies $c_-(N) = 1$. In appropriate coordinates, the linear action of f on $\mathbb{T}^2$ is given by product of an expanding map of $\mathbb{T}^1$ with the identity map of $\mathbb{T}^1$.

Proposition 6.1. *Suppose that $f \in \mathcal{B}_N$ with $\lambda_1(f) = d_{\rm top}(f)$, then $c_+(N) = \det N = d_{\rm top}(f)$ and $c_-(N) = 1$.*

Proof. Assume that there is some $f \in \mathcal{B}_N$ having $\lambda_1(f) = d_{\rm top}(f)$. We write $c_\pm \equiv c_\pm(N)$ for the eigenvalues of N.

Then,

$$c_- \cdot c_+ = \det N \leq d_{\mathrm{top}}(f) = \lambda_1(f) = c_+$$

using $0 < \det N \leq d_{\mathrm{top}}(f)$ and that $\lambda_1(f) = c_+$. This gives $0 < c_- \leq 1$.

Yet, $c_- + c_+ = \mathrm{tr} N \in \mathbb{N}$ and $c_+ = d_{\mathrm{top}}(f) \in \mathbb{N}$ so that $c_- \in \mathbb{N}$. Hence $c_- = 1$ and $c_+ = \det N$. □

We can construct non-monomial Blaschke products $f \in \mathcal{B}_N$ with $\lambda_1(f) = d_{\mathrm{top}}(f)$ by choosing the zeros σ to have many repeated values, as in §5.

Example 2. *Given zeros a_1, a_2, a_3, and b, consider the family*

$$f_{a_1,a_2,a_3,b}(z,w) =$$
$$\left(\theta_1 \frac{z-a_1}{1-\bar{a}_1 z} \cdot \frac{z-a_2}{1-\bar{a}_2 z} \cdot \frac{z-a_3}{1-\bar{a}_3 z} \cdot \left(\frac{w-b}{1-\bar{b}w}\right)^2,\ \theta_2 \frac{z-a_1}{1-\bar{a}_1 z} \cdot \frac{z-a_2}{1-\bar{a}_2 z} \cdot \left(\frac{w-b}{1-\bar{b}w}\right)^3\right)$$

corresponding to $N = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$.

One can directly check that $d_{\mathrm{top}}(f_{a_1,a_2,a_3,b}) = 5 = \det N$. Meanwhile, Theorem 1.2 gives $\lambda_1(f) = c_+(N) = 5$.

Question 4. *Suppose $f \in \mathcal{B}_N$ with $\lambda_1(f) = d_{\mathrm{top}}(f)$. Is the action of f on $\mathbb{T}^2$ given by a skew product between an expanding map and a neutral map?*

Appendix A. Entropy on $\mathbb{T}^2$

Proposition A.1. *Let $f \in \mathcal{B}_N$ be a Blaschke product diffeomorphism and $c_+(N)$ the leading eigenvalue of N. Then $h_{\mathrm{top}}(f_{|\mathbb{T}^2}) = \log(c_+(N)) > 0$.*

Proof. Central to the proof is that for any Blaschke product diffeomorphism f, the restriction $f_{|\mathbb{T}^2}$ has a global dominated splitting on all of $\mathbb{T}^2$; see [23, Cor 3.3]. This places serious restrictions on the dynamics. In particular, [23, Thm 3.10] gives that generic Blaschke product diffeomorphisms are Axiom-A with a very restricted behavior on the limit set. If we prove that $h_{\mathrm{top}}(f_{|\mathbb{T}^2}) = \log(c_+(N))$ for these maps, it will follow for all Blaschke product diffeomorphisms by the continuity of topological entropy for C^∞ surface diffeomorphisms [20, Theorem 6].

Let $f \in \mathcal{B}_N$ be such a generic Blaschke product diffeomorphism and let f_0 the monomial map associated to N. To simplify notation we will write $f \equiv f_{|\mathbb{T}^2}$ and similarly for f_0. Since f_0 is linear Anosov map induced by N we have $h_{\mathrm{top}}(f_0) = \det(c_+(N))$.

According to [23, Lemmas 3.11 and 3.12] there is a continuous surjective $\pi : \mathbb{T}^2 \to \mathbb{T}^2$ homotopic to the identity semiconjugating f to f_0 (and similarly for the lifts $\tilde{f}$ and $\tilde{f}_0$ to $\mathbb{R}^2$). By construction, we have that $\pi(x) = \pi(y)$ if and only if $\mathrm{dist}(\tilde{f}^n(x), \tilde{f}^n(y))$ remains bounded for all n. Because of the semiconjugacy π, we have $h_{\mathrm{top}}(f) \geq h_{\mathrm{top}}(f_0)$.

Theorem 3.10 from [23] gives that the limit set of f consists of a unique non-trivial homoclinic class $\mathcal{H}$, and possibly a finite number isolated of periodic points.

By "unique homoclinic class" we mean that, given any saddle periodic point p, the closure of all all transverse intersections of $W^s(p)$ with $W^u(p)$ is either $\mathcal{H}$ or $\emptyset$.

The topological entropy of f is concentrated on the limit set, so in this case $h_{\rm top}(f) = h_{\rm top}(f_{|\mathcal{H}})$. To complete the proof, we will check that $h_{\rm top}(f_{|\mathcal{H}}) \leq h_{\rm top}(f_0)$.

Consider the restriction $\pi_{|\mathcal{H}} : \mathcal{H} \to \pi(\mathcal{H})$ is a semiconjugacy onto its image. Applying Bowen's Formula [4, Theorem 17], we find

$$h_{\rm top}(f_{|\mathcal{H}}) \leq h_{\rm top}(f_{0|\pi(\mathcal{H})}) + \sum_{z \in \pi(\mathcal{H})} h_{\rm top}\left(f_{|\text{orbit of } \pi_{|\mathcal{H}}^{-1}(z)}\right). \tag{13}$$

It suffices to check that for any $z \in \pi(\mathcal{H})$ we have $h_{\rm top}\left(f_{|\text{orbit of } \pi_{|\mathcal{H}}^{-1}(z)}\right) = 0$. To do this, we will show that if $x \neq y \in \pi_{|\mathcal{H}}^{-1}(z)$ then both are in the stable set of some periodic attracting interval J. Note that the entropy of a diffeomorphism of an interval is 0.

Suppose that $\mathcal{H}$ is generated by the periodic point p. Since $\mathcal{H}$ is uniformly hyperbolic, there are local stable and unstable manifolds of a uniform length over all of $\mathcal{H}$. Furthermore, for any $h \in \mathcal{H}$, global manifolds can be formed, e.g. $W^s(h) = \cup f^{-n}(W^s_{\rm loc}(f^n(h))$. Notice that $W^u(p)$ accumulates to h, intersecting $W^s(h)$ transversely on at least one side $W^{s+}(h_1)$. It follows from the λ-Lemma [21] that $W^{s+}(h)$ is unbounded and accumulates to every other point $h_2 \in \mathcal{H}$. In particular, for any pair of points $h_1 \neq h_2$ in $\mathcal{H}$ there is a compact connected arc $I \subset W^{s+}(h_1)$ intersecting $W^u_{\rm loc}(h_2)$. Because $W^u(p)$ intersects I, the λ-lemma implies that $f^{-n}(I)$ has length arbitrarily large. Further, since $\tilde{f}$ inherits a dominated splitting on $\mathbb{R}^2$, this implies that the lifts $\tilde{f}^{-n}(I)$ will have arbitrarily large diameter.

We apply the above discussion to the pair $x \neq y \in \pi_{|\mathcal{H}}^{-1}(z)$, letting $x' \in W^{s+}(x) \cap W^u_{\rm loc}(y)$, $I \subset W^{s+}(x)$ the arc connecting x to x', and $I' \subset W^u_{\rm loc}(y)$ be the arc connecting x' to y. Notice that $x' \neq y$ because the discussion from the previous paragraph would give that $\tilde{f}^{-n}(x)$ and $\tilde{f}^{-n}(y)$ become arbitrarily far apart in $\mathbb{R}^2$. Similarly, that $f^n(I')$ remains a finite length because $x' \in W^s(x)$ and $\tilde{f}^n(x)$ and $\tilde{f}^n(y)$ remain at finite distance in $\mathbb{R}^2$.

We can now apply the Denjoy Property from [22, §2.4] to the interval I'. Since I' is part of $W^u_{\rm loc}(y)$, it is tangent to the center-unstable linefield F from the dominated splitting. In the language of [22, §2.4], I' is a δ-E-arc, where δ is the bound on the length of $f^n(I')$. Since there is a global dominated splitting for f on all of $\mathbb{T}^2$, Theorem 2.3 from [22] applies for δ-E-arcs for any $\delta > 0$. This gives that $\omega(I')$ is either a periodic closed curve, a periodic closed arc J (with $I' \subset W^s(J)$), or a periodic point which is either a sink or a saddle-node. By [23, Thm 3.10] there can be no periodic closed curves under f. A sink is impossible because $x, y \in \mathcal{H}$ cannot be in the basin of attraction of a sink. A saddle-node is impossible because f is Axiom-A. Therefore, $I' \subset W^s(J)$ for some periodic closed arc J. In particular $x', y \in I' \subset W^s(J)$ and $x' \in W^s(x)$ giving $x \in W^s(J)$. □

Remark 7. *If the set of all Blaschke product diffeomorphisms within $\mathcal{B}_N$ is connected, a much simpler proof of Proposition A.1 would follow directly from* [22, Thm. E]. *However, this is presently unknown.*

Proposition A.2. *Let $f \in \mathcal{B}_N$ be a Blaschke product diffeomorphism. Then, there is a unique measure of maximal entropy μ_{tor} for $f_{|\mathbb{T}^2}$.*

Note that here the meaning of "maximal" is with respect to invariant measures supported on $\mathbb{T}^2$. We have already observed in Proposition 4.3 that the generic Blaschke products $f \in \hat{B}_N$ have an invariant measure μ of higher entropy $\log(d_{\text{top}}(f)) > \log(c_+(N))$ that does not charge $\mathbb{T}^2$.

Proof. From [23, Thm 3.10] it follows that generic Blaschke product diffeomorphisms are Axiom-A on $\mathbb{T}^2$ with a unique non-trivial homoclinic class. From [22, Thm E], the restriction of any Blaschke product diffeomorphism to its limit set is conjugate to the restriction of one of these Axiom-A maps to its limit set. Therefore, on one hand, we conclude that any Blaschke product diffeomorphism has a unique non-trivial homoclinic class in the torus so that the topological entropy in $\mathbb{T}^2$ (which by Proposition A.1 is $\log(c_+(N))$) is equal to the topological entropy of the diffeomorphisms restricted to this homoclinic class. On the other hand, since the homoclinic class is conjugate to a hyperbolic one, it follows that it has a unique measure of maximal entropy with support in the homoclinic class. $\square$

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