TOPOLOGY OF FATOU COMPONENTS FOR ENDOMORPHISMS OF $\mathbb{CP}^k$: LINKING WITH THE GREEN'S CURRENT

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Abstract. Little is known about the global topology of the Fatou set $U(f)$ for holomorphic endomorphisms $f : \mathbb{CP}^k \to \mathbb{CP}^k$, when $k > 1$. Classical theory describes $U(f)$ as the complement in $\mathbb{CP}^k$ of the support of a dynamically-defined closed positive $(1,1)$ current. Given any closed positive $(1,1)$ current S on $\mathbb{CP}^k$, we give a definition of linking number between closed loops in $\mathbb{CP}^k \setminus \mathrm{supp} S$ and the current S. It has the property that if $lk(\gamma, S) \neq 0$, then γ represents a non-trivial homology element in $H_1(\mathbb{CP}^k \setminus \mathrm{supp} S)$.

As an application, we use these linking numbers to establish that many classes of endomorphisms of $\mathbb{CP}^2$ have Fatou components with infinitely generated first homology. For example, we prove that the Fatou set has infinitely generated first homology for any polynomial endomorphism of $\mathbb{CP}^2$ for which the restriction to the line at infinity is hyperbolic and has disconnected Julia set. In addition we show that a polynomial skew product of $\mathbb{CP}^2$ has Fatou set with infinitely generated first homology if some vertical Julia set is disconnected. We then conclude with a section of concrete examples and questions for further study.

1. Introduction

Our primary interest in this paper is the topology of the Fatou set for holomorphic endomorphisms of $\mathbb{CP}^k$ (written as $\mathbb{P}^k$ in the remainder of the paper). We develop a type of linking number that in many cases allows one to conclude that a given loop in the Fatou set is homologically non-trivial. One motivation is to find a generalization of the fundamental dichotomy for polynomial (or rational) maps of the Riemann sphere: the Julia set is either connected, or has infinitely many connected components. Further, this type of result paves the way to an exploration of a potentially rich algebraic structure to the dynamics on the Fatou set.

Given a holomorphic endomorphism $f : \mathbb{P}^k \to \mathbb{P}^k$, the *Fatou set* $U(f)$ is the maximal open set on which the iterates $\{f^n\}$ form a normal family. The *Julia set* $J(f)$ is the complement, $J(f) = \mathbb{P}^k \setminus U(f)$. The standard theory [14, 21, 36] gives a convenient description of these sets in terms the *Green's current* T. Specifically, T is a dynamically defined closed positive $(1,1)$ current with the property that $J(f) = \mathrm{supp}(T)$. We provide relevant background about the Green's current in Section 2. Throughout this paper we assume the degree of f is at least two (i.e. that the components of a lift of f to $\mathbb{C}^{k+1}$, with no common factors, have degree at least two).

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Motivated by this description of the Fatou set, in Section 3 we define a linking number $lk(\gamma, S)$ between a closed loop $\gamma \subset \mathbb{P}^k \setminus \text{supp } S$ and a closed positive $(1,1)$ current S. In Proposition 3.2 we will show that it depends only on the homology class of γ, and that it defines a homomorphism

$$lk(\cdot, S) : H_1(\mathbb{P}^k \setminus \text{supp } S) \to \mathbb{R}/\mathbb{Z}.$$

In particular, a non-trivial linking number in $\mathbb{R}/\mathbb{Z}$ proves that the homology class of γ is non-trivial. The techniques are based on a somewhat similar theory in [32].

This linking number can also be restricted to loops within any open $\Omega \subset \mathbb{P}^k \setminus \text{supp}S$, giving a homomorphism $lk(\cdot, S) : H_1(\Omega) \to \mathbb{R}/\mathbb{Z}$. If Ω is the basin of attraction for an attracting periodic point of a holomorphic endomorphism $f : \mathbb{P}^k \to \mathbb{P}^k$ and S is the Green's current, we will show in Proposition 3.8 that the image of this homomorphism is contained in $\mathbb{Q}/\mathbb{Z}$. This provides a natural setting to show that, under certain hypotheses, the Fatou set $U(f)$ has infinitely generated first homology:

Theorem 1.1. *Suppose that $f : \mathbb{P}^k \to \mathbb{P}^k$ is a holomorphic endomorphism and $\Omega \subset U(f)$ is a union of basins of attraction of attracting periodic points for f. If there are $c \in H_1(\Omega)$ with linking number $lk(c, T) \neq 0$ arbitrarily close to 0 in $\mathbb{Q}/\mathbb{Z}$, then $H_1(\Omega)$ is infinitely generated.*

(We prove Theorem 1.1 in Section 3.) Note that the hypotheses of Theorem 1.1 are satisfied if there are piecewise smooth loops $\gamma \subset \Omega$ with $lk(\gamma, T) \neq 0$ arbitrarily close to 0 in $\mathbb{Q}/\mathbb{Z}$. In our applications, we often find a loop γ_0 with nontrivial linking number, and then take an appropriate sequence of iterated preimages γ_n under f^n so that $lk(\gamma_n, T) \to 0$ in $\mathbb{Q}/\mathbb{Z}$.

In order to apply this theory to specific examples, one needs a detailed knowledge of the geometry of the Green's Current T. In the second half of the paper we consider two situations in which it can be readily applied to provide examples of endomorphism f of $\mathbb{P}^2$ having Fatou set $U(f)$ with infinitely generated homology.

The first situation is for polynomial endomorphisms of $\mathbb{P}^2$, that is, holomorphic maps of $\mathbb{P}^2$ that are obtained as the extension a polynomial map $f(z, w) = (p(z, w), q(z, w))$ on $\mathbb{C}^2$. Such mappings (and their generalizations to $\mathbb{P}^k$) were studied in [3]. Given a polynomial endomorphism $f : \mathbb{P}^2 \to \mathbb{P}^2$, the line at infinity, denoted by Π, is totally invariant and superattracting. Therefore the restriction of T to Π can be understood using the dynamics of the resulting rational map of $f_{|\Pi}$ and its Julia set J_Π. In Section 4 we prove the following theorem.

Theorem 1.2. *Suppose that f is a polynomial endomorphism of $\mathbb{P}^2$ with restriction $f_{|\Pi}$ to the line at infinity Π. If $f_{|\Pi}$ is hyperbolic and J_Π is disconnected, then the Fatou set $U(f)$ has infinitely generated first homology.*

This theorem provides for many examples of polynomial endomorphisms f of $\mathbb{P}^2$ with interesting homology of $U(f)$. We present one concrete family in Example 4.6.

We then consider the special family of polynomial endomorphisms known as polynomial skew products. While Theorem 1.2 applies to certain polynomial skew products, we develop additional sufficient criteria for $U(f)$ to have interesting homology.

A polynomial skew product is a polynomial endomorphism having the form $f(z, w) = (p(z), q(z, w))$, where p and q are polynomials. We assume that $\deg(p) = \deg(q) = d$ and $p(z) = z^d + O(z^{d-1})$ and $q(z) = w^d + O_z(w^{d-1})$, where we have normalized leading coefficients. Since f preserves the family of vertical lines $\{z\} \times \mathbb{C}$, one can analyze f via the collection of one variable fiber maps $q_z(w) = q(z,w)$, for each $z \in \mathbb{C}$. In particular, one can define fiber-wise filled Julia sets K_z and Julia sets $J_z := \partial K_z$ with the property that $w \in \mathbb{C} \setminus K_z$ if and only if the orbit of (z,w) escapes vertically to a superattracting fixed point $[0:1:0]$ at infinity.

For this reason, polynomial skew products provide an accessible generalization of one variable dynamics to two variables and have been previously studied by many authors, including Jonsson in [25] and DeMarco, together with the first author of this paper, in [11]. In Section 5 we provide the basic background on polynomial skew products and prove:

Theorem 1.3. *Suppose $f(z,w) = (p(z), q(z,w))$ is a polynomial skew product.*

- *If J_{z_0} is disconnected for any $z_0 \in J_p$, then $W^s([0:1:0])$ has infinitely generated first homology.*
- *Otherwise, $W^s([0:1:0])$ is homeomorphic to an open ball.*

The first statement is obtained by using Theorem 1.1, while the second is obtained using Morse Theory.

For any endomorphism there is also the measure of maximal entropy $\mu = T \wedge T$. Thus another candidate for the name "Julia set" is $J_2 := \mathrm{supp}(\mu)$. The Julia set that is defined as the complement of the Fatou set is sometimes denoted by J_1, to distinguish it from J_2.

The condition from Theorem 1.3 that for some $z_0 \in \mathbb{C}$, J_{z_0} is disconnected might seem somewhat unnatural. A seemingly more natural condition might be that J_2 is disconnected, since for polynomial skew products it is known (see [25]) that $J_2 = \overline{\bigcup_{z \in J_p} J_z}$. However, in Example 6.1 we present certain polynomial skew products with J_2 connected, but with the Fatou set having infinitely generated first homology. (These examples are obtained by applying Theorem 1.3 to examples from [25] and [11].) In fact, some of these examples persist over an open set within a one-variable holomorphic family of polynomial skew products. Therefore, for polynomial skew products, connectivity of the fiber Julia sets J_z is at least as important as the connectivity of J_2 to understanding the homology of the Fatou set.

In Section 6.2 we provide an example of a family of polynomial skew products f_a depending on a single complex parameter a with the following property: if a is in the Mandelbrot set $\mathcal{M}$, then the Fatou set $U(f_a)$ is homeomorphic to the union of three open balls, while if a is outside of $\mathcal{M}$ then $H_1(U(f_a))$ is infinitely generated.

Since neither of the sufficient conditions from Theorems 1.2 and 1.3 extend naturally to general endomorphisms of $\mathbb{P}^k$, it remains a mystery what is an appropriate condition for endomorphism to have non-simply connected Fatou set. We conclude Section 6, and this paper, with a discussion of a few potential further applications of the techniques of this paper to holomorphic endomorphisms of $\mathbb{P}^k$.

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2. The Green's current T

We provide a brief reminder of the properties of the Green's current that will be needed later in this paper. We refer the reader who would like to see more details to [14, 21, 36]. While the following construction works more generally for generic (algebraically stable) rational maps having points of indeterminacy, we restrict our attention to globally holomorphic maps of $\mathbb{P}^k$.

Suppose that $f : \mathbb{P}^k \to \mathbb{P}^k$ is holomorphic and that the Jacobian of f does not identically vanish on $\mathbb{P}^k$. Then f lifts to a polynomial map $F : \mathbb{C}^{k+1} \to \mathbb{C}^{k+1}$ each of whose coordinates is a homogeneous polynomial of degree d and so that the coordinates do not have a common factor. It is a theorem that

$$G(z) = \lim_{n\to\infty} \frac{1}{d^n} \log ||F^n(z)|| \tag{1}$$

converges to a plurisubharmonic[1] function $G : \mathbb{C}^{k+1} \to [-\infty, \infty)$ called the *Green's function associated to* f. Since f is globally well-defined on $\mathbb{P}^k$ we have that $F^{-1}(0) = 0$. It has been established that G is Holder continuous and locally bounded on $\mathbb{C}^{k+1} \setminus \{0\}$.

If $\pi : \mathbb{C}^{k+1} \setminus \{0\} \to \mathbb{P}^k$ is the canonical projection, there is a unique positive closed $(1,1)$ current T on $\mathbb{P}^k$ satisfying $\pi^* T = \frac{1}{2\pi} dd^c G$. (This normalization is not uniform–many authors do not divide by 2π.) More explicitly, consider any open set $V \subset \mathbb{P}^k$ that is "small enough" so that a holomorphic section $\sigma : V \to \mathbb{C}^{k+1}$ of π exists. Then, on V we have that T is given by $T = \frac{1}{2\pi} dd^c(G \circ \sigma)$. Choosing appropriate open sets covering $\mathbb{P}^k$ and sections of π on each of them, the result extends to all of $\mathbb{P}^k$ producing a single closed positive $(1,1)$ current on $\mathbb{P}^k$ independent of the choice of open sets and sections used. See [34, Appendix A.4]. By construction, the Green's current satisfies the invariance $f^*T = d \cdot T$. (See Section 3.3 for the definition of the pull-back f^*T.)

Recall that the Fatou set $U(f)$ is the maximal open set in $\mathbb{P}^k$ where the family of iterates $\{f^n\}$ form a normal family and that the Julia set of f is given by $J(f) = \mathbb{P}^k \setminus U(f)$. A major motivation for studying the Green's current is the following.

Theorem 2.1. *Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic endomorphism and let T be the Green's current corresponding to f. Then, $J(f) = \operatorname{supp} T$.*

See, for example, [14, Proposition 4.5] or [36, Theorem 2.2].

Remark 2.2. If f is a polynomial endomorphism, another form of Green's function, given by

$$G_{\text{affine}}(z) = \lim \frac{1}{d^n} \log_+ ||f^n(z)|| \tag{2}$$

is often considered in the literature. (Here $\log_+ = \max\{\log, 0\}$.) The result is again a PSH function $G : \mathbb{C}^k \to [0, \infty)$.

We can relate G_{affine} to G in the following way. Consider the open set $V = \mathbb{C}^k \subset \mathbb{P}^k$. Using the section $\sigma(z_1, \cdots, z_k) = (z_1, \cdots, z_k, 1)$, we find $G_{\text{affine}}(z_1, \cdots, z_k) = G \circ \sigma(z_1, \cdots, z_k)$ because $||F^k \circ \sigma||$ only differs from $||f^k||$ by a bounded amount for each iterate k.

Therefore, if f is a polynomial endomorphism of $\mathbb{P}^k$, one can compute T on $\mathbb{C}^k$ using the formula $T = \frac{1}{2\pi} dd^c G_{\text{affine}}$.

[1] We will often use the abbreviation PSH in place of plurisubharmonic and we use the convention that PSH functions cannot be identically equal to $-\infty$.

Remark 2.3. Note that formulae (1) and (2) are independent of the norm $\|\cdot\|$ that is used since any two norms are equivalent up to a multiplicative constant.

Remark 2.4. When $k = 1$, the resulting Green's current is precisely the measure of maximal entropy μ_f whose support is the Julia set $J(f) \subset \mathbb{P}^1$. If f is a polynomial, then μ_f also coincides with the harmonic measure on $K(f)$, taken with respect to the point at infinity.

3. Linking with a closed positive $(1,1)$ current in $\mathbb{P}^k$.

Suppose that S is an (appropriately normalized) closed positive $(1,1)$ current on $\mathbb{P}^k$ and $\gamma \subset \mathbb{P}^k \setminus \mathrm{supp}(S)$ is a piecewise smooth closed loop. We will define a linking number $lk(\gamma, S) \in \mathbb{R}/\mathbb{Z}$, depending only on the homology class $[\gamma] \in H_1(\mathbb{P}^k \setminus \mathrm{supp}(S))$.

3.1. **Classical linking numbers in $\mathbb{S}^3$.** Classically one considers the linking number of two oriented loops c and d in $\mathbb{S}^3$. The linking number $lk(c, d) \in \mathbb{Z}$ is found by taking any oriented surface Γ with oriented boundary c and defining $lk(c, d)$ to be the signed intersection number of Γ with d as in Figure 3.1. For this and many equivalent definitions of linking number in $\mathbb{S}^3$ see [33, pp. 132-133], [8, pp. 229-239], and [31, Problems 13 and 14].

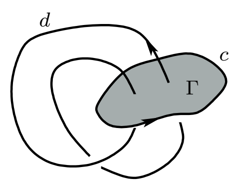

Figure 1. Here $lk(c, d) = +2$.

To see that this linking number is well-defined notice that assigning $lk(c, d) = [\Gamma] \cdot [d]$, where $\cdot$ indicates the intersection product on $H_*(\mathbb{S}^3, c)$, coincides with the classical definition. (For background on the intersection product on homology, see [9, pages 366-372].) If Γ' is any other 2-chain with $\partial\Gamma' = c$ then $\partial(\Gamma - \Gamma') = [c] - [c] = 0$ and $(\Gamma - \Gamma')$ represents a homology class in $H_2(\mathbb{S}^3)$. Since $H_2(\mathbb{S}^3) = 0$, $[\Gamma - \Gamma'] = 0$ forcing $[\Gamma - \Gamma'] \cdot [d] = 0$. Therefore: $[\Gamma] \cdot [d] = [\Gamma'] \cdot [d]$, so that $lk(c, d)$ is well defined.

3.2. **Generalization.** Given any closed positive $(1,1)$ current S on $\mathbb{P}^k$ and any piecewise smooth two chain σ in $\mathbb{P}^k$ with $\partial\sigma$ disjoint from supp S, we can define

$$\langle \sigma, S \rangle = \int_\sigma \eta_S$$

where η_S is a smooth approximation of S within it's cohomology class in $\mathbb{P}^k - \partial\sigma$, see [16, pages 382-385]. The resulting number $\langle \sigma, S \rangle$ will depend only on the cohomology class of S and the homology class of σ within $H_2(\mathbb{P}^k, \partial\sigma)$. (Note that if S is already a smooth form, one need not require that $\partial\sigma$ be disjoint from supp S.)

Notice that $H_2(\mathbb{P}^k)$ is generated by the class of any complex projective line $L \subset \mathbb{P}^k$. Since S is non-trivial, $\langle L, S \rangle \neq 0$, so that after an appropriate rescaling we can assume that $\langle L, S \rangle = 1$. In the remainder of the section we assume this normalization. (It is satisfied by the Green's Current from Section 2.)

What made the linking numbers in $\mathbb{S}^3$ well-defined, independent of the choice of Γ, is that $H_2(\mathbb{S}^3) = 0$. One cannot make the immediately analogous definition that $lk(\gamma, S) = \langle \Gamma, S\rangle$ in $\mathbb{P}^k$, since $H_2(\mathbb{P}^k) \neq 0$ implies that $\langle \Gamma, S\rangle$ can depend on the choice of Γ. For example, given Γ with $\partial\Gamma = \gamma$ then $\partial\Gamma' = \gamma$ for $\Gamma' = \Gamma + L$, however $\langle \Gamma', S\rangle - \langle \Gamma, S\rangle = \langle L, S\rangle = 1 \neq 0$.

There is a simple modification: Given any Γ and Γ' both having boundary γ, $[\Gamma' - \Gamma] \in H_2(\mathbb{P}^k)$ so that $[\Gamma' - \Gamma] \sim k \cdot [L]$ for some $k \in \mathbb{Z}$. Since S is normalized, this gives that $\langle \Gamma', S\rangle = \langle \Gamma, S\rangle \pmod 1$.

Definition 3.1. *Let S be a normalized closed positive $(1, 1)$ current on $\mathbb{P}^k$ and let γ be a piecewise smooth closed curve in $\mathbb{P}^k \setminus \mathrm{supp}(S)$. We define the* linking number $lk(\gamma, S)$ *by*

$$lk(\gamma, S) := \langle \Gamma, S\rangle \pmod 1$$

where Γ is any piecewise smooth two chain with $\partial\Gamma = \gamma$.

Unlike linking numbers between closed loops in $\mathbb{S}^3$, it is often the case that that $\langle \Gamma, S\rangle \notin \mathbb{Z}$, resulting in non-zero linking numbers (mod 1). See Subsection 3.4 for an explicit example.

Proposition 3.2. *If γ_1 and γ_2 are homologous in $H_1(\mathbb{P}^k \setminus \mathrm{supp}\ S)$, then $lk(\gamma_1, S) = lk(\gamma_2, S)$.*

Proof. Let Γ be any piecewise smooth two chain contained in $\mathbb{P}^k \setminus \mathrm{supp}\ S$ with $\partial\Gamma = \gamma_1 - \gamma_2$. Then, since $\mathbb{P}^k \setminus \mathrm{supp}\ S$ is open and Γ is compact subset, Γ is bounded away from the support of S. Consequently for any smooth approximation η_S of S supported in a sufficiently small neighborhood of S, we have $lk(\gamma_1, S) - lk(\gamma_2, S) = \int_\Gamma \eta_T = 0$. □

Corollary 3.3. *If $\gamma \in \mathbb{P}^k \setminus \mathrm{supp}\ S$ with $lk(\gamma, S) \neq 0$, then γ is a homologically non-trivial loop in $\mathbb{P}^k \setminus \mathrm{supp}\ S$.*

Since $lk(\gamma, S)$ depends only on the homology class of γ and the pairing $\langle \cdot, S\rangle$ is linear in the space of chains σ (having $\partial\sigma$ disjoint from supp S), the linking number descends to a homomorphism:

$$lk(\cdot, S) : H_1(\mathbb{P}^k \setminus \mathrm{supp}\ S) \to \mathbb{R}/\mathbb{Z}.$$

Similarly $lk(\cdot, S) : H_1(\Omega) \to \mathbb{R}/\mathbb{Z}$ for any open $\Omega \subset \mathbb{P}^k \setminus \mathrm{supp}\ S$.

Remark 3.4. (Topological versus Geometric linking numbers.) The classical linking number, and also Definition 3.1, depend only on the homology class of the loop γ (in the complement of some other loop of the support of some current, respectively.)

A linking number depending on the geometry of γ is given by

$$\widehat{lk}(\gamma, T) := \langle \Gamma, S - \Omega\rangle \in \mathbb{R},$$

where $\partial\Gamma = \gamma$ and Ω is (normalization of) the Kähler form defining the Fubini-Study metric on $\mathbb{P}^k$. Given any Γ and Γ' both having boundary γ we have that $\langle \Gamma - \Gamma', T - \Omega\rangle = 0$, since S and Ω are cohomologous. (In the language of [32, p. 132], we say that $T - \Omega$ is in the "linking kernel of $\mathbb{P}^k$".)

Because supp $\Omega = \mathbb{P}^k$, the statement of Proposition 3.2 does not apply. Rather, $\widehat{lk}(\gamma, S)$ depends on the geometry of $\gamma \subset \mathbb{P}^k \setminus \mathrm{supp}\ S$. In fact, similar linking numbers were used in [17, 18] to determine if a given real-analytic γ has the appropriate geometry to be the boundary of a positive holomorphic 1-chain (with bounded mass).

Remark 3.5. (Other manifolds.) Suppose that M is some other compact complex manifold with $H_2(M)$ of rank k, generated by $\sigma_1, \ldots, \sigma_k$. If $\langle \sigma_1, S\rangle, \ldots, \langle \sigma_k, S\rangle$ are rationally related, then S can be appropriately rescaled so that Definition 3.1 provides a well-defined linking number between any piecewise smooth closed curve $\gamma \in M \setminus \mathrm{supp} S$ and S. If $H_2(M)$ has rank $k > 1$, this provides a rather restrictive cohomological condition on S. (It is similar to the restriction of being in the "linking kernel" described in [32].)

3.3. **Invariance and restriction properties of $\langle \cdot, \cdot \rangle$.** Suppose that Ω, Λ are open subsets of $\mathbb{C}^j$ and $\mathbb{C}^k$, and $f : \Omega \to \Lambda$ is a (possibly ramified) analytic mapping. Let S be a closed positive $(1,1)$ current given on Λ by $S = dd^c u$ for some PSH function u. If $f(\Omega)$ is not contained in the polar locus of u, then the *pull-back of S under f* is defined by pulling back the potential: $f^*(S) := dd^c(u \circ f)$. Since $u \circ f$ is not identically equal to $-\infty$, it is also a PSH function, and $f^*(S)$ is a well-defined closed positive $(1,1)$ current.

Suppose that M and N are complex manifolds and that S is a closed positive $(1,1)$ current on N. If $f : M \to N$ is a holomorphic map with $f(M)$ not entirely contained in the polar locus of S, then the pull-back f^*S can be defined by taking local charts and local potentials for S. See [34, Appendix A.7] and [21, p. 330-331] for further details.

Proposition 3.6. *Suppose that S is a closed positive $(1,1)$ current on N and $f : M \to N$, with $f(M)$ not contained in the polar locus of S. If σ is a piecewise smooth two chain in M with $\partial\sigma$ disjoint from* supp f^*S*, then $\langle f_*\sigma, S\rangle = \langle \sigma, f^*S\rangle$.*

Proof. Since $f(M)$ is not contained in the polar locus of S, f^*S is well-defined. Since $\partial\sigma$ is disjoint from $\mathrm{supp} f^*S$, $\partial f(\sigma)$ is disjoint from $\mathrm{supp} S$. Let η_S be a smooth approximation of S in the same cohomology class as S and having support disjoint from $\partial f(\sigma)$. Then, $\langle f_*\sigma, S\rangle = \int_{f_*\sigma} \eta_S = \int_\sigma f^*\eta_S = \langle \sigma, f^*S\rangle$, since $f^*\eta_S$ is a smooth approximation of f^*S. □

In the case that M is an analytic submanifold of N not entirely contained in the polar locus of S, the restriction of S to M is defined by $S|_M := \iota^*S$, where $\iota : M \to N$ is the inclusion. When computing linking numbers, we will often choose Γ within some one-complex dimensional curve M in N, with M not contained in the polar locus of S. In that case $S|_M$ is a positive measure on M and we can use the following:

Corollary 3.7. *Let S be a positive closed $(1,1)$ current on N and M be an analytic curve in N that is not entirely contained in the polar locus of S. If Γ is a piecewise smooth two chain in M with $\iota(\partial\Gamma)$ disjoint from* supp S*, then*

$$\langle \iota(\Gamma), S\rangle = \int_\Gamma S|_M. \tag{3}$$

Proof. Proposition 3.6 gives $\langle \iota(\Gamma), S\rangle \equiv \langle \iota_*\Gamma, S\rangle = \langle \Gamma, \iota^*S\rangle = \langle \Gamma, S|_M\rangle$. Any positive $(1,1)$ current on M is a positive measure. Thus, $\int_\Gamma S|_M$ is defined, and coincides with the result obtained by first choosing a smooth approximation to $S|_M$. Thus $\langle \Gamma, S|_M\rangle = \int_\Gamma S|_M$. □

In the remainder of the paper, we will not typically distinguish between Γ and $\iota(\Gamma)$.

3.4. **Linking with the Green's Current.** We conclude the section with some observations specific to the Green's current T, including the proof of Theorem 1.1, as well as an example illustrating the definitions given above. It is worth noting that the Green's current has empty polar locus, since G is locally bounded on $\mathbb{C}^{k+1} \setminus \{0\}$, so that the hypotheses of Proposition 3.6 and Corollary 3.7 are easy to check.

Proposition 3.8. *Suppose that $f : \mathbb{P}^k \to \mathbb{P}^k$, $W^s(\zeta) \subset U(f)$ is the basin of attraction of some attracting periodic cycle ζ, and T is the Green's Current of f. Then*

$$lk(\cdot, T) : H_1(W^s(\zeta)) \to \mathbb{Z}[1/d]/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}.$$

Proof. Suppose that ζ is of period N. Then, the basin of attraction $W^s(\zeta)$ contains a union of small open balls $B_0, \ldots, B_{N-1}$ centered at each point $\zeta, \ldots, f^{N-1}(\zeta)$ of the orbit ζ. Since $H_1(W^s(\zeta))$ is generated by the classes of piecewise smooth loops, it is sufficient to consider a single such loop γ. Since γ is a compact subset of $W^s(\zeta)$, there is some n so that $f^n(\gamma)$ is contained in $\cup B_i$, giving that $f^n(\gamma)$ has trivial homology class in $H_1(W^s(\zeta))$. In particular, $lk(f^n(\gamma), T) = 0 \pmod 1$, so that for any Γ with $\partial\Gamma = \gamma$ we have $\langle f^n(\Gamma), T\rangle = k$ for some integer k.

Recall that $f^*T = dT$, where d is the algebraic degree of f. Proposition 3.6 gives that $k = \langle f^n(\Gamma), T\rangle = \langle \Gamma, (f^*)^n T\rangle = d^n \langle \Gamma, T\rangle$. In particular, $lk(\gamma, T) \equiv k/d^n \pmod 1$. □

Using Proposition 3.8, Theorem 1.1 presents a general strategy for showing that $H_1(U(f))$ is infinitely generated.

Proof of Theorem 1.1: Since Ω is a union of basins of attraction for attracting periodic points of f, Proposition 3.8 gives that $lk(\cdot, T) : H_1(\Omega) \to \mathbb{Q}/\mathbb{Z}$. There are homology classes $c \in H_1(\Omega)$ with $lk(c, T) \neq 0$ arbitrarily close to zero, so, since $lk(\cdot, T)$ is a homomorphism, the image of $lk(\cdot, T) : H_1(\Omega) \to \mathbb{Q}/\mathbb{Z}$ is dense in $\mathbb{Q}/\mathbb{Z}$. Because any dense subgroup of $\mathbb{Q}/\mathbb{Z}$ is infinitely generated, the image of $lk(\cdot, T)$ is infinitely generated, hence $H_1(\Omega)$ is, as well. □

Example 3.9. Consider the polynomial skew product $(z, w) \mapsto (z^2, w^2 + 0.3z)$, for which the Fatou set consists of the union of basins of attraction for three super-attracting fixed points: $[0 : 1 : 0]$, $[0 : 0 : 1]$, and $[1 : 0 : 0]$. In Figure 2 we show a computer generated image of the intersection of $W^s([0 : 1 : 0])$ (lighter grey) and $W^s([0 : 0 : 1])$ (dark grey) with the vertical line $z = z_0 = 0.99999$. In terms of the fiber-wise Julia sets that were mentioned in the introduction, K_{z_0} is precisely the closure of the dark grey region and J_{z_0} is its boundary.

We will see in Proposition 5.1 that $T|_{z=z_0}$ is precisely the harmonic measure on K_{z_0}. Using this knowledge, and supposing that the computer image is accurate, we illustrate how the above definitions can be used to show that the smooth loop γ shown in the figure represents a non-trivial homology class in $H_1(W^s([0 : 1 : 0]))$.

Suppose that we use the two chain Γ_1 that is depicted in the figure to compute $lk(\gamma, T)$. The harmonic measure on K_{z_0} is supported in J_{z_0} and equally distributed between the four symmetric pieces with total measure of K_{z_0} is 1. Therefore (using Corollary 3.7) we see that $lk(\gamma, T) = \int_{\Gamma_1} T|_{z=z_0} = \frac{1}{4} \pmod 1$, because Γ_1 covers exactly 1 these 4 pieces of K_{z_0}.

If instead we use Γ_2, the disc "outside of γ" within the projective line $z = z_0$ with the orientation chosen so that $\partial\Gamma_2 = c$ as depicted, then $lk(\gamma, T) = \int_{\Gamma_2} T|_{z=z_0} = -\frac{3}{4} \pmod 1$ (because Γ_2 covers 3 of the 4 symmetric pieces of K_{z_0}, but with the opposite orientation than that of Γ_1). However, $-\frac{3}{4} \pmod 1 = \frac{1}{4} \pmod 1$, so we see that the computed linking number does come out the same.

Since $lk(\gamma, T) \neq 0 \pmod 1$, Corollary 3.3 gives that it is impossible to have any 2-chain Λ within $W^s([0 : 1 : 0])$ (even outside of the vertical line $z = z_0$) so that $\partial\Lambda = c$. Thus $[\gamma] \neq 0 \in H_1(W^s([0 : 1 : 0]))$.

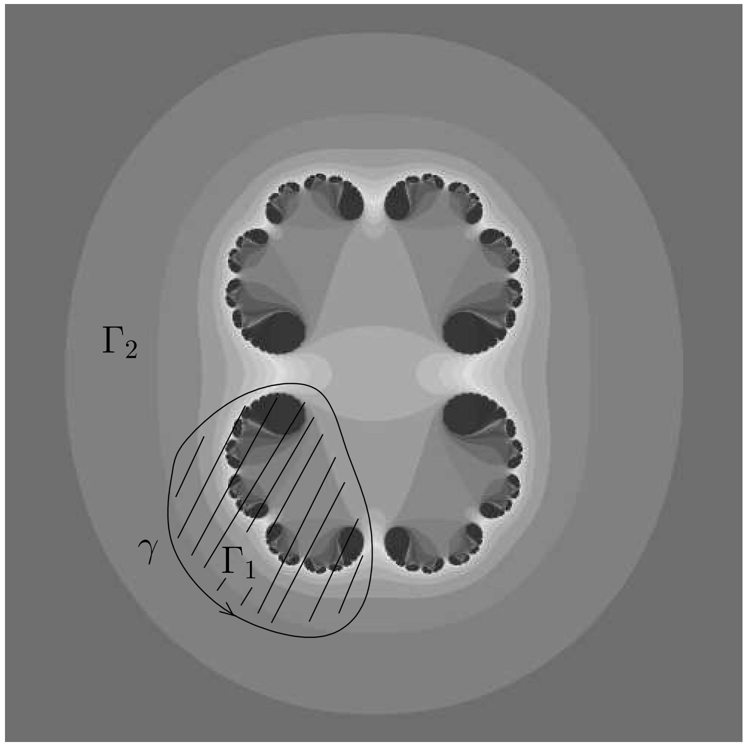

FIGURE 2. Both choices Γ_1 (inside of γ) and Γ_2 (outside of γ) yield the same $lk(\gamma, T)$.

4. Application to Polynomial Endomorphisms of $\mathbb{P}^2$

Having developed the linking numbers in Section 3, Theorem 1.2 will be a consequence of the following well-known result:

Theorem 4.1. [1, Thm. 5.7.1] *Let $g : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map. Then, if $J(g)$ is disconnected, it contains uncountably many components, and each point of $J(g)$ is an accumulation point of infinitely many distinct components of $J(g)$.*

Let us begin by studying the Fatou set of one-dimensional maps:

Proposition 4.2. *If $g : \mathbb{P}^1 \to \mathbb{P}^1$ is a hyperbolic rational map with disconnected Julia set $J(g)$, then the Fatou set $U(g)$ has infinitely generated first homology.*

Remark 4.3. When reading the proof of Proposition 4.2, it is helpful to keep in mind two examples. The first is the polynomial $r(z) = z^3 - 0.48z + (0.706260 + 0.502896i)$ for which one of the critical points escapes to infinity, while the other is in the basin of attraction for a cycle of period 3. The result is a filled Julia set with infinitely many non-trivial connected components, each of which is homeomorphic to the Douady's rabbit. (See [30].)

The second example are maps of the form $f(z) = z^n + \lambda/z^h$, which were considered in [28]. For suitable n, h, and λ the Julia set is a Cantor set of nested simple closed curves.

Proof of Proposition 4.2: Since g is hyperbolic, $U(g)$ consists of the basins of attraction of finitely many attracting periodic points. Therefore, according to Theorem 1.1, it is sufficient to find elements of $H_1(U(g))$ having non-zero linking numbers with $T = \mu_g$ that are arbitrarily close to 0.

Theorem 4.1 will allow us to find a sequence of piecewise smooth two chains $\Gamma_1, \Gamma_2, \ldots$ so that $0 < \langle \Gamma_{n-1}, \mu_G \rangle < \langle \Gamma_n, \mu_G \rangle < 1$ and $\partial \Gamma_n \subset U(g)$, as follows.

For each n, Γ_n will be a union of disjoint positively-oriented closed discs in $\mathbb{P}^1$, each counted with weight one. Since $J(g)$ is disconnected, we can find a piecewise smooth oriented loop $\gamma_1 \subset U(g)$ that separates $J(g)$. Let Γ_1 be the positively-oriented disc in $\mathbb{P}^1$ having γ_1 as its oriented boundary. Since μ_g is normalized and γ_1 separates $J(g) = \text{supp}(\mu_g)$, we have $0 < \langle \Gamma_1, \mu_g \rangle < 1$. Now suppose that $\Gamma_1, \ldots, \Gamma_{n-1}$ have been chosen. Since $\langle \Gamma_{n-1}, \mu_g \rangle < 1$, we have $J(g) \cap (\mathbb{P}^1 \setminus \Gamma_{n-1}) \neq \emptyset$. Then, according to Theorem 4.1, there is more than one component of $J(g) \cap (\mathbb{P}^1 \setminus \Gamma_{n-1})$, so we can choose an oriented loop $\gamma_n \subset U(g) \cap (\mathbb{P}^1 \setminus \Gamma_{n-1})$ so that at least one component of $J(g) \cap (\mathbb{P}^1 \setminus \Gamma_{n-1})$ is on each side of γ_n. Then, we let Γ_n be the union of oriented discs in $\mathbb{P}^1$ consisting of the points inside of γ_n and any discs from Γ_{n-1} that are not inside of γ_n.

Considering the homology class $[\partial \Gamma_n - \partial \Gamma_{n-1}] \in H_1(U(g))$ we have that

$$lk([\partial \Gamma_n - \partial \Gamma_{n-1}], \mu_g) = \langle \Gamma_n, \mu_G \rangle - \langle \Gamma_{n-1}, \mu_G \rangle \pmod 1$$

is non-zero for each n. However, since

$$\sum_n \langle \Gamma_n, \mu_G \rangle - \langle \Gamma_{n-1}, \mu_G \rangle$$

is bounded by 1, we have that $lk([\partial \Gamma_n - \partial \Gamma_{n-1}], \mu_g) \to 0$ in $\mathbb{Q}/\mathbb{Z}$. Theorem 1.1 then gives that $H_1(U(g))$ is infinitely generated. $\square$

Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be a polynomial endomorphism given in projective coordinates by

$$f([Z : W : T]) = [P(Z, W, T) : Q(Z, W, T) : T^d]. \tag{4}$$

Since $f : \mathbb{P}^2 \to \mathbb{P}^2$ is assumed globally holomorphic, $P(Z, W, T), Q(Z, W, T)$, and T^d have no common zeros other than $(0, 0, 0)$.

The (projective) line at infinity $\Pi := \{T = 0\}$ is uniformly super-attracting and the restriction f_Π is given in homogeneous coordinates by

$$f_\Pi : ([Z : W]) \to [P_0(Z, W) : Q_0(Z, W)]. \tag{5}$$

where $P_0 := P(Z, W, 0)$ and $Q_0 := Q(Z, W, 0)$.

Let $U(f)$ be the Fatou set for f and $U(f_\Pi)$ the Fatou set for f_Π. The former is an open set in $\mathbb{P}^2$, while the latter is an open set in the line at infinity Π.

Lemma 4.4. *If f_Π is hyperbolic then $U(f_\Pi) \subset U(f)$.*

Proof. Since f_Π is hyperbolic, $U(f_\Pi)$ is in the union of the basins of attraction $W^s_\Pi(\zeta_i)$ of a finite number of periodic attracting points $\zeta_1, \ldots, \zeta_k$. The line at infinity Π is transversally superattracting, so each ζ_i is superattracting in the transverse direction to Π and (at least) geometrically attracting within Π. Let $W^s(\zeta_i) \subset \mathbb{P}^2$ be the basin of attraction for ζ_i under f. Then, $W^s_\Pi(\zeta_i) \subset W^s(\zeta_i)$, giving $U(f_\Pi) \subset U(f)$. $\square$

Let T be the Green's current for f and let μ_Π be the measure of maximal entropy for the restriction $f_{|\Pi}$.

Lemma 4.5. *The restriction $T_{|\Pi}$ coincides with μ_Π.*

Proof. Consider the lift $F_\Pi : \mathbb{C}^2 \to \mathbb{C}^2$ of the rational map $f_\Pi : \mathbb{P}^1 \to \mathbb{P}^1$. As observed in Remark 2.4,

$$G_\Pi(Z,W) = \lim \frac{1}{d^n} \log ||F_\Pi^n(Z,W)||$$

is the potential for μ_Π, meaning that $\pi^*\mu_\Pi = \frac{1}{2\pi} dd^c G_\Pi$.

The restriction $T_{|\Pi}$ is obtained by restricting of the potential G to $\pi^{-1}(\Pi) = \{(Z,W,0) \in \mathbb{C}^3\}$. Specifically, it is defined by $\pi^*(T_{|\Pi}) = \frac{1}{2\pi} dd^c(G(Z,W,0))$. Therefore, it suffices to show that $G(Z,W,0) = G_\Pi(Z,W)$. However, this follows directly from the fact that $F(Z,W,0) = F_\Pi(Z,W)$. (Here F is the lift of f to $\mathbb{C}^3$, as given by (4) when considered in non-projective coordinates $[Z,W,T]$.) □

Proof of Theorem 1.2. As in the proof of Proposition 4.2, we can find a sequence of 1-cycles c_n in $U(f_\Pi)$ having linking numbers with μ_Π arbitrarily close to 0 in $\mathbb{Q}/\mathbb{Z}$. Since $f_{|\Pi}$ is hyperbolic, Lemma 4.4 gives that each c_n is in the union of basins of attraction for finitely many attracting periodic points of f. In particular, $lk(c_i,T)$ is well-defined for each n. Lemma 4.5 gives that $T_{|\Pi} = \mu_\Pi$, so that $lk(c_n,T)$ (considering c_n in $\mathbb{P}^2$) coincides with $lk(c_n,\mu_\Pi)$ (considering c_n in the projective line Π). Therefore, $lk(c_n,T) \neq 0$ and $lk(c_n,T) \to 0$ in $\mathbb{Q}/\mathbb{Z}$. Theorem 1.1 gives that the union of these basins has infinitely generated first homology, and hence $U(f)$ does as well. □

Example 4.6. We embed the polynomial dynamics of $r(z)$ from Remark 4.3 as the dynamics on the line at infinity Π for a polynomial endomorphism of $\mathbb{P}^2$. Let $R(Z,W) = Z^3 - 0.48ZW^2 + (0.706260 + 0.502896i)W^3$ be the homogeneous form of r, and let $P(Z,W,T)$ and $Q(Z,W,T)$ be any homogeneous polynomials of degree 2. Then

$$f([Z:W:T]) = [R(Z,W) + T \cdot P(Z,W,T) : W^3 + T \cdot Q(Z,W,T) : T^3]$$

is a polynomial endomorphism with $f_\Pi = r$. In this case, Theorem 1.2 gives that the basin of attraction of $[1:0:0]$ for f has infinitely generated first homology.

Remark 4.7. Suppose that $f : \mathbb{P}^k \to \mathbb{P}^k$ is a holomorphic endomorphism having an invariant projective line Π. Lemma 4.5 can be extended to give that $T_{|\Pi} = \mu_\Pi$, where μ_Π is the measure of maximal entropy for the one-dimensional map $f_{|\Pi}$. If Π is at least geometrically attracting transversally, $f_{|\Pi}$ is hyperbolic, and $J(f_{|\Pi})$ is disconnected, then essentially the same proof as that of Theorem 1.2 gives that the Fatou set $U(f)$ has infinitely generated first homology.

Using this observation, one can inductively create sequences of polynomial endomorphisms $f_k : \mathbb{P}^k \to \mathbb{P}^k$, for every k, each having Fatou set with infinitely generated first homology. One begins with a hyperbolic polynomial endomorphism f_1 of the Riemann sphere $\mathbb{P}^1$ having disconnected Julia set. Then, for each k, one can let $f_k : \mathbb{P}^k \to \mathbb{P}^k$ be any polynomial endomorphism whose dynamics on the hypersurface $\mathbb{P}^{k-1}$ at infinity is given by f_{k-1}. (When $k = 2$, the construction of $f_2 : \mathbb{P}^2 \to \mathbb{P}^2$ is similar to that from Example 4.6.) The resulting maps each have a totally-invariant projective line Π that is transversally superattracting with $f_{k|\Pi} = f_1$ hyperbolic with disconnected Julia set. Thus, the Fatou set $U(f_k)$ has infinitely generated first homology.

5. Application to Polynomial skew products

A *polynomial skew product* is a polynomial endomorphism of the form

$$f(z,w) = (p(z), q(z,w))$$

with p and q polynomials of degree d where $p(z) = z^d + O(z^{d-1})$ and $q(z) = w^d + O_z(w^{d-1})$. (See Jonsson [25].) Theorem 1.2 can by applied to many polynomial skew products f to show that that $H_1(U(f))$ is infinitely generated; for example, $f(z,w) = (z^2, w^2 + 10z^2)$, which has J_Π a Cantor set. Next we will find alternative sufficient conditions under which a polynomial skew product has Fatou set with infinitely generated first homology, proving Theorem 1.3. This theorem will apply to many maps to which Theorem 1.2 does not apply; for example, $f(z,w) = (z^2, w^2 - 3z)$, for which J_Π is equal to the unit circle.

5.1. **Preliminary background on polynomial skew products.** The Green's current for any polynomial endomorphism can be computed in the affine coordinates on $\mathbb{C}^2$ as $T := \frac{1}{2\pi} dd^c G_{\text{affine}}$, where G_{affine} is the (affine) Green's function defined in Remark 2.2. The "base map" $p(z)$ has a Julia set $J_p \subset \mathbb{C}$ and, similarly, a Green's function $G_p(z) := \lim_{n\to\infty} \frac{1}{d^n} \log_+ ||p^n(z)||$. Furthermore, one can define a fiber-wise Green's function[2] by:

$$G_z(w) := G_{\text{affine}}(z,w) - G_p(z).$$

For each fixed z, $G_z(w)$ is a subharmonic function of w and one defines the fiber-wise Julia sets by $K_z := \{G_z(w) = 0\}$ and $J_z := \partial K_z$.

The extension of f to $\mathbb{P}^2$ is given by

$$f([Z:W:T]) = [P(Z,T) : Q(Z,W,T) : T^d], \tag{6}$$

where $P(Z,T)$ and $Q(Z,W,T)$ are the homogeneous versions of p and q. The point $[0:1:0]$ that is "vertically at infinity" with respect to the affine coordinates (z,w) is a totally-invariant super-attracting fixed point and $(z,w) \in W^s([0:1:0])$ if and only if $w \in \mathbb{C} \setminus K_z$.

Suppose that $(z,w) \in W^s([0:1:0])$ and $(z_n, w_n) := f^n(z,w)$. Then,

$$G_{\text{affine}}(z,w) = \lim \frac{1}{d^n} \log_+ \|f^n(z,w)\|_\infty = \lim \frac{1}{d^n} \log_+ |w_n| \text{ and} \tag{7}$$

$$G_z(w) = G_{\text{affine}}(z,w) - G_p(z) = \lim \frac{1}{d^n} \log_+ |w_n| - \lim \frac{1}{d^n} \log_+ |z_n|. \tag{8}$$

since $|w_n| > |z_n|$ for all n sufficiently large.

As mentioned in Section 3.3, we can restrict the current T to any analytic curve obtaining a measure on that curve. Of particular interest for skew products is the restriction μ_{z_0} of T to a vertical line $\{z_0\} \times \mathbb{P}$. The following appears as Jonsson [25] Proposition 2.1 (i), we repeat it here for completeness:

Proposition 5.1. *The restriction $T_{|z=z_0}$ of the Green's current T to a vertical line $(\{z_0\} \times \mathbb{P})$ coincides with the harmonic measure μ_{z_0} on K_{z_0}.*

[2]For the purist: the Green's functions G_p and G_z should also have the subscript "affine", but it is dropped here for ease of notation. See Section 2 for the distinction.

Proof. Notice that

$$\begin{aligned} T_{|z=z_0} &= \frac{1}{2\pi} dd^c G_{\text{affine}|z=z_0} = \frac{1}{2\pi} dd^c G_{\text{affine}}(z_0, w) \\ &= \frac{1}{2\pi} dd^c \left(G_{\text{affine}}(z_0, w) - G_p(z_0) \right) = \frac{1}{2\pi} dd^c G_{z_0}(w). \end{aligned}$$

According to [25, Thm 2.1], G_{z_0} is the Green's function for K_z with pole at infinity. We have thus obtained that μ_{z_0} is exactly the harmonic measure μ_{z_0} on K_{z_0}. $\square$

5.2. **Topology of the basin of attraction** $W^s([0:1:0])$**.**

Proposition 5.2. *If ζ is a totally-invariant (super)-attracting fixed point for a holomorphic $f : \mathbb{CP}^k \to \mathbb{CP}^k$, then $W^s(\zeta)$ is path-connected.*

A nearly identical statement is proven for $\mathbb{CP}^2$ in Theorem 1.5.9 from [22]. We refer the reader to their proof since it is nearly identical for $\mathbb{CP}^k$. In particular, for any skew product $W^s([0:1:0])$ is path connected.

Although $G_z(w)$ is subharmonic in w for any fixed z, it does not form a PSH function of both z and w. Consider the points $(z, w) \in W^s([0:1:0])$ for which $z \in J_p$. At these points G_{affine} is pluriharmonic, i.e. $dd^c G_{\text{affine}} = 0$, but $G_p(z)$ is not pluriharmonic, i.e. $dd^c G_p(z) > 0$. Therefore, at these points $dd^c G_z(w) < 0$, so $G_z(w)$ is not PSH.

Lemma 5.3. *The function $-G_z(w)$ is PSH at all points $(z, w) \in W^s([0:1:0]) \cap \mathbb{C}^2$ and it extends to a PSH function on all of $W^s([0:1:0])$. The resulting function is pluriharmonic on $W^s([0:1:0])$ except at points for which $Z/T \in J_p$.*

Proof. Since $-G_z(w) = G_p(z) - G_{\text{affine}}(z, w)$, with $G_{\text{affine}}(z, w)$ pluriharmonic in $W^s([0:1:0])$ and $G_p(z)$ PSH everywhere, the result is PSH in $W^s([0:1:0]) \cap \mathbb{C}^2$.

Jonsson proves in [25, Lemma 6.3] that $G_z(w)$ extends as a PSH function in a suitable neighborhood of $\Pi \setminus \{[0:1:0]\}$ and his proof immediately gives that the result is pluriharmonic in a (possibly smaller) neighborhood within $W^s([0:1:0])$ of $\Pi \setminus \{[0:1:0]\}$. Therefore, $-G_z(w)$ is also pluriharmonic in the same neighborhood.

Thus, $-G_z(w)$ extends to a PSH on $W^s([0:1:0]) \setminus \{[0:1:0]\}$ and, assigning $-\infty$ to $[0:1:0]$, gives the desired extension to all of $W^s([0:1:0])$. The result will be pluriharmonic except at $[0:1:0]$ and at the points in $W^s([0:1:0]) \cap \mathbb{C}^2$ where $dd^c(-G_z(w)) > 0$, that is the points where $Z/T \in J_p$. $\square$

Proof of Theorem 1.3: We first suppose that J_{z_0} is disconnected for some $z_0 \in J_p$. Let $z_1, z_2, \ldots$ be any sequence of iterated preimages of z_0 so that $p^n(z_n) = z_0$.

Consider the vertical line $\{z_0\} \times \mathbb{C}$. Since J_{z_0} is disconnected, so is K_{z_0}, and we can choose two disjoint positively-oriented piecewise smooth loops $\eta_1, \eta_2 \subset \{z_0\} \times (\mathbb{C} \setminus K_{z_0})$ each enclosing a proper subset of K_{z_0}.

Perturbing η_1, η_2 within $\{z_0\} \times (\mathbb{C} \setminus K_{z_0})$, if necessary, we can suppose that none of the $d-1$ critical values of $f|_{\{z_1\} \times \mathbb{C}} : \{z_1\} \times \mathbb{C} \to \{z_0\} \times \mathbb{C}$ (counted with multiplicity) are on η_1 or η_2. Since the regions enclosed by η_1 and η_2 are disjoint, at least one of them contains at most $d-2$ of these critical values. Let γ_0 be this curve.

Since $\gamma_0 \subset \{z_0\} \times (\mathbb{C} \setminus K_{z_0})$, $\gamma_0 \subset W^s([0:1:0])$. Because γ_0 is compact, it is bounded away from $\text{supp}(T)$, and the linking number $lk(\gamma_0, T)$ is a well defined invariant of the homology class $[\gamma]$ within $H_1(W^s([0:1:0]))$. We let Γ_0 be the closed disc in $(\{z_0\} \times \mathbb{C})$ having γ_0 as its oriented boundary. Since Γ_0 contains some proper subset of K_{z_0} (and hence of J_{z_0}) with $\operatorname{supp}(\mu_{z_0}) = J_{z_0}$, we have that

$$0 < \langle \Gamma_0, T \rangle = \int_{\Gamma_0} \mu_{z_0} < 1.$$

Therefore, $lk(\gamma_0, T) = \langle \Gamma_0, T \rangle \pmod 1 \neq 0 \pmod 1$, giving that $[\gamma_0]$ is non-trivial.

Consider the preimages $D_1, \ldots, D_j$ of Γ_0 under $f|_{\{z_1\}\times\mathbb{C}} : \{z_1\} \times \mathbb{C} \to \{z_0\} \times \mathbb{C}$. Since at most $d-2$ critical values of the degree d ramified cover $f|_{\{z_1\}\times \cup D_i}$ are contained in Γ_0, it is a consequence of the Riemann-Hurwitz Theorem that the Euler characteristic of $\cup D_i$ is greater than or equal to 2. Because each D_i is a domain in $\mathbb{C}$, at least two components D_1 and D_2 are discs.

The total degree of $f|_{\{z_1\}\times\mathbb{C}} : \cup D_i \to \Gamma_0$ is d, so $f|_{\{z_1\}\times\mathbb{C}} : D_i \to \Gamma_0$ a ramified covering of degree $k_i \leq d-1$ for each i. Proposition 3.6 and the basic invariance $f^*T = d \cdot T$ for the Green's current give that

$$\langle D_i, T \rangle = \frac{1}{d} \langle D_i, f^*T \rangle = \frac{1}{d} \langle f_* D_i, T \rangle = \frac{1}{d} \langle k_i \Gamma_0, T \rangle \leq \frac{d-1}{d} \langle \Gamma_0, T \rangle \tag{9}$$

for each i.

As before, we can perturb the boundaries of D_1 and D_2 within $\{z_1\} \times (\mathbb{C} \setminus K_{z_1})$ so that none of the critical values of $f|_{\{z_2\}\times\mathbb{C}}$ lie on their boundaries and so that D_1 and D_2 remain disjoint. (It will not affect the pairings given by (9)). At least one of the discs D_1, D_2 contains at most $d-2$ critical values of $f|_{\{z_2\}\times C}$. We let Γ_1 be that disc and $\gamma_1 = \partial\Gamma_1$. Then

$$0 < \langle \Gamma_1, T \rangle \leq \frac{d-1}{d} \langle \Gamma_0, T \rangle \leq \frac{d-1}{d}.$$

Continuing in the same way, we can find a sequence of discs $\Gamma_0, \Gamma_1, \ldots$ so that

- $\Gamma_n \subset \{z_n\} \times \mathbb{C}$,
- $\gamma_n = \partial\Gamma_n \subset W^s([0:1:0])$,
- Γ_n contains at most $d-2$ critical values of $f|_{\{z_{n+1}\}\times\mathbb{C}}$ (counted with multiplicity), and
- $\langle \Gamma_n, T \rangle \leq \frac{d-1}{d} \langle \Gamma_{n-1}, T \rangle$.

Consequently,

$$0 < \langle \Gamma_n, T \rangle \leq \left(\frac{d-1}{d} \right)^n,$$

giving that $lk(\gamma_n, T) \to 0$ in $\mathbb{Q}/\mathbb{Z}$. Therefore, Theorem 1.1 gives that $H_1(W^s([0:1:0]))$ is infinitely generated.

We will now show that if J_z is connected for every $z \in J_p$, then $W^s([0:1:0])$ is homeomorphic to an open ball. Consider the local coordinates $z' = Z/W$, $t' = T/W$, chosen so that $(z', t') = (0,0)$ corresponds to $[0:1:0]$. In these coordinates

$$f(z', t') = \left(\frac{P(z', t')}{Q(z', 1, t')}, \frac{t'^d}{Q(z', 1, t')} \right),$$

where P and Q are the homogeneous versions of p and q appearing in Equation (6). The assumption that $q(z) = w^d + O_z(w^{d-1})$ and $p(z) = z^d + O(z^{d-1})$ imply that we have the expansion

$$f(z', t') = (P(z', t'), t'^d) + g(t', z'),$$

with $(P(z', t'), t'^d)$ non-degenerate of degree d and $g(t', z')$ consisting of terms of degree $d+1$ and larger.

Therefore, we can construct a potential function[3] for the superattracting point $(0, 0)$:

$$h(z', t') := \lim_{n\to\infty} \frac{1}{d^n} \log \|f^n(z', t')\|_\infty. \tag{10}$$

The result is a continuous pluri-subharmonic function [22] with logarithmic singularity at $(z', t') = (0, 0)$ having the property that $(z', t') \in W^s([0 : 1 : 0]) \setminus \{[0 : 1 : 0]\}$ if and only if $h(z', t') < 0$. In particular,

$$h : W^s([0 : 1 : 0]) \setminus \{[0 : 1 : 0]\} \longrightarrow (-\infty, 0)$$

is proper.

If we let $(z'_n, t'_n) = f^n(z', t')$, then Equation (10) simplifies to

$$h(z', t') = \begin{cases} \lim \frac{1}{d^n} \log |t'_n| & \text{if } z'/t' \in K_p \text{ and} \\ \lim \frac{1}{d^n} \log |z'_n| & \text{if } z'/t' \notin K_p. \end{cases} \tag{11}$$

since $z'_{n+1}/t'_{n+1} = p(z'_n/t'_n)$. Equation (8) gives that in the original affine coordinates (z, w) we have

$$h(z, w) = \begin{cases} \lim \frac{1}{d^n} \log |t'_n| = -\lim \frac{1}{d^n} \log |w_n| = -G_z(w) & \text{if } z \in K_p \text{ and,} \\ \lim \frac{1}{d^n} \log |z'_n| = \log |z| - \lim \frac{1}{d^n} \log |w_n| = -G_z(w) & \text{if } z \notin K_p, \end{cases} \tag{12}$$

which is harmonic on the intersection of any vertical line $\{z\} \times \mathbb{C}$ with $W^s([0 : 1 : 0])$ and pluriharmonic except when $z \in J_p$; see Lemma 5.3. A similar calculation shows that h coincides with the extension of $-G_z(w)$ described in Lemma 5.3 and that the restriction of h to Π is $-G_\Pi$. (Here, G_Π is the Green's function for the action f_Π of f on the line at infinity.)

Therefore, $h(z', t')$ is pluriharmonic on $W^s([0 : 1 : 0]) \setminus \{(z', w') : z'/w' \in J_p\}$ and the restriction of $h(z', t')$ to any line through $(0, 0)$ is harmonic on $W^s([0 : 1 : 0]) \setminus \{[0 : 1 : 0]\}$, with a logarithmic singularity at $(0, 0)$.

Since J_{z_0} is connected for every $z_0 \in J_p$, Proposition 6.3 from [25] gives that J_z is connected for every $z \in \mathbb{C}$ and also J_Π is connected, or, equivalently, that $G_z(w)$ (for any z) and G_Π have no (escaping) critical points. Therefore, the restriction of h to any complex line through $(0, 0)$ has no critical points in $W^s([0 : 1 : 0])$.

The sublevel set $W_a := h^{-1}([-\infty, a))$ is open for any $a \in (-\infty, 0)$ since $h : W^s([0 : 1 : 0]) \setminus \{[0 : 1 : 0]\} \to (-\infty, 0)$ is continuous with $h(z', t') \to -\infty$ if and only if $(z', t') \to (0, 0)$.

[3]The potential function h is sometimes also be called the Green's function of the point $(0, 0)$.

Equation 2.2 from [25] implies that

$$\begin{aligned} h(z',t') &= \log|t'| + G_p\left(\frac{z'}{t'}\right) + \eta(z',t') \text{ if } t' \neq 0, \text{ and} \\ h(z',t') &= \log|z'| + G_p^{\#}\left(\frac{t'}{z'}\right) + \eta(z',t') \text{ if } z' \neq 0, \end{aligned}$$

with $\eta(z',t')$ becoming arbitrarily small for (z',t') sufficiently small and $G_p^{\#}(x)$ obtained by extending $G_p(1/x) - \log(1/x)$ continuously through $x = 0$. Therefore, for a sufficiently negative, the intersection of W_a with any complex line through $(0,0)$ will be convex. In particular, W_a is an star-convex open subset in $\mathbb{C}^2$, implying that it is homeomorphic to an open ball. (See [7, Theorem 11.3.6.1].)

We define a new function $\widetilde{h}$ which agrees with h except in the interior of W_a, where we make a C^∞ modification (assigning values less than a) in order to remove the logarithmic singularity at $[0:1:0]$.

We will use $\widetilde{h}$ as Morse function to show that $W_b := h^{-1}([-\infty, b))$ is diffeomorphic to W_a for any $b \in (a, 0)$. The classical technique from Theorem 3.1 of [29] would use the normalization of $-\nabla\widetilde{h}$ to generate a flow whose time $(b-a)$ map gives the desired diffeomorphism. This will not work in our situation, since $\widetilde{h}$ is not differentiable at points for which $z'/w' \in J_p$. However, essentially the same proof works if we replace $-\nabla\widetilde{h}$ with any C^1 vector field V on $W^s([0:1:0])$ having no singularities in $\widetilde{h}^{-1}([a,b])$ and along which $\widetilde{h}$ is decreasing. Note that, as in [29], we need that $\widetilde{h}^{-1}([a,b])$ is compact, which follows from h being proper.

Let V be the the vector field parallel to each line through $(z',t') = (0,0)$, obtained within each line as minus the gradient of the restriction of $\widetilde{h}$ to that line. The restriction of $\widetilde{h}$ to each complex line through $(0,0)$ has no critical points in $\widetilde{h}^{-1}([a,b])$, so it is decreasing along V. Since h is pluriharmonic for points with $z'/t' \notin J_p$, it follows immediately that V is smooth there. To see that V is smooth in a neighborhood of points where $z'/t' \in J_p$, notice that

$$\nabla_w G_z(w) = \nabla_w G(z,w) - G_p(z) = \nabla_w G(z,w),$$

with $G(z,w)$ pluriharmonic on $W^s([0:1:0]) \cap \mathbb{C}^2$.

Therefore, for any $b \in (a,0)$, W_b is homeomorphic to W_a and thus to an open ball. One can then make a relatively standard construction, using these homeomorphisms for b increasing to 0, in order to show that $W^s([0:1:0]) = \cup_{b<0} W_b$ is homeomorphic to an open ball. □

6. Further applications

In this final section we discuss a few examples of maps to which we have applied the results of this paper, and then a few types of maps which we feel would be fruitful to study further with techniques similar to those of this paper.

6.1. **Relationship between connectivity of J_2 and the topology of the Fatou set for polynomial skew products.** For polynomial skew products, $J_2 = \text{supp}(\mu) = \text{supp}(T \wedge T) = \overline{\bigcup_{z \in J_p} J_z}$, which by [25] is also the closure of the set of repelling periodic points. Here we examine to what extent connectivity of J_2 affects the homology of the Fatou set U.

The following example shows that there are many polynomial skew products f with J_2 connected for which $H_1(U(f))$ is non-trivial (in fact infinitely generated.)

Example 6.1. Consider $f(z,w) = (z^2 - 2, w^2 + 2(2-z))$ which has J_2 connected and has J_z disconnected over $z = -2 \in J_p$, as shown in [25, Example 9.7]. Theorem 1.3 immediately applies, giving that $H_1(U(f))$ is infinitely generated.

In fact, examples of this phenomenon can appear "stably" within a one parameter family. Let $p_n(z) = z^2 + c_n$ be the unique quadratic polynomial with periodic critical point of least period n and c_n real. Then, [11, Theorem 6.1] yields that for n sufficiently large,

$$f_n(z,w) = (p_n(z), w^2 + 2(2-z))$$

is Axiom A with J_z disconnected for most $z \in J_{p_n}$ and with J_2 connected. Suppose that f_n is embedded within any holomorphic one-parameter family $f_{n,\lambda}$ of polynomial skew products. Then, Theorems 4.1 and 4.2 from [11] (see also, [24, Thm C]) give that all maps $f_{n,\lambda}$ within the same hyperbolic component as f_n also have J_2 connected, but J_z disconnected over most z in $J_{p_{n,\lambda}}$. (Here, $p_{n,\lambda}$ is the first component of $f_{n,\lambda}$.) An immediate application of Theorem 1.3 yields that $H_1(U(f_{n,\lambda}))$ is infinitely generated for all $f_{n,\lambda}$ within this hyperbolic component.

Next we consider the possibility of J_2 being disconnected, but f not satisfying the hypotheses of our Theorem 1.3.

Question 6.2. Is there a polynomial skew product f with J_2 disconnected, but all J_z's connected for all $z \in \mathbb{C}$, such that $H_1(U(f))$ is trivial? More generally, is there any endomorphism of $\mathbb{P}^2$ with J_2 disconnected, but with all Fatou components having trivial homology?

By [25, Proposition 6.6], in order for f to satisfy the hypotheses of this question, J_p would have to be disconnected. However, a simple product like $(z,w) \mapsto (z^2 - 100, w^2)$ does not suffice; note for this map, the basin of attraction of $[1:0:0]$, hence the Fatou set, has nontrivial homology. Not many examples of non-product polynomial skew products are understood, and the current list of understood examples contains no maps which satisfy the hypotheses of this question.

6.2. **A quadratic family of polynomial skew products.** We now consider the family of examples $f_a(z,w) = (z^2, w^2 + az)$, which are skew products over $p(z) = z^2$.

The geometry and dynamics in $J_p \times \mathbb{C}$ were explored in [11]. For example, there it is established that:

(1) [11, Theorem 5.1]: f_a is Axiom A if and only if $g_a(w) := w^2 + a$ is hyperbolic; and
(2) [11, Lemma 5.5]: J_2 can be described geometrically in the following manner: $J_{e^{it}}$ is a rotation of angle $t/2$ of $J_{\{z=1\}}$. That is, start with $J(g_a)$ in the fiber $J_{\{z=1\}}$, then as the base point $z = e^{it}$ moves around the unit circle $J_p = S^1$, the corresponding J_z's are rotations of $J(g_a)$ of angle $t/2$, hence the J_z's complete a half turn as z moves once around the base circle.

Due to the structure of J_2, the difference between f_a and the product $h_a(z,w) = (z^2, w^2 + a)$ is one "twist" in J_2. In [11] it is shown that f_a and h_a are in the same hyperbolic component if and only if a is in the main cardiod of the Mandelbrot set, $\mathcal{M}$.

Note that the extension of f_a to $\mathbb{P}^2$, given by $f_a([Z:W:T]) = [Z^2, W^2 + aZT : T^2]$, is symmetric under the involution $\mathcal{S}([Z:W:T]) = [T:W:Z]$.

Theorem 6.3. *The Fatou set of f_a is the union of the basins of attraction of three super-attracting fixed points: $[0:0:1], [0:1:0]$, and $[1:0:0]$, each of which is path-connected.*

Moreover:

- *If $a \notin \mathcal{M}$, then $W^s([0:1:0])$ has infinitely generated first homology.*
- *If $a \in \mathcal{M}$, then each of the three basins of attraction $W^s([0:1:0]), W^s([0:0:1])$ and $W^s([1:0:0])$ is homeomorphic to an open ball.*

Proof. For any a, the fiberwise Julia set J_0 is the unit circle $|z| = 1$. Proposition 4.2 from [32] can be modified to show that there is a local super-stable manifold $W^s_{\rm loc}(J_0)$ that is obtained as the image of a holomorphic motion of J_0 that is parameterized over $\mathbb{D}_\epsilon = \{|z| < \epsilon\}$, for $\epsilon > 0$ sufficiently small. The motion of $(0,w) \in J_0$ is precisely the connected component of local super-stable manifold of $(0,w)$ that contains $(0,w)$, which we will call the *superstable leaf of* w and denote by $W^s_{\rm loc}(w)$. By construction, f_a will map the superstable leaf of $(0,w)$ into the superstable leaf of $(0,w^2) = f_a(0,w)$. Moreover, the proof of Proposition 4.4 from [32] can also be adapted to show that $W^s_{\rm loc}(J_0)$ is the zero locus of a pluri-harmonic (hence real-analytic) function.

Pulling back $W^s_{\rm loc}(J_0)$ under iterates of f_a, we obtain a global separatrix $W^s(J_0)$ over the entire unit disc $\mathbb{D} = \{|z| = 1\}$. Note that $W^s(J_0)$ may not be a manifold, since ramification may occur at points where it intersects the critical locus of f_a. For $|z| < 1$, J_z is the intersection of $W^s(J_0)$ with $\{z\}\times\mathbb{C}$ and that K_z is the intersection of $W^s([0:0:1])\cup W^s(J_0)$ with $\{z\} \times \mathbb{C}$. Thus, any point (z,w) with $|z| < 1$ is in $W^s([0:0:1]) \cup W^s(J_0) \cup W^s([0:1:0])$.

Under the symmetry $\mathcal{S}$, each of the above statements about the super-stable manifold of J_0 corresponds immediately to a statement about the unit circle $J_\Pi = \{|Z/W| = 1\}$ in the line at infinity $\Pi = \{T = 0\}$. Moreover, any point in $\mathbb{P}^2$ with $|T| < |Z|$ is in $W^s([1:0:0]) \cup W^s(J_\Pi) \cup W^s([0:1:0])$. Therefore, the Fatou set of f_a is the union of basins of attraction for three superattracting fixed points $[1:0:0]$, $[0:1:0]$, and $[0:0:1]$. Since each of these fixed points is totally invariant, Proposition 5.2 gives that each of their basins of attraction is path connected.

The vertical Julia J_1 set over the fixed fiber $z = 1$ is precisely the Julia set of $w \mapsto w^2 + a$, which is connected if and only if $a \in \mathcal{M}$. In particular, if $a \notin \mathcal{M}$, it follows from Theorem 1.3 that $W^s([0:1:0])$ has infinitely generated first homology.

If $a \in \mathcal{M}$, then, for each $z \in J_p$, J_z is a rotation of the connected set J_1 and Theorem 1.3 gives that $W^s([0:1:0])$ is homeomorphic to an open ball. We will now use Slodkowski's Theorem on holomorphic motions [35] (see also [23, Section 5.2]) to show that $W^s([0:0:1])$ and $W^s([1:0:0]) = \mathcal{S}(W^s([0:0:1]))$ are homeomorphic to the open bidisc.

We will extend (in the parameter z) the holomorphic motion whose image is $W^s_{\rm loc}(J_0)$ to a holomorphic motion of J_0 parameterized by $z \in \mathbb{D}$, having the entire separatrix $W^s(J_0)$ as its image. Then, by Slodkowski's Theorem, this holomorphic motion extends (in the fiber w) from J_0 to a holomorphic motion of the entire Riemann sphere $\mathbb{P}^1$ that is also parameterized by $z \in \mathbb{D}$. Consequently, $W^s([0:0:1])$ will be the image of a holomorphic motion of the open disc $\{z = 0, |w| < 1\}$, parameterized by $z \in \mathbb{D}$.

Since $a \in \mathcal{M}$, it also follows from [25, Proposition 6.4] that for each $z \in \mathbb{C}$ the fiber-wise critical points

$$C_z := \{w \in \mathbb{C} \,:\, q_z'(w) = 0\}$$

are in K_z. We now check that they are disjoint from $W^s(J_0)$.

The union of these fiber-wise critical points is just the horizontal line $w = 0$ that stays on one side of $W^s(J_0)$, possibly touching at many points. Note, however that they are disjoint at $z = 0$. Consider the point z_0 (with $|z_0| < 1$) of smallest modulus where $w = 0$ and $W^s(J_0)$ touch. Then, there is a neighborhood of U of z_0 in $\mathbb{C}^2$ in which $W^s(J_0)$ is given by the zero set of a PSH function Ψ. Changing the sign of Ψ (if necessary) we can assume that $\Psi \leq 0$ for points in $K_z \cap U$. The restriction $\psi(z) = \Psi|_{w=0}$ is a non-positive harmonic function in a neighborhood of z_0 having $\psi(z_0) = 0$, but $\psi(z) < 0$ for z with $|z| < |z_0|$. This violates the maximum principle. Therefore, the fiber-wise critical points C_z are disjoint from $W^s(J_0)$ for every z.

Suppose that $\mathcal{D} \subset W^s(J_0)$ is the graph of a holomorphic function $\nu(z)$ defined on $\{|z| < r\}$, for some $0 < r < 1$. Then, since $W^s(J_0)$ is disjoint from the horizontal critical locus $w = 0$, the Implicit Function Theorem gives that $f_a^{-1}(\mathcal{D})$ is the union of two discs through the pre-images of $\nu(0)$, each given as the graph of a holomorphic function over $\{|z| < \sqrt{r}\}$.

Let $(0, w) \in J_0$ with preimages $(w_1, 0)$ and $(w_2, 0)$. Since $f_a(W^s_{\text{loc}}(w_{1,2})) \subset W^s_{\text{loc}}(w)$, the two discs from $f_a^{-1}(W^s_{\text{loc}}(w))$ form extensions of $W^s_{\text{loc}}(w_1)$ and $W^s_{\text{loc}}(w_2)$, as graphs of holomorphic functions of $|z| < \sqrt{\epsilon}$.

Therefore, by taking the preimages under f_a, the family of local stable discs can be extended, each as the graph of a holomorphic function over $|z| < \sqrt{\epsilon}$. Applied iteratively, we can extend them as the graphs of holomorphic functions over discs $|z| < r$ for any $r < 1$. In the limit we obtain global stable curves $W^s(w_0)$ through every $w_0 \in J_0$, each of which is the graph if a holomorphic function of $z \in \mathbb{D}$. Since the global stable curves of distinct points in J_0 are disjoint, their union gives $W^s(J_0)$ as the image of a holomorphic motion of J_0 parameterized by $z \in \mathbb{D}$. □

6.3. **Postcritically Finite Holomorphic Endomorphisms.** Until presenting the conjecture of the previous subsection, this paper has been about endomorphisms with complicated Fatou topology. The opposite extreme is that the Fatou topology may also be trivial in many cases. We suspect one simple case in which Fatou topology is trivial is when the map is postcritically finite (PCF).

Question 6.4. Does the Fatou set of a postcritically finite holomorphic endomorphism of $\mathbb{P}^2$ always have trivial homology?

A starting point for investigation into this question could be to attempt to establish it for the postcritically finite examples constructed by Sarah Koch [26, 27]. Heuristic evidence supports that the homology is trivial for Koch's maps. Her construction provides a class of PCF endomorphisms, containing an infinite number of maps, including the previously studied examples of [13] and [10].

6.4. **Other holomorphic endomorphisms of $\mathbb{P}^k$.** As we have demonstrated in Sections 4 and 5, given some information about the geometry of the support of T, we can apply the techniques of Sections 3 to study the Fatou set of a holomorphic endomorphism of $\mathbb{P}^2$. We would like to be able to apply this theorem to other holomorphic endomorphisms of $\mathbb{P}^k$.

However, specific examples of holomorphic endomorphisms that are amenable to analytic study are notoriously difficult to generate.

One family of endomorphisms which seem a potentially vast area of study are the Hénon-like endomorphisms. Introduced by Hubbard and Papadapol in [21], and studied a bit further by Fornæss and Sibony in [15], these are holomorphic endomorphisms arising from a certain perturbation of the Hénon diffeomorphisms. The Hénon diffeomorphisms have been deeply studied (e.g., by Bedford Lyubich, and Smillie, [2, 4], Bedford and Smillie [5, 6], Hubbard and Oberste-Vorth [19, 20], and Fornæss and Sibony [12]). A natural question which is thus far quite wide open is: how does the dynamics of a Hénon diffeomorphism relate to the dynamics of the perturbed Hénon endomorphism? Computer evidence suggests the dynamics of Hénon-like endomorphisms is rich and varied.

Specifically concerning the topology of the Fatou set, the main result of [5] is that connectivity of the Julia set is determined by connectivity of a slice Julia set in a certain unstable manifold. We ask whether this result would have implications for the related Hénon endomorphism, which would allow us to use linking numbers to establish some analog of Theorem 1.3 for Hénon endomorphisms.

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