

CHROMATIC ZEROS ON HIERARCHICAL LATTICES AND EQUIDISTRIBUTION ON PARAMETER SPACE

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ABSTRACT. Associated to any finite simple graph Γ is the *chromatic polynomial* $\mathcal{P}_\Gamma(q)$ whose complex zeros are called the *chromatic zeros* of Γ . A hierarchical lattice is a sequence of finite simple graphs $\{\Gamma_n\}_{n=0}^\infty$ built recursively using a substitution rule expressed in terms of a generating graph. For each n , let μ_n denote the probability measure that assigns a Dirac measure to each chromatic zero of Γ_n . Under a mild hypothesis on the generating graph, we prove that the sequence μ_n converges to some measure μ as n tends to infinity. We call μ the *limiting measure of chromatic zeros* associated to $\{\Gamma_n\}_{n=0}^\infty$. In the case of the Diamond Hierarchical Lattice we prove that the support of μ has Hausdorff dimension two.

The main techniques used come from holomorphic dynamics and more specifically the theories of activity/bifurcation currents and arithmetic dynamics. We prove a new equidistribution theorem that can be used to relate the chromatic zeros of a hierarchical lattice to the activity current of a particular marked point. We expect that this equidistribution theorem will have several other applications.

1. INTRODUCTION

Motivated by a concrete problem from combinatorics and mathematical physics, we will prove a general theorem about the equidistribution of certain parameter values for algebraic families of rational maps. We will begin with the motivating problem about chromatic zeros (Section 1.1) and then present the general equidistribution theorem (Section 1.2).

1.1. Chromatic zeros on hierarchical lattices. Let Γ be a finite simple graph. The *chromatic polynomial* $\mathcal{P}_\Gamma(q)$ counts the number of ways to color the vertices of Γ with q colors so that no two adjacent vertices have the same color. It is straightforward to check that the chromatic polynomial is monic, has integer coefficients, and has degree equal to the number of vertices of Γ . The chromatic polynomial was introduced in 1912 by G.D. Birkhoff in an attempt to solve the Four Color Problem [11, 12]. Although the Four Color Theorem was proved later by different means, chromatic polynomials and their zeros have become a central part of combinatorics.¹ For a comprehensive discussion of chromatic polynomials we refer the reader to the book [31].

A further motivation for study of the chromatic polynomials comes from statistical physics because of the connection between the chromatic polynomial and the partition function of the antiferromagnetic Potts Model; see, for example, [73, 5, 70] and [4, p.323-325].

We will call a sequence of finite simple graphs $\Gamma_n = (V_n, E_n)$, where the number of vertices $|V_n| \rightarrow \infty$, a “lattice”. The standard example is the \mathbb{Z}^d lattice where, for each $n \geq 0$, one defines Γ_n to be the graph whose vertices consist of the integer points in $[-n, n]^d$ and whose edges connect vertices at distance one in \mathbb{R}^d . For a given lattice, $\{\Gamma_n\}_{n=1}^\infty$, we are interested in whether the

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¹For example, a search on Mathscinet yields 333 papers having the words “chromatic polynomial” in the title.

sequence of measures

$$(1.1) \quad \mu_n := \frac{1}{|V_n|} \sum_{\substack{q \in \mathbb{C} \\ \mathcal{P}_{\Gamma_n}(q)=0}} \delta_q$$

has a limit μ , and in describing its limit if it has one. Here, δ_q is the Dirac measure which, by definition, assigns measure 1 to a set containing q and measure 0 otherwise. (In (1.1) zeros of $\mathcal{P}_{\Gamma_n}(q)$ are counted with multiplicity.) If μ exists, we call it the *limiting measure of chromatic zeros* for the lattice $\{\Gamma_n\}_{n=1}^\infty$.

This problem has received considerable interest from the physics community especially through the work of Shrock with and collaborators Biggs, Chang, and Tsai (see [64, 21, 10, 22, 66] for a sample) and Sokal with collaborators Jackson, Procacci, Salas and others (see [61, 62, 45] for a sample). Indeed, one of the main motivations of these papers is understanding the possible ground states (temperature $T = 0$) for the thermodynamic limit of the Potts Model, as well as the phase transitions between them. Most of these papers consider sequences of $m \times n$ grid graphs with $m \leq 30$ fixed and $n \rightarrow \infty$. This allows the authors to use transfer matrices and the Beraha-Kahane-Weiss Theorem [6] to rigorously deduce (for fixed m) properties of the limiting measure of chromatic zeros. The zeros typically accumulate to some real-algebraic curves in \mathbb{C} whose complexity increases as m does; see [64, Figures 1 and 2] and [61, Figures 21 and 22] as examples. Indeed, this behavior was first observed in the 1972 work of Biggs-Damerell-Sands [9] and then, more extensively, in the 1997 work of Shrock-Tsai [65]. Beyond these cases with m fixed, numerical techniques are used in [62] to make conjectures about the limiting behavior of the zeros as $m \rightarrow \infty$, i.e. for the \mathbb{Z}^2 lattice.

To the best of our knowledge, it is an open and very difficult question whether there is a limiting measure of chromatic zeros for the \mathbb{Z}^2 lattice. If such a measure does exist, rigorously determining its properties also seems quite challenging. For this reason, we will consider the limiting measure of chromatic zeros for *hierarchical lattices*. They are constructed as follows: start with a finite simple graph $\Gamma \equiv \Gamma_1$ as the *generating graph*, with two vertices labeled a and b , such that Γ is symmetric over a and b . For each $n > 1$, Γ_n retains the two marked vertices a and b from Γ , and we inductively obtain Γ_{n+1} by replacing each edge of Γ with Γ_n , using Γ_n 's marked vertices as if they were endpoints of that edge. A key example to keep in mind is the Diamond Hierarchical Lattice (DHL) shown in Figure 1. In fact, one can interpret the DHL as an anisotropic version of the \mathbb{Z}^2 lattice; see [13, Appendix E.4] for more details.

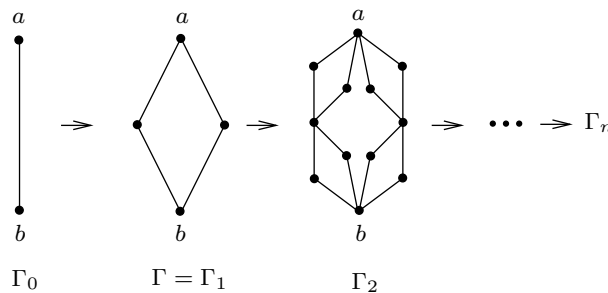


FIGURE 1. Diamond Hierarchical Lattice (DHL)

Several other possible generating graphs are shown in Figure 2, including a generalization of the DHL called the k -fold DHL.

Statistical physics on hierarchical lattices dates back to the work of Berker and Ostlund [7], followed by Griffiths and Kaufman [40], Derrida, De Seze, and Itzykson [29], Bleher and Žalys [15, 18, 16], and Bleher and Lyubich [14].

A graph Γ is called *2-connected* if Γ has three or more vertices and if there is no vertex whose removal disconnects the graph. Our main results about the limiting measure of chromatic zeros are:

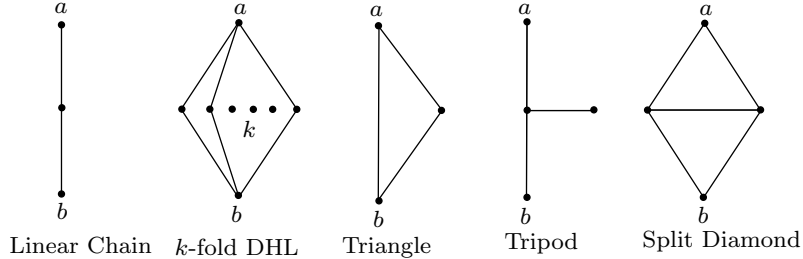


FIGURE 2. Several possible generating graphs. The k -fold DHL, Triangle, and Split Diamond are 2-connected, while the others are not.

Theorem A. *Let $\{\Gamma_n\}_{n=1}^\infty$ be a hierarchical lattice whose generating graph $\Gamma \equiv \Gamma_1$ is 2-connected. Then its limiting measure μ of chromatic zeros exists.*

Theorem B. *Let μ be the limiting measure of chromatic zeros for the k -fold DHL and suppose $k \geq 2$. Then, $\text{supp}(\mu)$ has Hausdorff dimension 2.*

Remark 1.1. *The set shown in Figure 3 has been studied from the perspective of holomorphic dynamics by several authors. We refer the reader to the works of Luo [46], Aspenberg-Yampolsky [3], Wang-Qiu-Yin-Qiao-Gao [71], and Yang-Zeng [74] for details.*

The technique for proving Theorems A and B comes from the connection between the anti-ferromagnetic Potts model in statistical physics and the chromatic polynomial; see, for example, [73, 5, 70] and [4, p.323-325].

For any graph Γ , let $Z_\Gamma(q, y)$ be the partition function (5.1) for the antiferromagnetic Potts model with q states and “temperature” y . We remark that Z_Γ is defined with multivariate edge variables y_e , here we set $y_e = y$ for all edges, and $Z_\Gamma(q, y)$ becomes a polynomial in both q and y by the Fortuin-Kasteleyn [35] representation (5.2). Then, by setting $y = 0$, one has:

$$(1.2) \quad \mathcal{P}_\Gamma(q) = Z_\Gamma(q, 0).$$

See Section 5 for more details.

Given a hierarchical lattice $\{\Gamma_n\}_{n=1}^\infty$ generated by $\Gamma = (V, E)$ let us write $Z_n(q, y) \equiv Z_{\Gamma_n}(q, y)$ for each $n \in \mathbb{N}$. The zero locus of $Z_n(q, y)$ is a (potentially reducible) algebraic curve in \mathbb{C}^2 . However, we will consider it as a divisor by assigning positive integer multiplicities to each irreducible component according to the order at which $Z_\Gamma(q, y)$ vanishes on that component. This divisor will be denoted by

$$\mathcal{S}_n := (Z_n(q, y) = 0),$$

where, in general, the zero divisor of a polynomial $p(x, y)$ will be denoted by $(p(x, y) = 0)$. Since Γ_0 is a single edge with its two endpoints, we have

$$\mathcal{S}_0 = (q(y + q - 1) = 0) = (q = 0) + (y + q - 1 = 0).$$

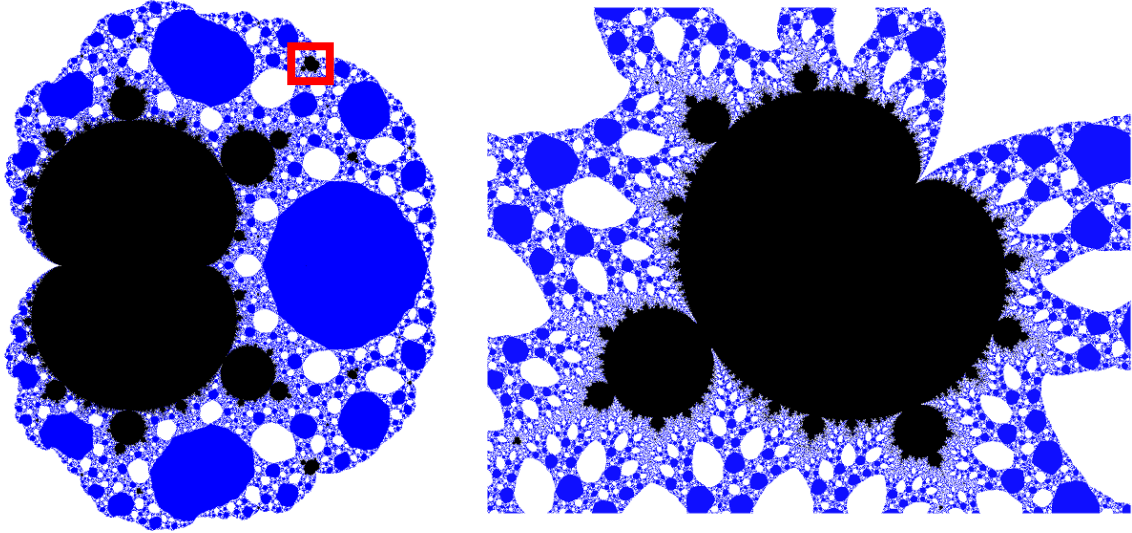


FIGURE 3. The support of the limiting measure of chromatic zeros for the DHL equals the union of boundaries of the black, blue, and white sets. Let $r_q(y)$ be the renormalization mapping for the DHL, given in (1.4). Points in white correspond to parameter values q for which $r_q^n(0) \rightarrow 1$, points in blue correspond to parameter values q for which $r_q^n(0) \rightarrow \infty$, and points in black correspond to parameter values for which $r_q^n(0)$ does neither. The region depicted on the left is approximately $-2 \leq \operatorname{Re}(q) \leq 4$ and $-3 \leq \operatorname{Im}(q) \leq 3$. The region on the right is a zoomed in view of the region shown in the red box on the left. See Section 7 for an explanation of the appearance of “baby Mandelbrot sets”, as on the right. Their appearance will imply Theorem B. Figures 3 and 4 were made using the Fractalstream software [1].

If Γ is 2-connected, there is a Migdal-Kadanoff renormalization procedure that takes Γ and produces a rational map

$$R \equiv R_\Gamma : \mathbb{C} \times \mathbb{P}^1 \rightarrow \mathbb{C} \times \mathbb{P}^1 \quad \text{be given by} \quad R(q, y) = (q, r_q(y)),$$

with the property that

$$(1.3) \quad \mathcal{S}_{n+1} = R^* \mathcal{S}_n \quad \text{for } n \geq 0.$$

Here, \mathbb{P}^1 denotes Riemann Sphere, R^* denotes the pullback of divisors, and $r_q : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is a degree $|E|$ rational map² depending on q . (Informally, one can think of the pullback on divisors R^* as being like the set-theoretic preimage, but designed to keep track of multiplicities.) The reader should keep in mind the case of the DHL for which

$$(1.4) \quad r_q(y) = \left(\frac{y^2 + q - 1}{2y + q - 2} \right)^2.$$

It will be derived in Section 5.

²Actually, the degree can drop below $|E|$ for finitely many values of q .

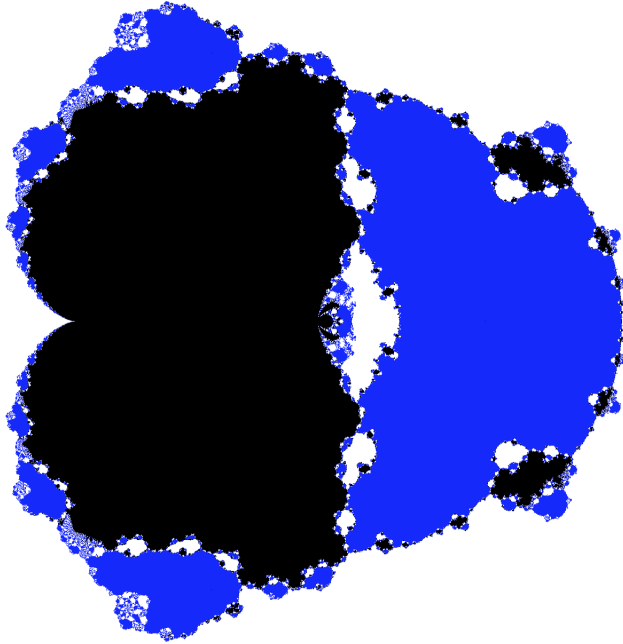


FIGURE 4. The support of the limiting measure of chromatic zeros for the hierarchical lattice generated by the split diamond (see Figure 2) equals the union of boundaries of the black, blue, and white sets. Let $r_q(y)$ denote the renormalization mapping generated by the split diamond, given in (8.2). The coloring scheme is the same as in Figure 3, but using this different mapping. The region depicted is approximately $-1 \leq \operatorname{Re}(q) \leq 5$ and $-3 \leq \operatorname{Im}(q) \leq 3$.

Because $R(q, y) = (q, r_q(y))$ is a skew product over the identity, for each $n \geq 0$ we have

$$\mathcal{S}_n = (R^n)^*((q = 0) + (y + q - 1 = 0)) = (q = 0) + (R^n)^*(y + q - 1 = 0).$$

The chromatic polynomial of a connected graph Γ has a simple zero at $q = 0$, which we can ignore when discussing the limiting measure of chromatic zeros. It corresponds to the divisor $(q = 0)$ above. Therefore, using (1.2), all of the chromatic zeros for Γ_n (other than $q = 0$) are given by

$$(1.5) \quad \tilde{\mathcal{C}}_n = (R^n)^*(y + q - 1 = 0) \cap (y = 0),$$

where each intersection point is assigned its Bezout multiplicity.

Since we want to normalize and then take limits as n tends to infinity, we re-write (1.5) in terms of currents (see [67, 30] for background). We find

$$(1.6) \quad \tilde{\mu}_n := \frac{1}{|V_n|}[\tilde{\mathcal{C}}_n] = (\pi_1)_* \left(\frac{1}{|V_n|} (R^n)^*[y + q - 1 = 0] \wedge [y = 0] \right),$$

where $\pi_1 : \mathbb{C} \times \mathbb{P}^1 \rightarrow \mathbb{C}$ defined by $\pi_1(q, y) = q$ is the projection map, the square brackets denote the current of integration over a divisor, and \wedge denotes the wedge product of currents. Since $[y = 0]$ is the current of integration over a horizontal line, the wedge product is just the horizontal slice of $\frac{1}{|V_n|} (R^n)^*[y + q - 1 = 0]$ at height $y = 0$. Since the wedge product results in a measure on $\mathbb{C} \times \mathbb{P}^1$,

we compose with the projection $(\pi_1)_*$ to obtain a measure on \mathbb{C} . (In the previous two paragraphs we have used tildes on \tilde{C}_n and $\tilde{\mu}_n$ to denote that we have dropped the simple zero at $q = 0$.)

If the generating graph Γ is 2-connected, then we will see in Proposition 6.2 that there are at most finitely many parameters q such that $y = 1 - q$ is an exceptional point for r_q . It then follows quickly from the one-dimensional equidistribution theorems of Lyubich [49, 50] and Freire-Lopez-Mañé [36] that the following convergence holds:

$$\frac{1}{|E_n|} (R^n)^* [y + q - 1 = 0] \rightarrow \hat{T},$$

where \hat{T} is the *fiber-wise Green current* for the family of rational maps $r_q(y)$. In Proposition 5.4 we'll see that $\alpha := \lim_{n \rightarrow \infty} \frac{|E_n|}{|V_n|}$ exists so that

$$\hat{T}_n := \frac{1}{|V_n|} (R^n)^* [y + q - 1 = 0] \rightarrow \alpha \hat{T}.$$

However:

First Main Technical Issue: $\hat{T}_n \rightarrow \alpha \hat{T}$ does not necessarily imply $\hat{T}_n \wedge [y = 0] \rightarrow \alpha \hat{T} \wedge [y = 0]$.

This issue will be handled using the notion of activity currents which were introduced by DeMarco in [25] to study bifurcations in families of rational maps (they are sometimes called bifurcation currents). Since then, they have been studied by Berteloot, DeMarco, Dujardin, Favre, Gauthier, Okuyama and many others. We refer the reader to the surveys by Berteloot [8] and Dujardin [32] for further details.

We can re-write (1.6) as

$$\tilde{\mu}_n := \frac{1}{|V_n|} [(r_q^n \circ a)(q) = b(q)],$$

where $a, b : \mathbb{C} \rightarrow \mathbb{P}^1$ are the two marked points

$$a(q) = 0 \quad \text{and} \quad b(q) = 1 - q.$$

(Special care must be taken at the finitely many parameters q for which $\deg_y(r_q(y)) < |E|$. It is the Second Main Technical Issue for proving Theorem A and it will be explained in the next subsection.)

Meanwhile, the *activity current* of the marked point a is defined by

$$T_a := \lim_{n \rightarrow \infty} \frac{1}{|E_n|} (r_q^n \circ a)^* \hat{\omega},$$

where $\hat{\omega}$ is the fiberwise Fubini-Study $(1, 1)$ form on $\mathbb{C} \times \mathbb{P}^1$. Therefore, proving Theorem A reduces to proving the convergence

$$(1.7) \quad \tilde{\mu}_n = \frac{1}{|V_n|} [(r_q^n \circ a)(q) = b(q)] \rightarrow \alpha T_a.$$

It will be a consequence of Theorems C and C' that are presented in the next subsection.

1.2. Equidistribution in parameter space. Let V be a connected projective algebraic manifold. An *algebraic family of rational maps of degree d* is a rational mapping

$$f : V \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$$

such that, there exists an algebraic hypersurface $V_{\text{deg}} \subset V$ (possibly reducible) with the property that for each $\lambda \in V \setminus V_{\text{deg}}$ the mapping

$$f_\lambda : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \quad \text{defined by} \quad f_\lambda(z) = f(\lambda, z)$$

is a rational map of degree d . A *marked point* is a rational map $a : V \dashrightarrow \mathbb{P}^1$. (We will denote the indeterminacy locus of a by $I(a)$. It is a proper subvariety of codimension at least two.)

Our result will depend heavily on a theorem from arithmetic dynamics due to Silverman [68, Theorem E] and this will require us to assume that the manifold V , the family f , and the marked points a and b are defined over the algebraic numbers $\overline{\mathbb{Q}}$. In other words, every polynomial in the definitions of these objects has coefficients in $\overline{\mathbb{Q}}$.

Convention. *Throughout the paper an algebraic family of rational maps $f : V \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ defined over $\overline{\mathbb{Q}}$ will mean that both V and f are defined over $\overline{\mathbb{Q}}$.*

Theorem C. *Let $f : V \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ be an algebraic family of rational maps of degree $d \geq 2$ defined over $\overline{\mathbb{Q}}$ and let $a, b : V \dashrightarrow \mathbb{P}^1$ be two marked points defined over $\overline{\mathbb{Q}}$. Extending V_{deg} , if necessary, we can suppose $I(a) \cup I(b) \subset V_{\text{deg}}$.*

Suppose that:

- (i) *There is no iterate n satisfying $f_\lambda^n a(\lambda) \equiv b(\lambda)$.*
- (ii) *The marked point $b(\lambda)$ is not persistently exceptional for f_λ .*

Then we have the following convergence of currents on $V \setminus V_{\text{deg}}$

$$(1.8) \quad \frac{1}{d^n} [(f_\lambda^n \circ a)(\lambda) = b(\lambda)] \rightarrow T_a,$$

where T_a is the activity current of the marked point $a(\lambda)$.

The precise definition of activity current will be given in Section 2.

The following version of Theorem C holds on all of V , without removing V_{deg} , an essential feature for our application to Theorem A.

Theorem C'. *Let $f : V \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ be an algebraic family of rational maps of degree $d \geq 2$ defined over $\overline{\mathbb{Q}}$ and let $a, b : V \dashrightarrow \mathbb{P}^1$ be two marked points defined over $\overline{\mathbb{Q}}$. Suppose that*

- (i) *There is no iterate n satisfying $f_\lambda^n a(\lambda) \equiv b(\lambda)$.*
- (ii) *The marked point $b(\lambda)$ is not persistently exceptional for f_λ .*

Consider the rational map

$$F : V \times \mathbb{P}^1 \dashrightarrow V \times \mathbb{P}^1 \quad \text{defined by} \quad F(\lambda, z) = (\lambda, f(\lambda, z)).$$

Then the following sequence of currents on V

$$(1.9) \quad (\pi_1)_* \left(\frac{1}{d^n} (F^n)^* [z = b(\lambda)] \wedge [z = a(\lambda)] \right)$$

converges and the limit equals T_a when restricted to $V \setminus V_{\text{deg}}$. Here, $\pi_1 : V \times \mathbb{P}^1 \rightarrow V$ is the projection onto the first coordinate $\pi_1(\lambda, z) = \lambda$.

Remark 1.2. *We have phrased Theorems C and C' in their natural level of generality. However, in most applications that we have in mind (in particular to the chromatic zeros), one can use $V = \mathbb{P}^m$ and define everything in the usual affine coordinates $\mathbb{C}^m \subset \mathbb{P}^m$ in the following ways:*

- (i) $f(\lambda, z) = \frac{P(\lambda, z)}{Q(\lambda, z)}$ with $P, Q \in \overline{\mathbb{Q}}[\lambda, z]$ and having no common factors of positive degree in $\overline{\mathbb{Q}}[\lambda, z]$, and

- (ii) $a(\lambda) = \frac{R(\lambda)}{S(\lambda)}$ with $R, S \in \overline{\mathbb{Q}}[\lambda]$ and having no common factors of positive degree in $\overline{\mathbb{Q}}[\lambda]$ (and similarly for $b(\lambda)$).

The reader can keep in mind the simple case of the renormalization mapping for the DHL (1.4) in which case everything is defined over $\mathbb{Q} \subset \overline{\mathbb{Q}}$. Here $V = \mathbb{P}^1$,

- (i) $r(q, y) = \left(\frac{y^2 + q - 1}{2y + q - 2} \right)^2$,
(ii) $a(q) \equiv 0$, and $b(q) = 1 - q$.

The degree of this family is $d = 4$ and $V_{\text{deg}} = \{0, \infty\}$ because the degree of $r_q(y)$ drops when $q = 0$ and $q = \infty$ but at no other values of q .

The proofs of Theorem C and C' will closely follow the strategy that Dujardin-Favre use in [33, Theorem 4.2]. However:

Second Main Technical Issue: The proof of [33, Theorem 4.2] requires a technical ‘‘Hypothesis (H)’’ that is not satisfied for the Migdal-Kadanoff renormalization mapping (1.4) for the DHL (and presumably not satisfied for many other hierarchical lattices). Indeed, $q = 0 \in V_{\text{deg}}$ for this mapping and there are active parameters accumulating to $q = 0$. One sees this in Figure 3 where $q = 0$ is the ‘‘main cusp’’ on the left side of the black region.

Our assumption that the family and the marked points are defined over $\overline{\mathbb{Q}}$ allows us to avoid Hypothesis (H). Note that, using quite different techniques, Okuyama has proved in [57, Theorem 1] a version of [33, Theorem 4.2] without Hypothesis (H). His proof requires the marked point to be critical, but does not require working over $\overline{\mathbb{Q}}$.

Once Theorem C is proved, one can extend the convergence (1.8) across V_{deg} by an application of the compactness theorem for families of plurisubharmonic functions [42, Theorem 4.1.9], thus proving Theorem C'. Note that a similar statement to Theorem C' is found in the work of Gauthier-Vigny [38, Corollary 3.1]. The proof there also uses such compactness to extend a given convergence across various ‘‘bad’’ parameters that are analogous to our V_{deg} .

1.3. Brief History of Migdal-Kadanoff Renormalization. The renormalization mapping $r_q(y)$ given in (1.4) for the DHL and its variants for other hierarchical lattices date back to the early 1980s. More specifically, Migdal [55, 56] and Kadanoff [47] described approximate renormalization equations for the Ising Model on the \mathbb{Z}^d . Berker-Ostland [7], Bleher-Žalys [16], Kaufman-Griffiths [40], Derrida-De Seze-Itzykson [29], Kinzel-Domany [48], Andelman-Berker [2], and others noticed that these equations became exact on suitable hierarchical lattices and that the setting extends to the Potts model. Equation (1.4) plays a prominent role in several of the papers referenced above. Study of Potts Models on Hierarchical Lattices continues to be an active area of physics [43].

1.4. Recent works on interplay between holomorphic dynamics and statistical physics. The present work lies in the context of several recent papers where holomorphic dynamics has played a role in studying problems from statistical physics. We describe a sample of them here.

To the best of our knowledge each of the previous works mentioned in Section 1.3 focuses on zeros of the partition function in the complex temperature plane (or sometimes the complex magnetic field plane) but not in the q -plane for fixed complex temperature. Studying the zeros of the partition function in the complex q -plane requires quite different techniques. To the best of our knowledge, the first time they were studied for hierarchical lattices is by Royle-Sokal in Appendix B of [60], where the accumulation loci of chromatic zeros for the leaf joined trees are studied. Because they are interested in the accumulation loci instead of the limiting measure of chromatic zeros, they are able to use a classical Proposition of Lyubich [51, Proposition 3.5] to deduce their results.

The novelty of our paper is that we also use holomorphic dynamics to study the chromatic zeros, but for a different type of hierarchical lattices. Moreover, we are interested in the limiting measure of chromatic zeros rather than the accumulation locus of them. This requires us to prove a new theorem in holomorphic dynamics (Theorem C) which can be interpreted as a quantitative version of the Proposition of Lyubich [51, Proposition 3.5].

One can interpret a rooted Cayley Tree as a type of hierarchical lattice, and this allows one to apply a renormalization theory that is similar to the Migdal-Kadanoff version used in this paper, in order to study statistical physics on such trees. This led to holomorphic dynamics playing an important role in proof of the Sokal Conjecture by Peters and Regts [59] and also in their work on the location of Lee-Yang zeros for bounded degree graphs [58]. The same renormalization theory was also recently used in combination with techniques from dynamical systems by He, Ji, and the authors of the present paper to characterize the limiting measure of Lee-Yang zeros for the Cayley Tree [23].

Meanwhile, holomorphic dynamics has been used by Bleher, Lyubich, and the second author of the present paper to characterize the limiting measure of Lee-Yang zeros for the DHL [13] and also to describe the limit behavior of the Lee-Yang-Fisher zeros for the DHL [17].

Finally, let us note that in the recent paper [20] Chang-Roeder-Shrock study the accumulation loci of q -plane zeros for the Diamond Hierarchical lattice and various fixed values of temperature y , both in ferromagnetic and antiferromagnetic regimes. The key technique in that paper is again [51, Proposition 3.5].

1.5. Relationship to a conjecture of Sokal. For any connected graph Γ let $\Delta(\Gamma)$ denote the maximal degree of a vertex of Γ . There is a conjecture of Sokal which asserts that $|P_\Gamma(q)| > 0$ for all complex q satisfying $\text{Re}(q) > \Delta(\Gamma)$. (See, for example, [44, Conjecture 21].)

The techniques in our paper do not give insight into this conjecture because our hypothesis that the generating graph be 2-connected leads to the marked vertices a and b having degree two or larger. This results in $\Delta(\Gamma_n)$ becoming unbounded as n tends to infinity. We use 2-connectivity of the generating graph to guarantee that the renormalization mapping does not have common factors (of positive degree) in the numerator and denominator. (See Sections 5.4 and 5.5.) We do not presently see how to work around this hypothesis, however it may be quite interesting for future study.

An additional challenge is that our techniques are about the limiting measure of chromatic zeros and hence would not detect regions in the q plane where there are a negligible proportion chromatic zeros, in the limit as n tends to infinity.

1.6. Structure of the paper. In Section 2 we present background on activity currents and describe the Dujardin-Favre classification of the passive locus, that will play an important role in the proofs of Theorems C and C'. Theorems C and C' are proved in Section 3 and Section 4.

We return to the problem of chromatic zeros in Section 5 by providing background on their connection with the Potts Model from statistical physics. We also set up the renormalization mapping $r_q(y)$ associated to any hierarchical lattice having 2-connected generating graph. We prove Theorem A in Section 6 by verifying the hypotheses of Theorem C'.

For the k -fold DHL with $k \geq 2$, one can check that the critical points $y = \pm\sqrt{1-q}$ satisfy $r_q(\pm\sqrt{1-q}) \equiv 0 \equiv a(q)$. Therefore, a result of McMullen [53, Corollary 1.6] gives that $\text{supp}(T_a)$ has Hausdorff dimension 2. This is explained in Section 7, where we prove Theorem B.

We conclude the paper with Section 8 where we discuss the chromatic zeros associated with the hierarchical lattices generated by each of the graphs shown in Figure 2. We also provide a more detailed explanation of Figures 3 and 4.

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2. BASICS IN ACTIVITY CURRENTS

2.1. Holomorphic Families, Marked Points, and Active/Passive Dichotomy. Let Λ be a connected complex manifold. A *holomorphic family of rational maps of degree $d \geq 2$* is a holomorphic map $f : \Lambda \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ such that $f_\lambda := f(\lambda, \cdot) : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is a rational map of degree d for every $\lambda \in \Lambda$.

Associated with f_λ is the skew product mapping

$$(2.1) \quad F : \Lambda \times \mathbb{P}^1 \rightarrow \Lambda \times \mathbb{P}^1 \quad \text{given by} \quad F(\lambda, z) = (\lambda, f_\lambda(z)).$$

Note that it is conventional in the literature to denote by $f_\lambda^n(z)$ the second component of $F^n(\lambda, z)$.

A *marked point* is a holomorphic map $a : \Lambda \rightarrow \mathbb{P}^1$. The marked point $a : \Lambda \rightarrow \mathbb{P}^1$ is called *passive* at $\lambda_0 \in \Lambda$ if the family $\{f_\lambda^n a(\lambda)\}$ is normal in some neighborhood of λ_0 , otherwise a is said to be *active* at λ_0 . The set of all parameters where a is active is called the *active locus* of a .

Remark 2.1. *Historically in holomorphic dynamics one considers marked critical points, i.e. marked points $a(\lambda)$ that are critical points of f_λ for every parameter λ . In this paper it will be crucial to consider non-critical marked points. Fortunately, several results from the classical literature carry over to our setting. We will review in this section the results that we need and carefully check that the marked points need not be critical.*

Let us note that recently there has been considerable interest in the dynamical properties of non-critical marked points, with some of the motivations coming from problems in arithmetic dynamics. As a sample of such recent papers, we refer the reader to [27, 28, 37, 34] and the references therein.

2.2. Activity Current for Holomorphic Families. The active locus naturally supports a closed positive $(1, 1)$ current T_a called the *activity current* of $a(\lambda)$, introduced by Laura DeMarco in [26]. (We refer the reader to [24] for background on currents, plurisubharmonic (PSH) functions, and Monge-Ampère operators.)

The construction of T_a can be done in a coordinate-free manner, however we will first express it in local coordinates, which are simpler for explicit calculations and allow us to check that the marked point need not be critical.

Suppose Λ is an open subset of \mathbb{C}^m . We can choose a lift $\tilde{f} : \Lambda \times \mathbb{C}^2 \rightarrow \mathbb{C}^2$ which is holomorphic and so that for each $\lambda \in \Lambda$,

$$(2.2) \quad \tilde{f}_\lambda(z, w) := \tilde{f}(\lambda, z, w) = (P_\lambda(z, w), Q_\lambda(z, w)),$$

where P_λ, Q_λ are both homogeneous polynomials of degree d . Similarly, the marked point $a : \Lambda \rightarrow \mathbb{P}^1$ can be lifted to a holomorphic map

$$\tilde{a} : \Lambda \rightarrow \mathbb{C}^2 \setminus \{(0, 0)\}.$$

The choices of lifts \tilde{f} and \tilde{a} are unique up to a non-vanishing scaling factor that depends holomorphically on λ .

The function $G_n : \Lambda \times \mathbb{C}^2 \rightarrow [-\infty, \infty)$ given by

$$G_n(\lambda, (z, w)) := \frac{1}{d^n} \log \|\tilde{f}_\lambda^n(z, w)\|$$

is PSH in (λ, z, w) , where $\|\cdot\|$ is the Euclidean norm on \mathbb{C}^2 . Over any compact subset of Λ there is a constant $C > 0$ such that

$$C^{-1} \|(z, w)\|^d \leq \|\tilde{f}_\lambda(z, w)\| \leq C \|(z, w)\|^d.$$

Using this one can check that the G_n converge uniformly on compact subsets of $\Lambda \times (\mathbb{C}^2 \setminus \{(0, 0)\})$. This implies that the limit $G : \Lambda \times \mathbb{C}^2 \rightarrow [-\infty, \infty)$ is PSH and continuous off of $\Lambda \times \{(0, 0)\}$.

Let $\text{pr} : \mathbb{C}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{P}^1$ be the canonical projection. For any sufficiently small open $U \subset \mathbb{P}^1$ there is a holomorphic $s : U \rightarrow \mathbb{C}^2 \setminus \{(0, 0)\}$ such that $\text{pr} \circ s(u) = u$ for all $u \in U$. One defines the *fiberwise Green current* \hat{T} on $\Lambda \times \mathbb{P}^1$ locally on $\Lambda \times U$ by

$$(2.3) \quad \hat{T} := \text{dd}^c G(\lambda, s(u)).$$

Here, $\text{dd}^c := \frac{i}{\pi} \partial \bar{\partial}$ is the Monge-Ampère operator and d is the exterior derivative (not to be confused with the degree d of a rational map). In (2.3) dd^c is taken with respect to (λ, u) .

Now consider the functions

$$(2.4) \quad H_n(\lambda) := G_n(\lambda, \tilde{a}(\lambda)) \quad \text{and} \quad H(\lambda) := G(\lambda, \tilde{a}(\lambda)).$$

Both are PSH functions and the locally uniform convergence G_n to G described in the previous paragraph implies locally uniform convergence of H_n to H . We define the *activity current* by

$$T_a := \text{dd}^c H(\lambda).$$

When defining \hat{T} and T_a we have made choices of lifts \tilde{f}_λ , \tilde{a} , and $s : U \rightarrow \mathbb{C}^2 \setminus \{(0, 0)\}$. One can check that a different choice results in adding a pluriharmonic function to $G(\lambda, s(u))$ and/or to $H(\lambda)$ and hence does not affect the definitions of \hat{T} and T_a .

If Λ is a complex manifold that is not an open subset of \mathbb{C}^m then one defines T_a using the above formula in local coordinates. The definition is compatible under change of coordinates.

Theorem 2.2. (DeMarco [26]) *The support of the activity current T_a coincides with the active locus of a .*

Indeed, in [26, Theorem 9.1] DeMarco proves the equivalence of the following two statements:

- (i) The functions $\{\lambda \mapsto f_\lambda^n(a(\lambda)) : n \geq 0\}$ forms a normal family in a neighborhood of λ_0 .
- (ii) For any holomorphic lift $\tilde{a}(\lambda)$ the function $G(\lambda, \tilde{a}(\lambda))$ is pluriharmonic in a neighborhood of λ_0 .

Although we do not include the proof here, let us note that it is explicit, concise and does not require the marked point $a(\lambda)$ to be critical.

2.3. Coordinate-free description of activity current T_a . Let ω be the Fubini-Study (1,1) form on \mathbb{P}^1 and let $\hat{\omega} = \pi_2^* \omega$, where $\pi_2(\lambda, z) = z$ is the projection onto the second coordinate.

Proposition 2.3. *We have*

$$(2.5) \quad \hat{T} = \lim_{n \rightarrow \infty} \frac{1}{d^n} (F^n)^* \hat{\omega} \quad \text{and} \quad T_a = \lim_{n \rightarrow \infty} \frac{1}{d^n} (f_\lambda^n \circ a)^* \omega,$$

where $F(\lambda, z) = (\lambda, f_\lambda(z))$ is the skew product associated with f_λ .

Proof. This is a consequence of the convergence of the potentials $G_n(\lambda, (z, w))$ to $G(\lambda, (z, w))$ that was discussed in the previous subsection and the fact that

$$\text{pr}^*\omega = \text{dd}^c \log \|(z, w)\|,$$

where $\text{pr} : \mathbb{C}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{P}^1$ is the canonical projection; see, for example, [39, p. 30]. \square

There is an equivalent, alternative, description of the activity current T_a given in [33, Proposition 3.1]. We state it here because it ties nicely with the discussion presented in the introduction of our paper, but we note that it's not actually needed in our proofs of Theorems C and C'. Let v_n, v_∞ be the local potentials of $d^{-n}(F^n)^*\widehat{\omega}$ and \widehat{T} respectively. In the proof of [33, Proposition 3.1] one sees that all the v_n 's and v_∞ are continuous, and particularly $v_n \rightarrow v_\infty$ locally uniformly. (This corresponds to the properties and convergence of $G_n(\lambda, (z, w))$ to $G(\lambda, (z, w))$ described in Section 2.2.) We therefore have the following corollary:

Corollary 2.4. *For any marked point $a : \Lambda \rightarrow \mathbb{P}^1$, we have the following convergence of intersection of currents:*

$$(2.6) \quad \frac{1}{d^n} (F^n)^*\widehat{\omega} \wedge [z = a(\lambda)] \rightarrow \widehat{T} \wedge [z = a(\lambda)].$$

Let $\Gamma := \{(\lambda, a(\lambda))\} \subset \Lambda \times \mathbb{P}^1$. Since $\pi_1 : \Gamma \rightarrow \Lambda$ is a biholomorphism, one can check that

$$(2.7) \quad T_a = (\pi_1)_* \left(\widehat{T} \wedge [z = a(\lambda)] \right).$$

2.4. Activity Current for Algebraic Families. Let $f : V \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ be an algebraic family of rational maps of degree d . In this case, one can delete V_{deg} to obtain a holomorphic family $f : (V \setminus V_{\text{deg}}) \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ and the construction from the previous subsection defines the activity current T_a for $f : (V \setminus V_{\text{deg}}) \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$. We will now show that there is a natural extension of T_a through the hypersurface V_{deg} .

As the construction is local, we can suppose V is a precompact open subset of \mathbb{C}^m and proceed as in Section 2.2. We choose a homogeneous lift $\tilde{f} : V \times \mathbb{C}^2 \rightarrow \mathbb{C}^2$ as in (2.2) and so that

$$(2.8) \quad \sup_{\lambda \in V, \|(z, w)\|=1} |\tilde{f}_\lambda(z, w)| = 1.$$

In this case P_λ, Q_λ have degree $d' \leq d$, with equality iff $\lambda \in V \setminus V_{\text{deg}}$.

Similarly, the marked point $a : V \dashrightarrow \mathbb{P}^1$ can be lifted to a holomorphic map

$$\tilde{a} : V \rightarrow \mathbb{C}^2.$$

Note that unlike in the holomorphic families we can have $\tilde{a}(\lambda) = (0, 0)$ for some $\lambda \in V_{\text{deg}}$, corresponding to indeterminate points of a .

We now use Equation (2.4) to define a PSH function $H_n : V \rightarrow [-\infty, \infty)$ using the lifts chosen above. However, a-priori, we only know that H_n converges on $V \setminus V_{\text{deg}}$ to the PSH function $H : V \setminus V_{\text{deg}} \rightarrow [-\infty, \infty)$. The next two propositions show that the convergence extends (in an appropriate way) to all of V .

Proposition 2.5. *Suppose $V \subset \mathbb{C}^m$ is open. The pointwise limit*

$$H(\lambda) := \lim_{n \rightarrow \infty} H_n(\lambda)$$

exists and is PSH in V . When restricted to $V \setminus V_{\text{deg}}$, the current $\tilde{T}_a := \text{dd}^c H$ is identically equal to the activity current T_a .

Proof. Fix a parameter $\lambda \in V$ so that $f_\lambda : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is a rational map with degree $d' \leq d$. Using (2.8) and the homogeneity of \tilde{f}_λ ,

$$\|\tilde{f}_\lambda(z, w)\| \leq \|(z, w)\|^{d'}, \text{ which implies } \|\tilde{f}_\lambda^{n+1}(z, w)\| \leq \|\tilde{f}_\lambda^n(z, w)\|^{d'},$$

so the maps H_n satisfy

$$H_{n+1}(\lambda) = \frac{1}{d^{n+1}} \log \|(\tilde{f}_\lambda^{n+1} \circ \tilde{a})(\lambda)\| \leq \frac{1}{d^{n+1}} \log \|(\tilde{f}_\lambda^n \circ \tilde{a})(\lambda)\|^{d'} \leq \frac{1}{d^n} \log \|(\tilde{f}_\lambda^n \circ \tilde{a})(\lambda)\| = H_n(\lambda),$$

which implies that $\{H_n\}_{n=1}^\infty$ is a decreasing sequence of PSH functions, so it either converges to a PSH limit function H or to $-\infty$ identically. The latter is impossible since $H_n(\lambda)$ converges to a finite value for any $\lambda \in V \setminus V_{\text{deg}}$. \square

Denote by $L_{\text{loc}}^1(V)$ the space of locally integrable functions in V . Since the convergence given by Proposition 2.5 is only pointwise we will check the following.

Proposition 2.6. *Suppose $V \subset \mathbb{C}^m$ is open. The sequence of PSH functions H_n converges to H in $L_{\text{loc}}^1(V)$. Equivalently, the sequence of currents $\text{dd}^c H_n$ converges to $\tilde{T}_a = \text{dd}^c H$.*

Proof. Assume on the contrary that H_n does not converge to H in $L_{\text{loc}}^1(V)$. Then the compactness theorem for PSH functions [42, Theorem 4.1.9] implies that there is a subsequence H_{n_k} and some PSH function $H' \neq H$ in $L_{\text{loc}}^1(V)$ such that $H_{n_k} \rightarrow H'$ in $L_{\text{loc}}^1(V)$. Then there is a set $\Omega \subset V$ of positive measure in which $H'(\lambda) \neq H(\lambda)$ for all λ . In particular, since V_{deg} is a hypersurface and hence has zero measure, we can find a compact set $K \subset \Omega \setminus V_{\text{deg}}$ in which $H_{n_k} \rightarrow H'$. However, by Corollary 2.4, $H_n \rightarrow H$ uniformly in K , which is a contradiction. \square

2.5. Classification of the Passivity Locus.

Dujardin-Favre Classification of Passivity Locus [33, Theorem 4]. *Let $f : \Lambda \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be a holomorphic family and let $a(\lambda)$ be a marked point. Assume $U \subset \Lambda$ is a connected open subset where $a(\lambda)$ is passive. Then exactly one of the following cases holds:*

- (i) $a(\lambda)$ is never preperiodic in U . In this case the closure of the orbit of $a(\lambda)$ can be followed by a holomorphic motion.
- (ii) $a(\lambda)$ is persistently preperiodic in U .
- (iii) There exists a persistently attracting (possibly superattracting) cycle attracting $a(\lambda)$ throughout U and there is a closed subvariety $U' \subsetneq U$ such that the set of parameters $\lambda \in U \setminus U'$ for which $a(\lambda)$ is preperiodic is a proper closed subvariety in $U \setminus U'$.
- (iv) There exists a persistently irrationally neutral periodic point such that $a(\lambda)$ lies in the interior of its linearization domain throughout U and the set of parameters $\lambda \in U$ for which $a(\lambda)$ is preperiodic is a proper closed subvariety in U .

There are two differences between the statement above and the statement given by Dujardin and Favre in [33, Theorem 4]. In that paper:

- (1) the authors suppose the marked point is critical, and
- (2) in Part (iii) the authors claim that the set of parameters for which $a(\lambda)$ is preperiodic is a proper closed subvariety of U , without first removing U' .

For this reason we present the proof of the classification of the passivity locus from [33], carefully verifying that the marked point need not be critical. (In Remark 2.9 we will also clarify the issue about Part (iii) of the statement.)

Proof of the classification of the passivity locus is a consequence of the following theorem in local holomorphic dynamics which we cite verbatim from [33, Theorem 1.1]:

Theorem 2.7. *Let f_λ be any holomorphic family of holomorphic maps parameterized by a connected complex manifold Λ . Suppose that each f_λ is defined on the unit disc with values in \mathbb{C} , and leaves the origin fixed, i.e. $f_\lambda(0) = 0$. Let $\lambda \mapsto p(\lambda)$ be any holomorphic map such that $p(\lambda_0) = 0$ for some parameter λ_0 .*

Assume that for all $n \in \mathbb{N}$, the function $f_\lambda^n p(\lambda)$ is well-defined and takes its values in the unit disk. Then one of the following three cases holds.

- (1) *For every $\lambda \in \Lambda$, the point 0 is attracting or superattracting, and $p(\lambda)$ lies in the (immediate) basin of attraction of 0.*
- (2) *The point p is periodic for all parameters, i.e. $f_\lambda^\ell p(\lambda) = p(\lambda)$ for some ℓ and all $\lambda \in \Lambda$.*
- (3) *The multiplier of f_λ at 0 is constant and equals $\exp(2i\pi\theta)$, with $\theta \in \mathbb{R} \setminus \mathbb{Q}$. The map f_λ is linearizable and $p(\lambda)$ lies in the interior of the domain of linearization of f_λ .*

Note that in the above theorem there is no assumption about $p(\lambda)$ being a critical point or being some iterate of a critical point.

We will also need the following lemma which is adapted from of [32, p. 524-525].

Lemma 2.8. *Let f_λ be any holomorphic family of holomorphic maps parameterized by a connected complex manifold Λ . Then, after passing to a branched cover of Λ , any fixed point of f_λ can be followed holomorphically over all of Λ .*

More specifically, suppose that z_0 is a fixed point of f_{λ_0} for some $\lambda_0 \in \Lambda$. Then, there is a holomorphic branched cover $\tau : \tilde{\Lambda} \rightarrow \Lambda$ and a holomorphic map $z_0 : \tilde{\Lambda} \rightarrow \mathbb{P}^1$ with the following properties. If we let

$$(2.9) \quad \tilde{f}_{\tilde{\lambda}}(z) := f_{\tau(\tilde{\lambda})}(z)$$

then

- (1) *$z_0(\tilde{\lambda})$ is a fixed point of $\tilde{f}_{\tilde{\lambda}}$ for every $\tilde{\lambda} \in \tilde{\Lambda}$ and*
- (2) *For any $\tilde{\lambda}_0$ with $\tau(\tilde{\lambda}_0) = \lambda_0$ we have $z_0(\tilde{\lambda}_0) = z_0$.*

Proof. Consider the analytic hypersurface

$$\{(\lambda, z) \in \Lambda \times \mathbb{P}^1 : f_\lambda(z) = z\}$$

and let $\hat{\Lambda}$ be the irreducible component containing (λ_0, z_0) . Let $\pi_1 : \hat{\Lambda} \rightarrow \Lambda$ and $\pi_2 : \hat{\Lambda} \rightarrow \mathbb{P}^1$ be projection onto the first and second coordinates, respectively. A-priori, $\hat{\Lambda}$ could have singularities, but we can let $\tilde{\Lambda}$ be the desingularization of it and let $\tau : \tilde{\Lambda} \rightarrow \Lambda$ and $z_0 : \tilde{\Lambda} \rightarrow \mathbb{P}^1$ be the lifts of π_1 and π_2 , respectively, to $\tilde{\Lambda}$. Properties (1) and (2) then follow if we let $\tilde{f}_{\tilde{\lambda}}$ be defined as in (2.9). \square

Proof of the classification of the passive locus. What follows is very close to what is presented in [33].

Let $U \subset \Lambda$ be a connected open set on which $a(\lambda)$ is passive. Suppose that neither Cases (i) or (ii) of the statement hold, in order to prove that Case (iii) or (iv) holds. Then, there is some parameter $\lambda_0 \in U$ for which $a(\lambda_0)$ is preperiodic for f_{λ_0} , but not persistently in U . Passing to a suitable iterate, we can suppose that $a(\lambda_0)$ is prefixed, i.e. that there is some iterate $n_0 \geq 0$ such that $f_{\lambda_0}^{n_0}(a(\lambda_0)) = w$ with w a fixed point of f_{λ_0} .

A-priori the fixed point w may not vary holomorphically with λ (it is the case when $f'_{\lambda_0}(w) = 1$). However, it follows from Lemma 2.8 that we can replace Λ by a branched covering $\tilde{\Lambda}$ so that w depends holomorphically on $\tilde{\lambda}$ and so that $a(\tilde{\lambda})$ continues to be passive for the lifted family $\tilde{f}_{\tilde{\lambda}}$ for all $\tilde{\lambda} \in \tau^{-1}(U)$. To keep notation simple we will suppose that w already varied holomorphically over Λ .

Let $v \neq w$ be a repelling fixed point for f_{λ_0} . As in the previous paragraph, after passing to a branched cover of Λ , we can also suppose that v varies holomorphically over all of Λ .

Conjugating by a suitable holomorphically varying Möbius transformation we can therefore suppose that $w(\lambda) = 0$ and $v(\lambda) = \infty$ for all $\lambda \in \Lambda$. Let $p(\lambda) := f_\lambda^{n_0}(a(\lambda))$ and note that $p(\lambda_0) = w = 0$ but that this does not hold on all of U .

We claim that for every $\lambda \in U$ and every $n \geq 0$ we have that $f_\lambda^n(p(\lambda)) \neq \infty$. Suppose for contradiction that there is some $\lambda_1 \in U$ and some n_1 such that $f_{\lambda_1}^{n_1}(p(\lambda_1)) = \infty$. Since $a(\lambda)$ and hence $p(\lambda)$ is passive on U we can then find a small neighborhood \mathcal{N} of λ_1 so that

$$f_\lambda^{n_1+n}(p(\lambda)) \in \mathbb{D}_1(\infty)$$

for all $\lambda \in \mathcal{N}$ and all $n \geq 0$. (Here, $\mathbb{D}_1(\infty)$ is the disc of radius 1 centered at infinity.) It then follows from Theorem 2.7 that for all $\lambda \in \mathcal{N}$ the fixed point at infinity is attracting or that it has a constant multiplier equal to $\exp(2i\pi\theta)$ with $\theta \in \mathbb{R} \setminus \mathbb{Q}$. In the first case, we would have that $f_\lambda^{n_1+n}(p(\lambda)) \rightarrow \infty$ on \mathcal{N} . Since $p(\lambda)$ is passive on $U \supset \mathcal{N}$ this would need to happen on all of U contrary to the fact that $f_{\lambda_0}^k(p(\lambda_0)) = 0$ for all $k \geq 0$. In the second case the multiplier of infinity at λ_0 would also equal $\exp(2i\pi\theta)$ contrary to the hypothesis that $v(\lambda_0) = \infty$ is a repelling fixed point for f_{λ_0} .

Since $\lambda \mapsto f_\lambda^n(p(\lambda))$ forms a normal family as mappings into \mathbb{P}^1 and none of the mappings hit infinity for any $\lambda \in U$ they form a normal family of mappings into \mathbb{C} (over all $\lambda \in U$). Normality implies that for any precompact open $V \subset U$ we can find a disc $D_R(0)$ of some radius $R > 0$ such that

$$f_\lambda^n(p(\lambda)) \in \mathbb{D}_R(0)$$

for all $n \geq 0$. It then follows from Theorem 2.7 that for all $\lambda \in V$ the fixed point at 0 is attracting or that it has a constant multiplier equal to $\exp(2i\pi\theta)$ with $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Since $V \subset U$ was arbitrary this holds on all of U . In particular, we are in Cases (iii) or (iv) from the statement of the Classification of the passive locus.

It remains to prove the statements about preperiodic parameters in these two cases. Suppose we are in Case (iii) so that $z = 0$ is attracting or superattracting for all $\lambda \in U$. For each $\lambda \in U$ let $m(\lambda)$ denote the local multiplicity of 0 for f_λ . Let $m_0 := \min\{m(\lambda) : \lambda \in U\}$ and let

$$U' := \{\lambda \in U : m(\lambda) > m_0\}.$$

Suppose $\lambda_0 \in U \setminus U'$, and choose a neighborhood W of λ_0 such that its closure \overline{W} is compactly contained in $U \setminus U'$. Then, there exists $\epsilon > 0$ such that:

- (A) $f_\lambda(\mathbb{D}_\epsilon(0))$ is compactly contained in $\mathbb{D}_\epsilon(0)$, and
- (B) for each $\lambda \in W$ and each $z \in \mathbb{D}_\epsilon(0) \setminus \{0\}$ we have that $f_\lambda(z) \neq 0$,
i.e. 0 is the only preimage of 0 under f_λ within $\mathbb{D}_\epsilon(0)$.

Since \overline{W} is compact and $f_\lambda^n(a(\lambda)) \rightarrow 0$ for all $\lambda \in \overline{W}$ there exists $k > 0$ such that for all $\lambda \in W$ we have that $f_\lambda^k(a(\lambda)) \subset \mathbb{D}_\epsilon(0)$. Then, using (B) above, the set of preperiodic parameters in W is

$$\{\lambda \in W : f_\lambda^n(a(\lambda)) = 0 \text{ for some } 0 \leq n \leq k\},$$

which is a closed subvariety of W . We conclude that that assertions from Case (iii) of the classification of the passive locus hold for all $\lambda \in U$.

Suppose we are in Case (3) of Theorem 2.7 so that $z = 0$ has multiplier $\exp(2i\pi\theta)$, with $\theta \in \mathbb{R} \setminus \mathbb{Q}$. The map f_λ is linearizable and $p(\lambda)$ lies in the interior of the domain of linearization of f_λ . In this case, as explained in [33], by making U smaller (if necessary), there is a uniform $\epsilon > 0$ such that for all $\lambda \in U$ the disc $D_\epsilon(0)$ is contained in the linearization domain for f_λ . It then follows that $p(\lambda)$ is preperiodic for f_λ if and only if $p(\lambda) = 0$. Therefore, in U the marked point $a(\lambda)$ is pre-periodic

if and only $f^{n_0}(a(\lambda)) = 0$. Since n_0 is fixed, this is an analytic condition on λ . We conclude that the assertions from Case (iv) of the the classification of the passive locus hold for $\lambda \in U$. \square

Remark 2.9. *One should note that the statement in [33] claims that in Case (iii) the set λ such that $a(\lambda)$ is preperiodic is a closed subvariety of U itself, without first removing a proper closed subvariety U' . Unfortunately, that is not true, even if the marked point is critical. Fortunately this claim about preperiodic parameters is not used anywhere later in their paper.*

Consider the following holomorphic family of polynomial mappings

$$f_\lambda(z) = z(z - \lambda)(z - 1/2)$$

where $\lambda \in \mathbb{C}$. The critical points of f_λ are

$$c_\pm(\lambda) = \frac{(1 + 2\lambda) \pm \sqrt{4\lambda^2 - 2\lambda + 1}}{6},$$

which vary holomorphically in a neighborhood $\mathbb{D}_r(0)$ of $\lambda = 0$, for some $r > 0$. Notice that $c_-(0) = 0$ and $c_+(0) = \frac{1}{3}$. Consider the marked critical point $c(\lambda) := c_+(\lambda)$. One can check that

- (1) There exists $0 < \epsilon < r$ such that $\lambda \in \mathbb{D}_\epsilon(0)$ implies that $f_\lambda^n(c(\lambda)) \rightarrow 0$ with $|f_\lambda^n(c(\lambda))| < 1/2$ for all $n \geq 0$, and
- (2) There exists an infinite sequence $\{\lambda_k\}_{k=1}^\infty$ in $\mathbb{D}_\epsilon(0) \setminus \{0\}$ with $\lambda_k \rightarrow 0$ such that for each k there is an iterate n_k with $f_{\lambda_k}^{n_k}(c(\lambda_k)) = 0$.

Therefore, the set of preperiodic parameters $\lambda \in \mathbb{D}_\epsilon(0)$ is not a closed subvariety, but they are in $\mathbb{D}_\epsilon(0) \setminus \{0\}$.

3. PROOF OF THEOREM C

Our proof of Theorem C will closely follow the strategy that Dujardin-Favre use to prove Theorem 4.2 in [33] and we will assume some of the basic results from their proof.

3.1. Strategy for Proof of Theorem C. Let $f : V \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ be an algebraic family of rational maps of degree d defined over $\overline{\mathbb{Q}}$. Let $a, b : V \dashrightarrow \mathbb{P}^1$ be marked points and assume, without loss of generality, that the indeterminacy $I(a) \cup I(b) \subset V_{\text{deg}}$. Let T_a be the activity current of $a(\lambda)$ and suppose that all hypotheses of Theorem C are satisfied.

Proposition 3.1. *The following convergence of currents*

$$(3.1) \quad \frac{1}{d^n} [(f_\lambda^n \circ a)(\lambda) = b(\lambda)] \rightarrow T_a$$

holds in $V \setminus V_{\text{deg}}$ if and only if there is a dense set of parameters $\lambda \in V_{\text{good}} \subset V \setminus V_{\text{deg}}$ such that

$$(3.2) \quad h_n(\lambda) := \frac{1}{d^n} \log \text{dist}_{\mathbb{P}^1}(f_\lambda^n a(\lambda), b(\lambda)) \rightarrow 0,$$

where $\text{dist}_{\mathbb{P}^1}$ denotes the chordal distance on \mathbb{P}^1 .

Proof. A direct adaptation of the first four paragraphs of the proof of Theorem 4.2 in [33] shows that (3.1) holds if and only if $h_n \rightarrow 0$ in $L_{\text{loc}}^1(V \setminus V_{\text{deg}})$. Let us summarize the key steps here.

This is a local statement, so we can suppose without loss of generality that V is an open subset of \mathbb{C}^m . Choose a lift $\tilde{f} : V \times \mathbb{C}^2 \rightarrow \mathbb{C}^2$ and denote the iterates of each $f_\lambda : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ by

$$\tilde{f}_\lambda^n(z, w) = \left(P_\lambda^{(n)}(z, w), Q_\lambda^{(n)}(z, w) \right).$$

Choose lifts $\tilde{a}, \tilde{b} : V \rightarrow \mathbb{C}^2$ and denote their coordinates by $\tilde{a}(\lambda) = (a_1(\lambda), a_2(\lambda))$ and $\tilde{b}(\lambda) = (b_1(\lambda), b_2(\lambda))$.

Using the formula for chordal distance on \mathbb{P}^1 we have:

$$(3.3) \quad h_n(\lambda) := \frac{1}{d^n} \log |P_\lambda^{(n)}(\tilde{a}(\lambda))b_2(\lambda) - Q_\lambda^{(n)}(\tilde{a}(\lambda))b_1(\lambda)|^2 - \frac{1}{d^n} \log \|(\tilde{f}_\lambda^n \circ \tilde{a})(\lambda)\|^2 - \frac{1}{d^n} \log \|\tilde{b}(\lambda)\|^2.$$

Note that the last term converges 0 and the second to last term converges to $2H(\lambda)$, both locally uniformly on $V \setminus V_{\text{deg}}$. (Here, $H(\lambda)$ is the local potential for T_a .)

Therefore we can conclude that $h_n \rightarrow 0$ in $L_{\text{loc}}^1(V \setminus V_{\text{deg}})$ if and only if

$$\frac{1}{d^n} \log |P_\lambda^{(n)}(\tilde{a}(\lambda))b_2(\lambda) - Q_\lambda^{(n)}(\tilde{a}(\lambda))b_1(\lambda)|^2 \rightarrow 2H(\lambda) \quad \text{in } L_{\text{loc}}^1(V \setminus V_{\text{deg}}).$$

The PSH functions on the left hand side are local potentials for the currents expressed in (1.8) and H is a local potential for T_a on $V \setminus V_{\text{deg}}$.

If h_n converges to 0 in $L_{\text{loc}}^1(V \setminus V_{\text{deg}})$ then clearly it converges on a dense set. Conversely, suppose h_n does not converge to 0 in $L_{\text{loc}}^1(V \setminus V_{\text{deg}})$, then, as in paragraph seven of the proof of Theorem 4.2 in [33] shows that (3.1), one uses Hartogs' Lemma [42, Theorem 4.1.9(b)] to find an open set $U \subset V \setminus V_{\text{deg}}$ and a subsequence n_k such that $h_{n_k}(\lambda) \rightarrow h(\lambda) < 0$ for all $\lambda \in U$. \square

The proof of Theorem C will then follow immediately from the following:

Proposition 3.2. *There is a dense set of parameters $\lambda \in V_{\text{good}} \subset V \setminus V_{\text{deg}}$ such that (3.2) holds.*

This will follow from the Dujardin-Favre classification of the passive locus and the following beautiful theorem:

Silverman's Theorem E [68]. *Let $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be a rational map of degree $d \geq 2$ defined over a number field K . Let $A, B \in \mathbb{P}^1(\overline{K})$ and assume that B is not exceptional for ϕ and that A is not preperiodic for ϕ . Then*

$$(3.4) \quad \lim_{n \rightarrow \infty} \frac{\delta(\phi^n A, B)}{d^n} = 0,$$

where $\delta(P, Q) = 2 - \log \text{dist}_{\mathbb{P}^1}(P, Q)$ is the logarithmic distance function³.

Remark that (3.4) holds if and only if $\lim_{n \rightarrow \infty} \frac{1}{d^n} \log \text{dist}_{\mathbb{P}^1}(\phi^n A, B) = 0$.

3.2. Proof of Proposition 3.2. We will need the following result:

Algebraic Points Are Dense. *Let $V \subset \mathbb{P}^n$ be a projective algebraic manifold that is defined over $\overline{\mathbb{Q}}$. Then, the set of points $a \in V$ that can be represented by homogeneous coordinates in $\overline{\mathbb{Q}}$ form a dense subset of V (in the complex topology). I.e. $V(\overline{\mathbb{Q}})$ is dense in V .*

We could not find this statement in the literature, but it can be proved by induction on $\dim(V)$. The base of the induction, when $\dim(V) = 0$, plays an important role in the theory of Kleinian Groups, see for example [52, Lemma 3.1.5].

Proof of Proposition 3.2. We will consider the active and passive loci separately. Let λ_0 be an active parameter, and $W \subset V$ be any open neighborhood containing λ_0 . We will find a parameter $\lambda_1 \in W$ at which (3.2) holds. We will do this by showing that there exists a parameter $\lambda_1 \in W$ such that iterates of $a(\lambda_1)$ under f_{λ_1} will eventually land on a repelling cycle disjoint from $b(\lambda_1)$. This will immediately imply (3.2) at λ_1 .

³In [68, Theorem E] a different logarithmic distance function was used. However, as mentioned in Section 3 of the referenced paper, the result still holds if we use $2 - \text{dist}_{\mathbb{P}^1}(\cdot, \cdot)$ instead.

Pick three distinct points in a repelling cycle of f_{λ_0} which is disjoint from $b(\lambda_0)$. By reducing W to a smaller neighborhood of λ_0 if necessary, we can ensure that the repelling cycle moves holomorphically as λ varies over W , and that $b(\lambda)$ is disjoint from the cycle for every $\lambda \in W$. Since the family $\{f_\lambda^n a(\lambda)\}_{n=1}^\infty$ is not normal in W , it cannot avoid all three points.

We now suppose λ_0 is in the passive locus for a and let U be the connected component of the passive locus containing λ_0 . Then, the Dujardin-Favre classification gives four possible behaviors for $f_\lambda^n(a(\lambda))$ in U .

In Cases (i), (iii), and (iv) the classification gives a (possibly empty) closed subvariety $U' \subsetneq U$ such that the set of parameters for which $a(\lambda)$ is preperiodic is contained in a proper closed subvariety $U_1 \subset (U \setminus U')$. Moreover, the hypothesis that marked point $b(\lambda)$ is not persistently exceptional gives that there is another proper closed subvariety $U_2 \subset U$ such that $b(\lambda)$ is not exceptional for $\lambda \in U \setminus U_2$. Then, $U \setminus (U' \cup U_1 \cup U_2)$ is an open dense subset of U . Since $V(\overline{\mathbb{Q}})$ is dense in V , see the beginning of this subsection, arbitrarily close to λ_0 is a point $\lambda_1 \in U \setminus (U' \cup U_1 \cup U_2)$ with coordinates in $\overline{\mathbb{Q}}$. Since there are only finitely many coefficients to consider, we can find a number field K so that $f_{\lambda_1} \in K(z)$ and the points $a(\lambda_1), b(\lambda_1) \in \mathbb{P}^1(K)$. Since $\lambda_1 \in U \setminus (U' \cup U_1 \cup U_2)$, the point $a(\lambda_1)$ has infinite orbit under f_{λ_1} , and the point $b(\lambda_1)$ is not exceptional for f_{λ_1} . Hence Silverman's Theorem E implies that (3.2) holds for the parameter λ_1 .

Finally suppose we are in Case (ii), so that the marked point $a(\lambda)$ is persistently preperiodic. By assumption there is no iterate n with $f_\lambda^n(a(\lambda)) \equiv b(\lambda)$, so there is a proper closed subvariety $U_1 \subset U$ such that for all $\lambda \in U \setminus U_1$ and all $n \geq 0$, we have $f_\lambda^n(a(\lambda)) \neq b(\lambda)$. It follows that for each $\lambda \in U \setminus U_1$, the quantities $\text{dist}_{\mathbb{P}^1}(f_\lambda^n(a(\lambda)), b(\lambda))$ are uniformly bounded in $n \geq 0$, which implies (3.2) for all $\lambda \in U \setminus U_1$. \square

3.3. Arithmetic proof of Proposition 3.2 under additional hypotheses. Under additional hypotheses we can prove Proposition 3.2 (and hence Theorems C and C') without appealing to the Dujardin-Favre classification of the passive locus. Instead we will require some technical results from arithmetic dynamics.

The additional hypotheses we need are:

- (iii) The parameter space is \mathbb{P}^1 .
- (iv) The marked point a is not passive on all of $\mathbb{P}^1 \setminus V_{\text{deg}}$.

For applications in chromatic zeros our parameter space is \mathbb{P}^1 so that Hypothesis (iii) will automatically hold (in fact, we typically think of it as $\mathbb{C} \subset \mathbb{P}^1$). Meanwhile, for the renormalization mappings associated with many hierarchical lattices one can check Hypothesis (iv) directly, but it does not hold for all such mappings (e.g. when the generating graph is a triangle, as discussed in Section 8.3).

Proposition 3.2 will follow from Silverman's Theorem E and the next statement (choosing K to be dense in \mathbb{C}), whose proof was communicated to us by Laura DeMarco and Niki Myrto Mavraki.

Proposition 3.3. *Suppose the hypotheses in Theorem C and additionally hypotheses (iii) and (iv) above. Then, for any number field K there are at most finitely many parameters $\lambda \in \mathbb{P}^1(K) \setminus V_{\text{deg}}$ such that the marked point $a(\lambda)$ is preperiodic under f_λ .*

We will need the following two results, which depend on having a one-dimensional parameter space. Denote the logarithmic absolute Weil height on $\overline{\mathbb{Q}}$ by $h : \overline{\mathbb{Q}} \rightarrow \mathbb{R}$. For a rational map $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ defined over K and a point $P \in \mathbb{P}^1(K)$, we denote the canonical height function associated to ϕ by $\widehat{h}_\phi(P)$. For more background on these definitions, see [69].

Call-Silverman Specialization [19, Theorem 4.1]. *Let (f, a) be a one-dimensional algebraic family of rational maps of degree $d \geq 2$ with a marked point a , both defined over a number field K .*

Then, for any sequence of parameters $\{\lambda_n\}_{n=1}^\infty \subset \mathbb{P}^1(K) \setminus V_{\text{deg}}$ such that $h(\lambda_n) \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} \frac{\widehat{h}_{f_{\lambda_n}}(a(\lambda_n))}{h(\lambda_n)} = \widehat{h}_f(a),$$

where $\widehat{h}_f(a)$ is the canonical height associated to the pair (f, a) .

The canonical height $\widehat{h}_f(a)$ was introduced in [19].

The pair (f, a) is called *isotrivial* if there exists a branched covering $W \rightarrow \mathbb{P}^1 \setminus V_{\text{deg}}$ and a family of holomorphically varying Möbius transformations M_λ such that $M_\lambda \circ f_\lambda \circ M_\lambda^{-1}$ is independent of $\lambda \in W$ and also $M_\lambda \circ a$ is a constant function of $\lambda \in W$.

Theorem 3.4. (DeMarco [27, Theorem 1.4]) *Suppose $f : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ is a non-isotrivial one-dimensional algebraic family of rational maps. Let $\widehat{h}_f : \mathbb{P}^1(\bar{k}) \rightarrow \mathbb{R}$ be a canonical height of f , defined over the function field $k = \mathbb{C}(\mathbb{P}^1)$. For each $a \in \mathbb{P}^1(\bar{k})$, the following are equivalent:*

- (1) *The marked point a is passive in all of $\mathbb{P}^1 \setminus V_{\text{deg}}$;*
- (2) *$\widehat{h}_f(a) = 0$;*
- (3) *(f, a) is preperiodic.*

Moreover, the set

$$\{a \in \mathbb{P}^1(k) : a \text{ is passive in all of } \mathbb{P}^1 \setminus V_{\text{deg}}\}$$

is finite.

Proof of Proposition 3.3. Assume on the contrary that there is a sequence of distinct parameters $\{\lambda_n\}_{n=1}^\infty \subset \mathbb{P}^1(K) \setminus V_{\text{deg}}$ such that $a(\lambda_n)$ is preperiodic for f_{λ_n} . It follows from Northcott property [69, Theorem 3.7] that the parameters λ_n satisfies $h(\lambda_n) \rightarrow \infty$. Meanwhile, since $a(\lambda_n)$ is preperiodic for f_{λ_n} , we have $\widehat{h}_{f_{\lambda_n}}(a(\lambda_n)) = 0$. Then Call-Silverman Specialization implies $\widehat{h}_f(a) = 0$, so by Theorem 3.4 the marked point a must be passive in all of $\mathbb{P}^1 \setminus V_{\text{deg}}$, which contradicts hypothesis (iv). \square

4. PROOF OF THEOREM C'

The following statement about convergence of sequences of PSH functions is probably standard, but we will include a proof because we cannot find an appropriate reference.

Proposition 4.1. *Let $\{\phi_n\}_{n=1}^\infty$ be a sequence of PSH functions in an open connected set $U \subseteq \mathbb{C}^m$ which is uniformly bounded above in compact sets. Suppose there is a PSH function ϕ in U such that $\phi_n \rightarrow \phi$ in $L^1_{\text{loc}}(U \setminus X)$, where $X \subset U$ is an analytic hypersurface. Then $\phi_n \rightarrow \phi$ in $L^1_{\text{loc}}(U)$.*

Proof. Assume by contradiction that ϕ_n does not converge to ϕ in $L^1_{\text{loc}}(U)$. Then there is an $\epsilon > 0$, a compact set K with positive Lebesgue measure, and a subsequence ϕ_{n_k} such that

$$\|\phi_{n_k} - \phi\|_{L^1(K)} > \epsilon \text{ for all } k.$$

Note that since $\phi_n \rightarrow \phi$ in $L^1_{\text{loc}}(U \setminus X)$, the compact set K must intersect X . By the hypotheses, the sequence ϕ_{n_k} satisfies the conditions for the compactness theorem for PSH functions [42, Theorem 4.1.9], so we can find a further subsequence (which we still denote by ϕ_{n_k}), and a PSH function $\tilde{\phi}$ in U such that

$$\phi_{n_k} \rightarrow \tilde{\phi} \text{ in } L^1_{\text{loc}}(U).$$

In particular $\phi_{n_k} \rightarrow \tilde{\phi}$ in $L^1(K)$, which implies $\tilde{\phi} \neq \phi$ in $L^1(K)$, so there exist $\delta > 0$ and a compact subset $K' \subset K$ with positive Lebesgue measure such that $|\tilde{\phi}(z) - \phi(z)| > \delta$ for all $z \in K'$.

Let X_ϵ be the ϵ -neighborhood of X in U , and let $X'_\epsilon := X_\epsilon \cap K'$. Choose $\epsilon > 0$ which satisfies $\text{Leb}(K') = 2\text{Leb}(X'_\epsilon)$, where Leb denotes Lebesgue measure. It follows that

$$(4.1) \quad \int_{K' \setminus X'_\epsilon} |\tilde{\phi} - \phi| \, d\text{Leb} > \delta \cdot \text{Leb}(K' \setminus X'_\epsilon) = \frac{\delta}{2} \text{Leb}(K') > 0.$$

Meanwhile, since $K' \setminus X'_\epsilon$ is a compact subset of U disjoint from X , we must have $\tilde{\phi} = \phi$ in $L^1(K' \setminus X'_\epsilon)$, which contradicts (4.1). \square

Proof of Theorem C'. This is a local statement, so we can suppose without loss of generality that V is an open subset of \mathbb{C}^m . In the proof of Theorem C we saw that

$$(4.2) \quad \frac{1}{d^n} \log |P_\lambda^{(n)}(\tilde{a}(\lambda))b_2(\lambda) - Q_\lambda^{(n)}(\tilde{a}(\lambda))b_1(\lambda)|^2 \rightarrow 2H(\lambda) \quad \text{in } L^1_{\text{loc}}(V \setminus V_{\text{deg}}).$$

Here, the notation is as in the proof of Proposition 3.1. Note that the PSH functions on the left hand side of (4.2) are defined on all of V and are potentials for the currents defined in Equation (1.9) from Theorem C'.

On any precompact open subset of V we can choose our lift $\tilde{a}(\lambda)$ sufficiently close to the origin $(0, 0) \in \mathbb{C}^2$ so that it is in the basin of attraction of $(0, 0)$ under $f_\lambda(z, w) = (P_\lambda(z, w), Q_\lambda(z, w))$. It then follows that the sequence of potentials on the left hand side of (4.2) is locally bounded above. Hence, Proposition 4.1 implies that the convergence extends to all of V . \square

5. THE POTTS MODEL, CHROMATIC ZEROS, AND MIGDAL-KADANOFF RENORMALIZATION

We first give a brief account of the antiferromagnetic Potts model on a graph Γ and its connection with the chromatic zeros of \mathcal{P}_Γ . Suitable references include [73, 5, 70], [4, p.323-325], and references therein. We then describe the Migdal-Kadanoff Renormalization procedure that produces a rational function $r_q(y)$ relating the zeros for the Potts Model on one level of a hierarchical lattice to the zeros for the next level. The remainder of the section is devoted to proving properties of the renormalization mappings $r_q(y)$.

5.1. Basic Setup. Fix a graph $\Gamma = (V, E)$ and fix an integer $q \geq 2$. A *spin configuration* of the graph Γ is a map

$$\sigma : V \rightarrow \{1, 2, \dots, q\}.$$

Fix the coupling constant $J < 0$. The energy $H_\Gamma(\sigma)$ associated with a configuration σ on Γ is defined as

$$H_\Gamma(\sigma) = -J \sum_{\{v_i, v_j\} \in E} \delta(\sigma(v_i), \sigma(v_j)) = -J\mathcal{E}(\sigma),$$

where $\delta(a, b) = 1$ if $a = b$ and 0 otherwise, and $\mathcal{E}(\sigma)$ is the number of edges whose endpoints are assigned the same spin under σ . Remark that since $J < 0$ it is energetically favorable to have different spins at the endpoints of each edge, if possible. This means that we are in the *antiferromagnetic* regime.

The Boltzmann distribution assigns a configuration σ on Γ probability proportional to the weight

$$W_\Gamma(\sigma) = \exp\left(-\frac{H_\Gamma(\sigma)}{\mathsf{T}}\right) = \exp\left(\frac{J\mathcal{E}(\sigma)}{\mathsf{T}}\right),$$

where $T > 0$ is the temperature of the system ⁴. The probability $\Pr(\sigma)$ of σ occurring is therefore

$$(5.1) \quad \Pr(\sigma) = W_\Gamma(\sigma)/Z_\Gamma \quad \text{where} \quad Z_\Gamma := \sum_{\sigma} W_\Gamma(\sigma),$$

and the sum is over all possible spin configurations. Some intuition for this distribution can be gained by considering the following two extreme cases: when T is near zero, configurations with minimum energy are strongly favored. Meanwhile for high temperature, all configurations occur with nearly equal probability.

Let us introduce the temperature-like variable $y := e^{J/T}$, so that $W_\Gamma(\sigma) = y^{\mathcal{E}(\sigma)}$. All the quantities above implicitly depend on q , y , and the graph Γ . The normalizing factor $Z_\Gamma(q, y)$ is called the *partition function* and given by

$$Z_\Gamma(q, y) := \sum_{\sigma} y^{\mathcal{E}(\sigma)}.$$

It turns out that $Z_\Gamma(q, y)$ is actually a polynomial in *both* q and y . To see this it will be helpful to express the partition function in terms of (q, v) where $v = y - 1$. For any subset of the edge set $A \subseteq E$ is a subgraph (V, A) . We have

$$(5.2) \quad Z_\Gamma(q, v) = \sum_{\sigma} \prod_{(i,j) \in E} [1 + v\delta(\sigma_i, \sigma_j)] = \sum_{A \subseteq E} q^{k(A)} v^{|A|}.$$

where $k(A)$ is the number of connected components of (V, A) , including isolated vertices. This is called the *Fortuin-Kasteleyn [35] representation* of $Z_\Gamma(q, v)$; see, for example, [70, Section 2.2]. (We will only express Z_Γ in terms of v instead of y in this paragraph and in Subsection 5.2.)

As discussed in the introduction, we will describe the zeros of $Z_\Gamma(q, y)$ as a divisor denoted

$$\mathcal{S} := (Z_\Gamma(q, y) = 0).$$

Remark that in the next subsection we will see that if Γ is 2-connected, then $Z_\Gamma(q, y) = q\tilde{Z}_\Gamma(q, y)$ with $\tilde{Z}_\Gamma(q, y)$ irreducible, implying \mathcal{S} is a *reduced divisor*, i.e. all multiplicities are one. Therefore, if Γ is 2-connected there is no harm in thinking of \mathcal{S} as a (reducible) algebraic curve.

To establish the connection between the chromatic polynomial $\mathcal{P}_\Gamma(q)$ and the partition function $Z_\Gamma(q, y)$ of the Potts model note that

$$\mathcal{P}_\Gamma(q) = \sum_{\substack{\sigma \text{ such that} \\ \mathcal{E}(\sigma)=0}} 1 = Z_\Gamma(q, 0).$$

Therefore, the chromatic zeros are given by the intersection:

$$\mathcal{C} := \mathcal{S} \cap (y = 0),$$

where Bezout intersection multiplicities and multiplicities of the divisor \mathcal{S} are taken into account.

5.2. Irreducibility of $\tilde{Z}_\Gamma(q, y)$ for 2-connected Γ . It follows from (5.2) that we can always factor $Z_\Gamma(q, v) = q\tilde{Z}_\Gamma(q, v)$ in the polynomial ring $\mathbb{C}[q, v]$. The goal of this subsection is to prove:

Proposition 5.1. *If Γ is 2-connected, then $\tilde{Z}_\Gamma(q, v)$ is irreducible in $\mathbb{C}[q, v]$. (The same holds in the (q, y) variables.)*

⁴We set the Boltzmann constant $k_B = 1$.

We will prove this proposition using the well-known relationship between $\tilde{Z}_\Gamma(q, y)$ and the *Tutte Polynomial* of Γ . It is defined as

$$(5.3) \quad \mathcal{T}_\Gamma(x, y) = \sum_{A \subseteq E} (x-1)^{k(A)-1} (y-1)^{|A|+k(A)-|V|},$$

where $k(A)$ has the same interpretation as in (5.2). The variables (x, y) in the Tutte Polynomial are related⁵ to the variables (q, v) in the Partition Function (5.2) by:

$$x = 1 + (q/v) \quad \text{and} \quad y = v + 1.$$

Comparing (5.3) with (5.2) we see the following relationship [70, Section 2.5] between $\mathcal{T}(x, y)$ and $Z_\Gamma(q, v)$:

$$(5.4) \quad \mathcal{T}_\Gamma(x, y) = (x-1)^{-1} (y-1)^{-|V|} Z_\Gamma((x-1)(y-1), y-1).$$

Proposition 5.1 will be a corollary to the following nice result by de Mier, Merino, and Noyi [54].

Irreducibility of Tutte Polynomials (Merino-Mier-Noyi [54]). *If Γ is a 2-connected graph, then $\mathcal{T}_\Gamma(x, y)$ is irreducible in $\mathbb{C}[x, y]$.*

Remark that the theorem proved in [54] is that the Tutte polynomial of a connected matroid is irreducible. However, that implies the result stated above because associated to any graph Γ is a matroid $M(\Gamma)$, called the *cycle matroid* of Γ , which has the following properties:

- (1) The Tutte polynomial of Γ equals the Tutte polynomial of $M(\Gamma)$; (see e.g. [70, Equation 1.4]),
- (2) Γ is 2-connected iff $M(\Gamma)$ is connected; (see e.g. [72, p. 78].)

Lemma 5.2. *$Z_\Gamma(q, v)$ vanishes to order exactly $|V|$ at the origin.*

Proof. For any subgraph (V, A) , it follows from a counting argument that $k(A) + |A| \geq |V|$. Moreover, for the subgraph (V, A_0) without any edges, the sum $k(A_0) + |A_0|$ is exactly $|V|$. Therefore the order of vanishing is exactly $|V|$ at the origin. \square

Proof of Proposition 5.1. By the Irreducibility of the Tutte Polynomial, it suffices to prove that if \tilde{Z}_Γ is reducible then so is \mathcal{T}_Γ . Suppose \tilde{Z}_Γ is reducible:

$$\tilde{Z}_\Gamma = A_1 \cdot A_2 \cdot B,$$

where A_1, A_2 are non-constant irreducible factors, and B can potentially be a unit. Denote by C_i the zero set of A_i .

Let $H : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ be the birational map $(x, y) \mapsto ((x-1)(y-1), y-1)$, so that by (5.4) we have

$$\mathcal{T}_\Gamma(x, y) = (y-1)^{-|V|+1} (\tilde{Z}_\Gamma \circ H).$$

Therefore, in order to prove that \mathcal{T}_Γ is reducible it suffices to find at least two irreducible factors of $\tilde{Z}_\Gamma \circ H$ each of which is not equal to $y-1$.

For $i = 1$ and 2 , although $H^{-1}(C_i)$ can possibly contain the line $E := \{(x, y) \in \mathbb{C}^2 : y = 1\}$, it cannot be the only irreducible component of $H^{-1}(C_i)$ because $H(E)$ is a single point $(0, 0)$. From this observation we now have to consider two separate cases.

- (i) If $A_1 \not\equiv A_2$, then the zero set of \mathcal{T}_Γ contains at least two distinct irreducible components, neither of which is the line E .

⁵Although the variable y appears in Equation (5.1) for the partition function and also in Equation (5.3) for the Tutte Polynomial, there is no conflict of notation because both satisfy $y = v + 1$.

- (ii) If $A_1 \equiv A_2$, then the zero set of \mathcal{T}_Γ contains an irreducible component of multiplicity at least two, which is not the line E .

In either case, we conclude that \mathcal{T}_Γ is reducible. □

5.3. Combinatorics of Hierarchical Lattices.

Proposition 5.3. *Suppose $\{\Gamma_n\}_{n=1}^\infty$ is a hierarchical lattice that is generated by a 2-connected generated graph $\Gamma = (V, E)$. Then, Γ_n is 2-connected for each $n \geq 0$.*

Proof. The proof is by induction on n . Since Γ_0 is a single edge with two vertices at its endpoints it is 2-connected. Suppose now that Γ_n is 2-connected for some $n \geq 0$ to show that Γ_{n+1} is 2-connected. Recall that Γ_{n+1} is built by replacing each edge of the generating graph Γ with a copy of Γ_n using the marked vertices a and b as endpoints. The vertices of Γ_{n+1} fall into two classes:

- (1) The $|V|$ vertices of Γ_{n+1} that come from the vertices of Γ . Each of them is a marked vertex a or b from some copy of Γ_n , and
- (2) The remaining vertices.

If the removal of a vertex of Type (1) disconnects Γ_{n+1} then, since each Γ_n is 2-connected, this would imply that removal of the corresponding vertex of Γ disconnects Γ . This is impossible because Γ is 2-connected. Meanwhile, if removal of a vertex of Type (2) disconnects Γ_{n+1} then its removal will also disconnect the unique copy of Γ_n that the vertex is contained in. This contradicts the induction hypothesis. □

Proposition 5.4. *Let $\Gamma_n = (V_n, E_n)$ be a hierarchical lattice generated by generating graph $\Gamma = (V, E)$. Then $|V_n|$ and $|E_n|$ grow at the same exponential rate as $n \rightarrow \infty$.*

Proof. Observe that for any $n \geq 1$,

$$|V_{n+1}| = |V_n| + |E_n| \cdot (|V| - 2) = |V_n| + |E|^n \cdot (|V| - 2).$$

It follows from induction that

$$|V_n| = |V| + (|V| - 2) \cdot \sum_{i=1}^{n-1} |E|^i = |V| + (|V| - 2) \cdot |E| \frac{|E|^{n-1} - 1}{|E| - 1},$$

which proves the assertion. □

5.4. Migdal-Kadanoff Renormalization for the DHL. Let $\{\Gamma_n = (V_n, E_n)\}_{n=0}^\infty$ be the Diamond Hierarchical Lattice (DHL). For each $n \geq 0$ the partition function $Z_n(q, y) \equiv Z_{\Gamma_n}(q, y)$ has zero divisor

$$\mathcal{S}_n := (Z_n(q, y) = 0).$$

Remark that Γ_0 is always a single edge with two vertices at its endpoints, so a simple calculation yields $Z_{\Gamma_0}(q, y) = q(y + q - 1)$ so that

$$\mathcal{S}_0 := (q(y + q - 1) = 0).$$

Associated to the hierarchical lattice $\{\Gamma_n\}_{n=0}^\infty$ is a Migdal-Kadanoff renormalization mapping that relates the zero divisor \mathcal{S}_{n+1} to the zero divisor \mathcal{S}_n .

Proposition 5.5. *For the DHL we have that for each $n \geq 0$*

$$\mathcal{S}_n = (R^n)^*(\mathcal{S}_0)$$

where $R : \mathbb{C} \times \mathbb{P}^1 \rightarrow \mathbb{C} \times \mathbb{P}^1$ is given by

$$(5.5) \quad R(q, y) = (q, r_q(y)), \quad \text{where} \quad r_q(y) = \left(\frac{y^2 + q - 1}{2y + q - 2} \right)^2.$$

As usual, the superscript $*$ denotes pullback of a divisor and we will denote points $y \in \mathbb{P}^1$ using the standard chart $\mathbb{C} \subset \mathbb{P}^1$.

The proof will be very similar to the derivation of the Migdal-Kadanoff renormalization transformation for the Ising Model on the DHL [13, Section 2.5] and it relies on the multiplicativity of the conditional partition functions which is proved in [13, Lemma 2.1], in the context of the Ising Model.

Proof. For each $n \geq 0$ consider the following conditional partition functions:

$$\mathcal{U}_n \equiv \mathcal{U}_n(q, y) := \sum_{\substack{\sigma \text{ such that} \\ \sigma(a)=\sigma(b)=1}} W(\sigma) \quad \text{and} \quad \mathcal{V}_n \equiv \mathcal{V}_n(q, y) := \sum_{\substack{\sigma \text{ such that} \\ \sigma(a)=1, \sigma(b)=2}} W(\sigma).$$

We claim for each $n \geq 0$ that

$$(5.6) \quad \mathcal{U}_{n+1} = (\mathcal{U}_n^2 + (q-1)\mathcal{V}_n^2) \quad \text{and} \quad \mathcal{V}_{n+1} = (2\mathcal{U}_n\mathcal{V}_n + (q-2)\mathcal{V}_n^2).$$

To show this, it will be helpful to depict them graphically as follows:

$$\mathcal{U}_n := Z_n \left(\begin{array}{c} \textcircled{1} \\ \text{---} \\ \textcircled{1} \end{array} \right), \quad \mathcal{V}_n := Z_n \left(\begin{array}{c} \textcircled{1} \\ \text{---} \\ \textcircled{2} \end{array} \right) = Z_n \left(\begin{array}{c} \textcircled{2} \\ \text{---} \\ \textcircled{1} \end{array} \right).$$

The ones and twos in the figure denote the spins at the marked vertices a and b . Let us graphically illustrate the derivation of the first equation from (5.6):

$$\begin{aligned} \mathcal{U}_{n+1} &= Z_{n+1} \left(\begin{array}{c} \textcircled{1} \\ \text{---} \\ \textcircled{1} \end{array} \right) \\ &= Z_{n+1} \left(\begin{array}{c} \textcircled{1} \\ \text{---} \\ \textcircled{1} \end{array} \right) + 2(q-1)Z_{n+1} \left(\begin{array}{c} \textcircled{1} \\ \text{---} \\ \textcircled{2} \end{array} \right) \\ &\quad + (q-1)Z_{n+1} \left(\begin{array}{c} \textcircled{1} \\ \text{---} \\ \textcircled{2} \end{array} \right) + (q-1)(q-2)Z_{n+1} \left(\begin{array}{c} \textcircled{1} \\ \text{---} \\ \textcircled{3} \end{array} \right) \\ &= \mathcal{U}_n^4 + 2(q-1)\mathcal{U}_n^2\mathcal{V}_n^2 + (q-1)\mathcal{V}_n^4 + (q-1)(q-2)\mathcal{V}_n^4. \end{aligned}$$

The numbers one, two, and three in the second row of the figure above are meant to denote the boundary conditions imposed on each of the four copies of Γ_n that are glued together to form Γ_{n+1} . The third line is obtained from the second using multiplicativity of the conditional partition functions. (Once the spins at those four vertices are fixed, the conditional partition function is the same as that of a disjoint union of the four copies of Γ_n , each with its corresponding boundary conditions.) The expression for \mathcal{V}_{n+1} in (5.6) can be obtained similarly.

In order to use an iteration on \mathbb{P}^1 instead of \mathbb{C}^2 it will be more convenient to iterate the ratio $y_n := \mathcal{U}_n/\mathcal{V}_n$, where $n \geq 0$. A simple calculation shows that $y_0 = y = e^{J/T}$. Using (5.6) we find that

$$y_{n+1} = \frac{\mathcal{U}_{n+1}}{\mathcal{V}_{n+1}} = \left(\frac{\mathcal{U}_n^2 + (q-1)\mathcal{V}_n^2}{2\mathcal{U}_n\mathcal{V}_n + (q-2)\mathcal{V}_n^2} \right)^2 = \left(\frac{y_n^2 + q - 1}{2y_n + q - 2} \right)^2 = r_q(y_n).$$

Therefore, $(q_n, y_n) = R^n(q, y)$ where $q_n = q$ for all n .

Note that

$$(5.7) \quad Z_n(q, y) = q\mathcal{U}_n + q(q-1)\mathcal{V}_n.$$

Since the generating graph Γ is 2 connected Proposition 5.3 implies that Γ_n is 2-connected for each $n \geq 0$. Therefore, Proposition 5.1 gives that $\tilde{Z}_n(q, y) = \mathcal{U}_n + (q-1)\mathcal{V}_n$ is irreducible, implying that \mathcal{U}_n and \mathcal{V}_n have no common factors of positive degree in q or y . Therefore,

$$(5.8) \quad \mathcal{S}_n = (Z_n(q, y) = 0) = (q_n(\mathcal{U}_n + (q_n - 1)\mathcal{V}_n) = 0) = (q_n(y_n + q_n - 1) = 0) = (R^n)^*\mathcal{S}_0,$$

where in the third equality we used that \mathcal{U}_n and \mathcal{V}_n have no common factors of positive degree. \square

The map $r_q(y)$ given in (5.5) is called the *Migdal-Kadanoff renormalization mapping* for the q -state Potts model on the DHL. Remark that this is an algebraic family of rational mappings of degree 4 defined over \mathbb{Q} . As a consequence of Proposition 5.5, the chromatic zeros for the DHL can be obtained dynamically:

$$(5.9) \quad \mathcal{C}_n = (R^n)^*(\mathcal{S}_0) \cap (y = 0)$$

and note that up to the simple zero at $q = 0$ we can use

$$(5.10) \quad \tilde{\mathcal{C}}_n = (R^n)^*(y + q - 1 = 0) \cap (y = 0).$$

When considering the limiting measure of chromatic zeros it suffices to consider $\tilde{\mathcal{C}}_n$.

5.5. Migdal-Kadanoff Renormalization for arbitrary hierarchical lattices. Now suppose $\Gamma_n = (V_n, E_n)$ is the hierarchical lattice generated by an arbitrary generating graph $\Gamma = (V, E)$. It is clear that we can repeat the procedure in Proposition 5.5 to produce a renormalization mapping $r_q(y)$ associated to the generating graph Γ , which is a rational map in y on the Riemann sphere of degree at most $|E|$, parameterized by polynomials in q with integer coefficients.

However, it is possible that the generic degree of $r_q(y)$ is strictly smaller than $|E|$. One such example is the Tripod shown in Figure 2 for which we have

$$\mathcal{U}_{n+1} = (\mathcal{U}_n + (q-1)\mathcal{V}_n) (\mathcal{U}_n^2 + (q-1)\mathcal{V}_n^2) \quad \text{and} \quad \mathcal{V}_{n+1} = (\mathcal{U}_n + (q-1)\mathcal{V}_n) (2\mathcal{U}_n\mathcal{V}_n + (q-2)\mathcal{V}_n^2).$$

The common factor of positive degree $(\mathcal{U}_n + (q-1)\mathcal{V}_n)$ is a consequence of the ‘‘horizontal’’ edge that is connected to the remainder of the generating graph Γ at a single vertex. When taking the ratios $y_n = \mathcal{U}_n/\mathcal{V}_n$ we lose track of these common factors resulting in the drop of generic degree:

$$R(q, y) = (q, r_q(y)) \quad \text{where} \quad r_q(y) = \frac{y^2 + q - 1}{2y + q - 2},$$

which has degree two even though Γ has three edges. This drop in generic degree results in $(R^n)^*\mathcal{S}_0 < (Z_n(q, y))$ for the hierarchical lattice generated by the Tripod.

This phenomenon can be avoided if the generating graph is 2-connected and the proof is exactly the same as for the DHL. We summarize:

Proposition 5.6. *Let $\{\Gamma_n\}_{n=0}^\infty$ be the hierarchical lattice generated by $\Gamma = (V, E)$. If Γ is 2-connected, then the associated renormalization mapping $R(q, y) = (q, r_q(y))$ has generic degree $|E|$ and satisfies*

$$\mathcal{S}_n = (R^n)^*(\mathcal{S}_0),$$

where $\mathcal{S}_n = (Z_n(q, y))$ and $\mathcal{S}_0 = (q(y + q - 1))$. Moreover, r_q is defined over \mathbb{Q} .

Several concrete examples are presented in Section 8.

6. PROOF OF THEOREM A

Let $\{\Gamma_n\}_{n=1}^\infty$ be a hierarchical lattice whose generating graph $\Gamma = (V, E)$ is 2-connected. Denote its Migdal-Kadanoff renormalization mapping by $R(q, y) = (q, r_q(y))$. Since Γ is 2-connected, Proposition 5.6 implies that the chromatic zeros for Γ_n (omitting the simple zero at $q = 0$) are given by $\tilde{\mathcal{C}}_n = (R^n)^*(y + q - 1 = 0) \cap (y = 0)$. Therefore, in the language of currents,

$$\tilde{\mu}_n := \frac{1}{|V_n|} \sum_{\substack{q \in \mathbb{C} \setminus \{0\} \\ \mathcal{P}_{\Gamma_n}(q) = 0}} \delta_q = (\pi_1)_* \left(\frac{1}{|V_n|} (R^n)^*[y + q - 1 = 0] \wedge [y = 0] \right),$$

where the zeros of $\mathcal{P}_{\Gamma_n}(q)$ are counted with multiplicities, as always. Since $\tilde{\mu}_n$ and μ_n (see (1.1)) differ by $1/|V_n|$ times a Dirac measure at $q = 0$, it suffices to prove that the sequence $\tilde{\mu}_n$ converges. Moreover, Proposition 5.4 allows us to replace the normalizing factor of $|V_n|$ with $|E_n|$. Therefore, it suffices to verify that $R = (q, r_q(y))$ and the marked points $a(q) = 0$ and $b(q) = 1 - q$ satisfy the hypotheses of Theorem C'.

By Proposition 5.6, the algebraic family r_q is defined over \mathbb{Q} . Hypotheses (i) and (ii) on the marked points will be verified in Propositions 6.1 and 6.2 below.

Proposition 6.1. *There are no iterates $n \geq 0$ satisfying $r_q^n(0) \equiv 1 - q$.*

Proof. Away from the finitely many points in V_{deg} , the chromatic zeros of Γ_n are solutions in q to $r_q^n(0) = 1 - q$. If there is some iterate $n \geq 0$ such that $r_q^n(0) \equiv 1 - q$, this will imply that Γ_n has infinitely many chromatic zeros, which is impossible because $\deg(\mathcal{P}_{\Gamma_n}) = |V_n|$. \square

Proposition 6.2. *The marked point $b(q) = 1 - q$ is not persistently exceptional for r_q .*

Proof. Assume by contradiction that the marked point $b(q) = 1 - q$ is persistently exceptional. Taking the second iterate, we can suppose it is a fixed point. Then by (5.8), the pullback of the divisor $(y = 1 - q)$ by the map R^2 satisfies

$$(\tilde{Z}_2(q, y)) = (R^2)^*(y = 1 - q) = |E|^2(y = 1 - q),$$

which implies that the partition function, $\tilde{Z}_2(q, y) = (y + q - 1)^{|E|^2}$, for Γ_2 is reducible. However, since the generating graph Γ is assumed to be 2-connected, Γ_2 is also 2-connected, so $\tilde{Z}_2(q, y)$ is irreducible by Proposition 5.1, which is a contradiction. \square

\square (Theorem A)

7. PROOF OF THEOREM B

This is the only section of the paper where we will use marked points that are critical. We will use the following famous result which appears as Corollary 1.6 from [53]:

Theorem 7.1 (McMullen [53]). *For any holomorphic family of rational maps over the unit disk Δ , the bifurcation locus $B(f) \subset \Delta$ is either empty or has Hausdorff dimension two.*

Although the above theorem states that the bifurcation locus, which is the union of the active loci of all the critical points, has Hausdorff dimension two (unless it is empty), one can check that the proof still applies to each individual marked critical point $c(\lambda)$, as long as it bifurcates. Indeed the proof of Theorem 7.1 consists of using activity of the marked point to construct a holomorphically-varying family of polynomial-like mappings, whose critical point is the marked one $c(\lambda)$. Associated to this family is the space of parameters λ for which the orbit of the critical point remains bounded (in the polynomial-like mapping). McMullen shows that this set is a quasiconformal image of the Mandelbrot set (or a higher degree generalization). The boundary of this “baby Mandelbrot set” has Hausdorff Dimension two [63], and, by definition, the marked point $c(\lambda)$ is active at such points.

Proof of Theorem B. Using an analogous proof to that of Proposition 5.5 one finds that the renormalization mapping for the k -fold DHL is

$$(7.1) \quad r_q(y) = \left(\frac{y^2 + q - 1}{2y + q - 2} \right)^k.$$

For this family of mappings we have $V_{\text{deg}} = \{0, \infty\}$. Since the generating graph is 2-connected Theorem A implies that the limiting measure of chromatic zeros exists for the k -fold DHL and the proof of Theorem A implies that on $\mathbb{C} \setminus V_{\text{deg}}$ it coincides with the activity measure for the marked point $a(q) \equiv 0$.

One can check that $c(q) := \sqrt{1-q}$ is a critical point for $r_q(y)$, which we can suppose is marked after replacing \mathbb{C} with a branched cover. A direct calculation shows that $r_q(c(q)) \equiv 0 \equiv a(q)$. Therefore, the activity loci of marked point $a(q)$ (and hence of our limiting measure of μ of chromatic zeros) coincides with the activity locus for the critical point $c(q)$.

It remains to check that these are non-empty and not entirely contained in the set of parameters for which the degree of $r_q(y)$ drops. Drop in degree of $r_q(y)$ corresponds to values of q for which numerator and denominator of $r_q(y)$ have a common zero. One can check that this only happens when $q = 0$.

One can also check by direct calculation that $y = 1$ and $y = \infty$ are both persistently superattracting fixed points for $r_q(y)$. One has that $r_q(0)$ is a degree $k \geq 2$ rational function of q and that $r_0(0) = (1/2)^k$. Therefore, there is some parameter $q_1 \neq 0$ for which $r_{q_1}(0) = 1$. On some open neighborhood of this parameter one has $r_q^n(0) \rightarrow 1$. Meanwhile, one has $r_2(0) = \infty$ and so there is an open neighborhood of $q = 2$ on which $r_q^n(0) \rightarrow \infty$. This implies that the marked point $a(q)$ cannot be passive on the connected set $\mathbb{C} \setminus \{0\}$ by the identity theorem.

Theorem 7.1 and the paragraph following it then give that the activity locus of $c(q)$ has Hausdorff Dimension equal to two. □

In the special case that $k = 2$, Laura DeMarco and Niki Myrto Mavraki observed the following:

Proposition 7.2. *Let $r_q(y)$ be the renormalization mapping for the 2-fold DHL given by (7.1) with $k = 2$. Then, $B(r_q) = \text{supp}(T_a)$.*

Proof. The critical points of the map $r_q(y)$ are $y = 1, 1 - q, \infty, \frac{2-q}{2}, \pm\sqrt{1-q}$. Three of them behave similarly: $y = 1$ and $y = \infty$ are superattracting fixed points, while $y = \frac{2-q}{2}$ is just a preimage of ∞ . Meanwhile, note that $\pm\sqrt{1-q}$ are both preimages of $y = 0$, so the bifurcation locus of the family is the union of the activity loci of the two marked points $y = 0, y = 1 - q$.

The map r_q commutes with

$$C_q(y) := \left(\frac{y + q - 1}{y - 1} \right)^2,$$

which satisfies $C_q(1 - q) = 0$ and $C_q(0) = r_q(1 - q) = (1 - q)^2$. Therefore, the activity loci of $y = 0$ and $y = 1 - q$ coincide, and it follows that the bifurcation locus of the family is equal to the activity locus of the non-critical marked point $y = 0$. \square

8. EXAMPLES

We conclude the paper with a discussion of the chromatic zeros associated with the hierarchical lattices generated by the graphs shown in Figure 2. We also provide a more detailed explanation of Figures 3 and 4.

8.1. Linear Chain. In this case, each graph Γ_n is a tree so that $\mathcal{P}_{\Gamma_n}(q) = q(q - 1)^{|V_n|}$. See, for example, [41]. Therefore, the limiting measure of chromatic zeros for the linear chain is a Dirac measure at $q = 1$.

Meanwhile, even though the generating graph is not 2-connected, the statement of Proposition 5.6 still applies with

$$r_q(y) = \frac{y^2 + q - 1}{2y + q - 2},$$

which is the same formula as for the k -fold DHL, except with exponent $k = 1$. One can check that r_q has $y = 1 - q$ as a persistent exceptional point, so that Theorem C' does not apply. Indeed, the activity locus for marked point $a(q) \equiv 0$ is the round circle $|q - 1/2| = 1/2$ while for each $n \geq 0$ the sequence of wedge products (1.9) is just the Dirac measure at $q = 1$.

8.2. k -fold DHL, where $k \geq 2$. In the proofs of Theorems A and B we already saw that the limiting measure μ of chromatic zeros exists for this lattice and that outside of $V_{\text{deg}} = \{q = 0, \infty\}$ it coincides with the activity measure for the marked point $a(q) \equiv 0$. Here, we will explain the claim the activity locus, and hence $\text{supp}(\mu)$, is the boundary between any two of the colors (blue, black, and white) in Figure 3.

The Migdal-Kadanoff renormalization mapping is given by (7.1). One can check that this mapping has $y = 1$ and $y = \infty$ as persistent superattracting fixed points. In Figure 3, the set q for which $r_q^n(0) \rightarrow 1$ is shown in white (i.e. not colored) and the set of q for which $r_q^n(0) \rightarrow \infty$ is shown in blue. Each of these corresponds to passive behavior for the marked point $a(q) \equiv 0$. Meanwhile, if there is some neighborhood N of $q_0 \in \mathbb{C} \setminus V_{\text{deg}}$ on which $r_q^n(0)$ does not have one of these two behaviors, then Montel's Theorem implies that $a(q)$ is also passive on N . Such points are colored black.

Conversely, if q_0 is on the boundary of two colors (blue, black, and white), then q_0 is an active parameter for the marked point $a(q)$. Indeed, if N is any neighborhood of q_0 then along any subsequence n_k we have that $r_q^{n_k}(0)$ will converge uniformly to 1 or ∞ the parts of N that are white or blue, respectively, and $r_q^{n_k}(0)$ will remain bounded away from 1 and ∞ on the black. Therefore, $r_q^n(0)$ cannot form a normal family on N .

8.3. **Triangles.** As the generating graph is 2-connected, Proposition 5.6 applies and one can compute that the Migdal-Kadanoff renormalization mapping is:

$$(8.1) \quad r_q(y) = y \left(\frac{y^2 + q - 1}{2y + q - 2} \right).$$

It is the same as for the linear chain, but with an extra factor of y . Notice that for this family of mappings $V_{\text{deg}} = \{q = 0, 2, \infty\}$. The proof of Theorem A applies and one concludes that on $\mathbb{C} \setminus V_{\text{deg}}$ the limiting measure of chromatic zeros μ coincides with the activity measure of the marked point $a(q) \equiv 0$. However, a curious thing happens: for every iterate n we have $r_q^n(0) = 0$ so that the marked point a is globally passive on $\mathbb{C} \setminus V_{\text{deg}}$. Therefore, μ is supported on V_{deg} . This illustrates why it was important to use Theorem C' (instead of just Theorem C) when proving Theorem A. Working inductively with (8.1) one can directly prove that μ is the Dirac measure at $q = 2$.

8.4. **Tripods.** As explained in Section 5.5, the Migdal-Kadanoff renormalization mapping for the tripod coincides with that of the linear chain, due to a common factor appearing in the numerator and denominator. This drop in degree makes r_q not useful for studying the chromatic zeros on the hierarchical lattice generated by the tripod. However, since each of the graphs Γ_n in this hierarchical lattice is a tree, the limiting measure of chromatic zeros exists and is a Dirac measure at $q = 1$, by the same reasoning as for the linear chain.

8.5. **Split Diamonds.** The split diamond is 2-connected and Theorem A implies that there is a limiting measure of chromatic zeros μ for the associated lattice. One can check that the Migdal-Kadanoff renormalization mapping for this generating graph is

$$(8.2) \quad r_q(y) = \frac{y^5 + 2(q-1)y^2 + (q-1)y + (q-1)(q-2)}{2y^3 + 2y^2 + 5(q-2)y + (q-2)(q-3)}.$$

As for the k -fold DHL, one can check that r_q has $y = 1$ and $y = \infty$ as persistent superattracting fixed points. Therefore, one can use the the same coloring scheme as for the k -fold DHL to make computer images of the activity locus of $a(q) \equiv 0$, and hence of $\text{supp}(\mu)$; See Figure 4. With some explicit calculations, one can rigorously verify that each of the three behaviors (white, blue, and black) actually occurs for $q \notin V_{\text{deg}}$.

REFERENCES

- [1] Fractalstream dynamical systems software. Written by Matthew Noonan. <https://code.google.com/archive/p/fractalstream/>.
- [2] David Andelman and A. Nihat Berker. Scale-invariant quenched disorder and its stability criterion at random critical points. *Phys. Rev. B*, 29:2630–2635, Mar 1984.
- [3] Magnus Aspenberg and Michael Yampolsky. Mating non-renormalizable quadratic polynomials. *Comm. Math. Phys.*, 287(1):1–40, 2009.
- [4] Rodney J. Baxter. *Exactly solved models in statistical mechanics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London, 1989. Reprint of the 1982 original.
- [5] Laura Beaudin, Joanna Ellis-Monaghan, Greta Pangborn, and Robert Shrock. A little statistical mechanics for the graph theorist. *Discrete Math.*, 310(13-14):2037–2053, 2010.
- [6] S. Beraha, J. Kahane, and N. J. Weiss. Limits of zeroes of recursively defined polynomials. *Proc. Nat. Acad. Sci. U.S.A.*, 72(11):4209, 1975.
- [7] A N Berker and S Ostlund. Renormalisation-group calculations of finite systems: order parameter and specific heat for epitaxial ordering. *Journal of Physics C: Solid State Physics*, 12(22):4961–4975, nov 1979.

- [8] François Berteloot. Bifurcation currents in holomorphic families of rational maps. In *Pluripotential theory*, volume 2075 of *Lecture Notes in Math.*, pages 1–93. Springer, Heidelberg, 2013.
- [9] N. L. Biggs, R. M. Damerell, and D. A. Sands. Recursive families of graphs. *J. Combinatorial Theory Ser. B*, 12:123–131, 1972.
- [10] Norman Biggs and Robert Shrock. $T = 0$ partition functions for Potts antiferromagnets on square lattice strips with (twisted) periodic boundary conditions. *J. Phys. A*, 32(46):L489–L493, 1999.
- [11] G. D. Birkhoff and D. C. Lewis. Chromatic polynomials. *Trans. Amer. Math. Soc.*, 60:355–451, 1946.
- [12] George D. Birkhoff. A determinant formula for the number of ways of coloring a map. *Ann. of Math. (2)*, 14(1-4):42–46, 1912/13.
- [13] P. Bleher, M. Lyubich, and R. Roeder. Lee–Yang zeros for the DHL and 2D rational dynamics, I. Foliation of the physical cylinder. *J. Math. Pures Appl. (9)*, 107(5):491–590, 2017.
- [14] P. M. Bleher and M. Yu. Lyubich. Julia sets and complex singularities in hierarchical Ising models. *Comm. Math. Phys.*, 141(3):453–474, 1991.
- [15] P. M. Bleher and E. Žalys. Existence of long-range order in the Migdal recursion equations. *Comm. Math. Phys.*, 67(1):17–42, 1979.
- [16] P. M. Bleher and E. Žalys. Asymptotics of the susceptibility for the Ising model on the hierarchical lattices. *Comm. Math. Phys.*, 120(3):409–436, 1989.
- [17] Pavel Bleher, Mikhail Lyubich, and Roland Roeder. Lee–Yang–Fisher zeros for the DHL and 2D rational dynamics, II. Global pluripotential interpretation. *J. Geom. Anal.*, 30(1):777–833, 2020.
- [18] P. M. Bleher and È. Zhalis. Limit Gibbs distributions for the Ising model on hierarchical lattices. *Litovsk. Mat. Sb.*, 28(2):252–268, 1988.
- [19] Gregory S. Call and Joseph H. Silverman. Canonical heights on varieties with morphisms. *Compositio Math.*, 89(2):163–205, 1993.
- [20] Shu-Chiuan Chang, Roland K. W. Roeder, and Robert Shrock. q -plane zeros of the Potts partition function on diamond hierarchical graphs. *J. Math. Phys.*, 61(7):073301, 32, 2020.
- [21] Shu-Chiuan Chang and Robert Shrock. Tutte polynomials and related asymptotic limiting functions for recursive families of graphs. *Adv. in Appl. Math.*, 32(1-2):44–87, 2004. Special issue on the Tutte polynomial.
- [22] Shu-Chiuan Chang and Robert Shrock. Zeros of the Potts model partition function on Sierpinski graphs. *Phys. Lett. A*, 377(9):671–675, 2013.
- [23] Ivan Chio, Caleb He, Anthony L. Ji, and Roland K. W. Roeder. Limiting measure of Lee–Yang zeros for the Cayley tree. *Comm. Math. Phys.*, 370(3):925–957, 2019.
- [24] Jean-Pierre Demailly. Monge–Ampère operators, Lelong numbers and intersection theory. In *Complex analysis and geometry*, Univ. Ser. Math., pages 115–193. Plenum, New York, 1993.
- [25] Laura DeMarco. Dynamics of rational maps: a current on the bifurcation locus. *Math. Res. Lett.*, 8(1-2):57–66, 2001.
- [26] Laura DeMarco. Dynamics of rational maps: Lyapunov exponents, bifurcations, and capacity. *Math. Ann.*, 326(1):43–73, 2003.
- [27] Laura DeMarco. Bifurcations, intersections, and heights. *Algebra Number Theory*, 10(5):1031–1056, 2016.
- [28] Laura DeMarco. Dynamical moduli spaces and elliptic curves. *Ann. Fac. Sci. Toulouse Math. (6)*, 27(2):389–420, 2018.
- [29] B. Derrida, L. de Seze, C. Itzykson, and and. Fractal structure of zeros in hierarchical models. *J. Statist. Phys.*, 33(3):559–569, 1983.

- [30] Tien-Cuong Dinh and Nessim Sibony. Dynamics in several complex variables: endomorphisms of projective spaces and polynomial-like mappings. In *Holomorphic dynamical systems*, volume 1998 of *Lecture Notes in Math.*, pages 165–294. Springer, Berlin, 2010.
- [31] F. M. Dong, K. M. Koh, and K. L. Teo. *Chromatic polynomials and chromaticity of graphs*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005.
- [32] Romain Dujardin. Bifurcation currents and equidistribution in parameter space. In *Frontiers in complex dynamics*, volume 51 of *Princeton Math. Ser.*, pages 515–566. Princeton Univ. Press, Princeton, NJ, 2014.
- [33] Romain Dujardin and Charles Favre. Distribution of rational maps with a preperiodic critical point. *Amer. J. Math.*, 130(4):979–1032, 2008.
- [34] Charles Favre and Thomas Gauthier. The arithmetic of polynomial dynamical pairs. Preprint: <https://arxiv.org/abs/2004.13801>.
- [35] C. M. Fortuin and P. W. Kasteleyn. On the random-cluster model. I. Introduction and relation to other models. *Physica*, 57:536–564, 1972.
- [36] Alexandre Freire, Artur Lopez, and Ricardo Mañé. An invariant measure for rational maps. *Bol. Soc. Brasil. Mat.*, 14(1):45–62, 1983.
- [37] Thomas Gauthier. Dynamical pairs with an absolutely continuous bifurcation measure. Preprint: <https://arxiv.org/abs/1810.02385>.
- [38] Thomas Gauthier and Gabriel Vigny. Distribution of points with prescribed derivative in polynomial dynamics. *Riv. Math. Univ. Parma (N.S.)*, 8(2):247–270, 2017.
- [39] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*. Wiley-Interscience [John Wiley & Sons], New York, 1978. Pure and Applied Mathematics.
- [40] Robert B. Griffiths and Miron Kaufman. Spin systems on hierarchical lattices. introduction and thermodynamic limit. *Phys. Rev. B*, 26:5022–5032, Nov 1982.
- [41] Frank Harary. *Graph theory*. Addison-Wesley Publishing Co., Reading, Mass.-Menlo Park, Calif.-London, 1969.
- [42] Lars Hörmander. *The analysis of linear partial differential operators. I*. Springer Study Edition. Springer-Verlag, Berlin, second edition, 1990. Distribution theory and Fourier analysis.
- [43] Ferenc Iglói and Loïc Turban. Disordered potts model on the diamond hierarchical lattice: Numerically exact treatment in the large- q limit. *Phys. Rev. B*, 80:134201, Oct 2009.
- [44] Bill Jackson. Zeros of chromatic and flow polynomials of graphs. *J. Geom.*, 76(1-2):95–109, 2003. Combinatorics, 2002 (Maratea).
- [45] Bill Jackson, Aldo Procacci, and Alan D. Sokal. Complex zero-free regions at large $|q|$ for multivariate Tutte polynomials (alias Potts-model partition functions) with general complex edge weights. *J. Combin. Theory Ser. B*, 103(1):21–45, 2013.
- [46] Luo Jiaqi. Combinatorics and holomorphic dynamics: Captures, matings, newton’s method, 1995. Thesis, Cornell University.
- [47] L. Kadanoff. Notes on migdal’s recursion formulas. *Annals of Physics*, 100:359–394, 1976.
- [48] Wolfgang Kinzel and Eytan Domany. Critical properties of random potts models. *Phys. Rev. B*, 23:3421–3434, Apr 1981.
- [49] M. Ju. Ljubich. Entropy properties of rational endomorphisms of the Riemann sphere. *Ergodic Theory Dynam. Systems*, 3(3):351–385, 1983.
- [50] M. Yu. Lyubich. Some typical properties of the dynamics of rational mappings. *Uspekhi Mat. Nauk*, 38(5(233)):197–198, 1983.
- [51] M. Yu. Lyubich. Investigation of the stability of the dynamics of rational functions. *Teor. Funktsii Funktsional. Anal. i Prilozhen.*, (42):72–91, 1984. Translated in *Selecta Math. Soviet.* **9** (1990), no. 1, 69–90.

- [52] Colin Maclachlan and Alan W. Reid. *The arithmetic of hyperbolic 3-manifolds*, volume 219 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2003.
- [53] Curtis T. McMullen. The Mandelbrot set is universal. In *The Mandelbrot set, theme and variations*, volume 274 of *London Math. Soc. Lecture Note Ser.*, pages 1–17. Cambridge Univ. Press, Cambridge, 2000.
- [54] C. Merino, A. de Mier, and M. Noy. Irreducibility of the Tutte polynomial of a connected matroid. *J. Combin. Theory Ser. B*, 83(2):298–304, 2001.
- [55] A. A. Migdal. Phase transitions in gauge and spin-lattice systems. *JETP*, 69:1457–1467, 1975.
- [56] A. A. Migdal. Recurrence equations in gauge field theory. *JETP*, 69:810–822, 1975.
- [57] Yūsuke Okuyama. Equidistribution of rational functions having a superattracting periodic point towards the activity current and the bifurcation current. *Conform. Geom. Dyn.*, 18:217–228, 2014.
- [58] H. Peters and G. Regts. Location of zeros for the partition function of the Ising model on bounded degree graphs. Preprint: <https://arxiv.org/abs/1810.01699>.
- [59] H. Peters and G. Regts. On a conjecture of sokal concerning roots of the independence polynomial. *Michigan Mathematical Journal*, 68(1):33–55, 2019.
- [60] G. F. Royle and A. Sokal. Linear bound in terms of maxmaxflow for the chromatic roots of series-parallel graphs. arXiv version, see <https://arxiv.org/abs/1307.1721>.
- [61] Jesús Salas and Alan D. Sokal. Transfer matrices and partition-function zeros for antiferromagnetic Potts models. I. General theory and square-lattice chromatic polynomial. *J. Statist. Phys.*, 104(3-4):609–699, 2001.
- [62] Jesús Salas and Alan D. Sokal. Transfer matrices and partition-function zeros for antiferromagnetic Potts models VI. Square lattice with extra-vertex boundary conditions. *J. Stat. Phys.*, 144(5):1028–1122, 2011.
- [63] Mitsuhiro Shishikura. The boundary of the Mandelbrot set has Hausdorff dimension two. *Astérisque*, (222):7, 389–405, 1994. Complex analytic methods in dynamical systems (Rio de Janeiro, 1992).
- [64] Robert Shrock. Chromatic polynomials and their zeros and asymptotic limits for families of graphs. *Discrete Math.*, 231(1-3):421–446, 2001. 17th British Combinatorial Conference (Canterbury, 1999).
- [65] Robert Shrock and Shan-Ho Tsai. Asymptotic limits and zeros of chromatic polynomials and ground-state entropy of potts antiferromagnets. *Phys. Rev. E*, 55:5165–5178, May 1997.
- [66] Robert Shrock and Shan-Ho Tsai. Ground-state entropy of the Potts antiferromagnet on cyclic strip graphs. *J. Phys. A*, 32(17):L195–L200, 1999.
- [67] Nessim Sibony. Dynamique des applications rationnelles de \mathbf{P}^k . In *Dynamique et géométrie complexes (Lyon, 1997)*, volume 8 of *Panor. Synthèses*, pages ix–x, xi–xii, 97–185. Soc. Math. France, Paris, 1999.
- [68] Joseph H. Silverman. Integer points, Diophantine approximation, and iteration of rational maps. *Duke Math. J.*, 71(3):793–829, 1993.
- [69] Joseph H. Silverman. *The arithmetic of dynamical systems*, volume 241 of *Graduate Texts in Mathematics*. Springer, New York, 2007.
- [70] Alan D. Sokal. The multivariate Tutte polynomial (alias Potts model) for graphs and matroids. In *Surveys in combinatorics 2005*, volume 327 of *London Math. Soc. Lecture Note Ser.*, pages 173–226. Cambridge Univ. Press, Cambridge, 2005.
- [71] XiaoGuang Wang, WeiYuan Qiu, YongCheng Yin, JianYong Qiao, and JunYang Gao. Connectivity of the Mandelbrot set for the family of renormalization transformations. *Sci. China Math.*, 53(3):849–862, 2010.

- [72] D. J. A. Welsh. *Matroid theory*. Academic Press [Harcourt Brace Jovanovich, Publishers], London-New York, 1976. L. M. S. Monographs, No. 8.
- [73] F. Y. Wu. The potts model. *Rev. Mod. Phys.*, 54:235–268, Jan 1982.
- [74] Fei Yang and Jinsong Zeng. On the dynamics of a family of generated renormalization transformations. *J. Math. Anal. Appl.*, 413(1):361–377, 2014.

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