DYNAMICS OF GROUPS OF AUTOMORPHISMS OF CHARACTER VARIETIES AND FATOU/JULIA DECOMPOSITION FOR PAINLEVÉ 6

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Abstract. We study the dynamics of the group of holomorphic automorphisms of the affine cubic surfaces

$$S_{A,B,C,D} = \{(x,y,z) \in \mathbb{C}^3 \,:\, x^2+y^2+z^2+xyz = Ax+By+Cz+D\},$$

where A, B, C, and D are complex parameters. We focus on a finite index subgroup $\mathcal{G}_{A,B,C,D} < \mathrm{Aut}(S_{A,B,C,D})$ whose action not only describes the dynamics of Painlevé 6 differential equations but also arises naturally in the context of character varieties. We define the Julia and Fatou sets of this group action and prove that there is a dense orbit in the Julia set. In order to show that the Julia set is "large" we consider a second dichotomy, between locally discrete and locally non-discrete dynamics. For an open set in parameter space, $\mathcal{P} \subset \mathbb{C}^4$, we show that there simultaneously exist an open set in $S_{A,B,C,D}$ on which $\mathcal{G}_{A,B,C,D}$ acts locally discretely and a second open set in $S_{A,B,C,D}$ on which $\mathcal{G}_{A,B,C,D}$ acts locally non-discretely. Their common boundary contains an invariant set $\mathscr{B}_{A,B,C,D}$ of topological dimension 3. After removing a countable union of real-algebraic hypersurfaces from $\mathcal{P}$ we show that $\mathcal{G}_{A,B,C,D}$ simultaneously exhibits a non-empty Fatou set and also a Julia set having non-trivial interior. The open set $\mathcal{P}$ contains a natural family of parameters previously studied by Dubrovin-Mazzocco.

The interplay between the Fatou/Julia dichotomy and the locally discrete/non-discrete dichotomy plays a major theme in this paper and seems bound to play an important role in further dynamical studies of holomorphic automorphism groups.

1. Introduction

1.1. **Setting.** Let A, B, C, and D be fixed complex parameters and consider the affine cubic surface

$$S_{A,B,C,D} = \{(x,y,z) \in \mathbb{C}^3 \,:\, x^2+y^2+z^2+xyz = Ax+By+Cz+D\}.$$

It is known that the corresponding family of projective surfaces contains all smooth cubic surfaces (see [31] or Section 4.2 for further detail). Note that every line parallel to the x-axis intersects the surface $S_{A,B,C,D}$ at two points (counted with multiplicity) and one can therefore define an involution $s_x : S_{A,B,C,D} \to S_{A,B,C,D}$ that switches them:

$$s_x\,(x,y,z) = (-x - yz + A, y, z)\,. \tag{1}$$

Two further involutions $s_y : S_{A,B,C,D} \to S_{A,B,C,D}$ and $s_z : S_{A,B,C,D} \to S_{A,B,C,D}$ are defined analogously by means of lines parallel to the y-axis and z-axis, respectively.

Consider the group

$$\mathcal{G}^{\pm} \equiv \mathcal{G}^{\pm}_{A,B,C,D} = \langle s_x, s_y, s_z \rangle \leq \mathrm{Aut}(S_{A,B,C,D}), \tag{2}$$

where $\mathrm{Aut}(S_{A,B,C,D})$ denotes the group of all (algebraic) holomorphic diffeomorphisms of $S_{A,B,C,D}$. For a generic choice of parameters one has $\mathcal{G}^{\pm}_{A,B,C,D} = \mathrm{Aut}(S_{A,B,C,D})$ and, in general, $\mathcal{G}^{\pm}_{A,B,C,D}$ is a subgroup of $\mathrm{Aut}(S_{A,B,C,D})$ of index at most 24; See [13, Theorem 3.1] and also [23].

Consider also the index two subgroup

$$\mathcal{G} \equiv \mathcal{G}_{A,B,C,D} = \langle g_x, g_y, g_z \rangle < \mathcal{G}^{\pm}.$$

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where $g_x = s_z \circ s_y$, $g_y = s_x \circ s_z$, and $g_z = s_y \circ s_x$.

The dynamics of the action of groups $\mathcal{G}^{\pm}_{A,B,C,D}$ and $\mathcal{G}_{A,B,C,D}$ on $S_{A,B,C,D}$ and their individual elements have several deep connections, including to the dynamics of mapping class groups on character varieties, to the monodromy of the Painlevé 6 differential equation, and to the aperiodic Schrödinger equation; see, for example, [9] for a nice description of these connections. For this reason there is an extensive body of previous works on this dynamical system. In Section 2 we will provide a sample, focusing on those that we consider most closely related to the present paper. We will also provide further details about the connections with dynamics on character varieties and with the Painlevé 6 differential equation in Section 3.

Main goal of our work: Our paper is devoted to questions about the "pointwise complex dynamics of the whole group", i.e. to the orbits of individual points, their closures, and more generally to the nature of subsets of the complex surface $S_{A,B,C,D}$ that are invariant under $\mathcal{G}^{\pm}$ and $\mathcal{G}$.

We focus on the "chaotic" part of the dynamics (i.e. the dynamics on the Julia set $\mathcal{J}_{A,B,C,D}$, as defined in Section 1.5) which is therefore complementary to the extensive work previously done by Bowditch, Tan and his collaborators, and others about the domains on which the dynamics is properly discontinuous. Indeed, our work can be seen as an effort to answer the questions posed by Bowditch immediately below Corollary 5.6 on p. 728 of [7]. Let us point out that the complex dynamics on the $SL(2,\mathbb{C})$ character varieties (i.e. of $\mathcal{G}_{A,B,C,D}$ on $S_{A,B,C,D}$) is described as being "extremely mysterious and non-trivial" by Goldman [30, p. 461] and by Tan, Wong, and Zhang [63, p. 762].

It is important to remark that we are considering the action of $\mathcal{G}_{A,B,C,D}$ on the (non-compact) affine surface $S_{A,B,C,D} \subset \mathbb{C}^3$. Indeed the elements of $\mathcal{G}_{A,B,C,D}$ extend only as birational mappings of the compactification $\overline{S}_{A,B,C,D} \subset \mathbb{CP}^3$, with both indeterminate and super-attracting/collapsing behavior at infinity (see Section 9). This lack of a good compact domain for $\mathcal{G}_{A,B,C,D}$ makes several aspects of this dynamical system rather challenging. It also seems to rule out ergodic-theoretic methods like those recently used by Cantat and Dujardin [12] to study automorphism groups of compact surfaces. This will be discussed further in Section 2.1.

1.2. **Some "preferred" parameters.** Throughout the paper we will refer to the following specific parameters and parametric families.

Markoff Parameters: $(A,B,C,D) = (0,0,0,0)$, as discussed in [7] and [59].

Picard Parameters: $(A,B,C,D) = (0,0,0,4)$. For these parameters, Picard proved that the Painlevé equation has explicit first integrals and countably many algebraic solutions. This is related to the curious fact that the action of $\mathcal{G}_{0,0,0,4}$ is semi-conjugate to an action on $(\mathbb{C}\setminus\{0\})^2$ by monomial mappings. In particular, for these parameters everything can be computed rather explicitly.

Punctured Torus Parameters: $(A,B,C,D) = (0,0,0,D)$ for any $D \in \mathbb{C}$. These parameters correspond to dynamics on the character variety of the once punctured torus; see, e.g. [9, Sec. 1.1]. For real D and the corresponding real surfaces, this is the family studied by Goldman [30].

Dubrovin-Mazzocco Parameters: This is a real 1-parameter family studied by Dubrovin and Mazzocco [22] which seems to play a significant role in several problems related to Mathematical-Physics and, in particular, on the study of Frobenius manifolds. In our notations, the Dubrovin-Mazzocco parameters correspond to

$$A(a) = B(a) = C(a) = 2a+4, \quad \text{and} \quad D(a) = -(a^2+8a+8) \tag{3}$$

for $a \in (-2,2)$.

Notice that both the Markoff and Picard parameters are included within the Punctured Torus Parameters. Meanwhile, the Picard parameters are in the closure of the Dubrovin-Mazzocco parameters, corresponding to $a=-2$, but the Markoff parameters are not.

1.3. **Two relevant dynamical dichotomies.** With the goal of producing interesting invariant sets and finding points with complicated orbit closures we introduce two dynamically invariant dichotomies.

A consequence of our results is that, in general, the action of $\mathcal{G}$ (or of $\mathcal{G}^{\pm}$) is genuinely non-linear, for example, it does not preserve any rigid geometric structure in the sense of Gromov [32]; see Remark 8.8. Whereas this was mostly expected, this action still appears to share some basic properties/issues with actions of countable subgroups of finite dimensional Lie groups. This typically happens on some (proper) open subsets of the surface $S_{A,B,C,D}$ and, for this reason, the notions of *locally discrete* vs. *locally non-discrete dynamics* of $\mathcal{G}$ will come in handy. Meanwhile, to deal with the non-linear nature of the global dynamics we will adapt the *Fatou/Julia theory* to the group $\mathcal{G}$. The core of this paper lies in the interplay between these two dichotomies.

1.4. **Locally non-discrete/discrete dichotomy.** Let M be a (possibly open) connected complex manifold and consider a group G of holomorphic diffeomorphisms of M. The group G is said to be *locally non-discrete* on an open set $U \subset M$ if there is a sequence of maps $\{f_n\}_{n=0}^{\infty} \in G$ satisfying the following conditions (see for example [55]):

(1) For every n, f_n is different from the identity.
(2) The sequence of maps f_n converges uniformly to the identity on compact subsets of U.

If there is no such sequence f_n on U we say that G is *locally discrete* on U.

Remark that for an action by a finite dimensional Lie group, local non-discreteness on some open set implies that the corresponding sequence of elements converges to the identity on all of M, i.e. that the action is globally non-discrete. However, in our context the non-linearity of the mappings allow for local non-discreteness to occur on a proper open subset $U \subset M$ in such a way that it does not extend beyond U.

For any choice of parameters (A, B, C, D) let

$$\mathcal{N}_{A,B,C,D} = \{p \in S_{A,B,C,D} \,:\, \mathcal{G}_{A,B,C,D} \text{ is locally non-discrete on an open neighborhood } U \text{ of } p\},$$

and let $\mathcal{D}_{A,B,C,D} = S_{A,B,C,D} \setminus \mathcal{N}_{A,B,C,D}$. We will refer to $\mathcal{N}_{A,B,C,D}$ as the "locally non-discrete locus" and to $\mathcal{D}_{A,B,C,D}$ as the "locally discrete locus". By definition, $\mathcal{N}_{A,B,C,D}$ is open, $\mathcal{D}_{A,B,C,D}$ is closed, and each is invariant under $\mathcal{G}_{A,B,C,D}$.

Note that $\mathcal{N}_{A,B,C,D}$ can be empty for certain parameter values; indeed it is for the Picard Parameters (Theorem D(ii), below). We do not know if $\mathcal{D}_{A,B,C,D}$ can be empty for any parameter values.

1.5. **Fatou/Julia dichotomy.** For any point $p \in S_{A,B,C,D}$ we denote the orbit of p under $\mathcal{G}$ by

$$\mathcal{G}(p) = \{\, \gamma(p) \;:\; \gamma \in \mathcal{G}\,\}.$$

The *Fatou set* of the group action $\mathcal{G}$ is defined as

$$\mathcal{F}_{A,B,C,D} = \{p \in S_{A,B,C,D} \,:\, \mathcal{G} \text{ forms a normal family in some open neighborhood of } p\}.$$

Naturally the condition of being a normal family means that every sequence of maps as indicated must have a convergent subsequence (for the topology of uniform convergence on compact subsets – compact-open topology). However, since $S_{A,B,C,D}$ is *open*, sequences of maps avoiding compact sets are expected to arise as well. It is then convenient to make the notion of converging subsequence accurate by means of the following definition: a sequence of maps (diffeomorphisms onto their images) f_i is said to *converge to infinity* on an open set $U \subset S_{A,B,C,D}$ if for every compact set $\overline{V} \subset U$ and every compact set $K \subset S_{A,B,C,D}$, there are only finitely many maps f_i such that $f_i(\overline{V}) \cap K \neq \emptyset$. Sequences converging to infinity are to be included in the definition of normal family used above. In particular, if the sequence formed by all elements of $\mathcal{G}$ converge to infinity on some open set $U \subset S_{A,B,C,D}$, then U is contained in the Fatou set of $\mathcal{G}$.

Remark 1.1. According to Proposition 10.2, any component of the Fatou set $\mathcal{F}_{A,B,C,D}$ is Kobayashi hyperbolic and, by exploiting this condition, it can be shown that our definition amounts to requiring the family to be normal if it is viewed as (continuous) maps from $S_{A,B,C,D}$ with values in its one-point compactification.

The *Julia set* of the group action $\mathcal{G}$ is defined as

$$\mathcal{J}_{A,B,C,D} = S_{A,B,C,D} \setminus \mathcal{F}_{A,B,C,D}.$$

It follows from the definitions that $\mathcal{F}_{A,B,C,D}$ is open while $\mathcal{J}_{A,B,C,D}$ is closed. Furthermore both sets are invariant under $\mathcal{G}$. (For some parameters $S_{A,B,C,D}$ may be singular, but this is not an issue: we will see in Remark 4.1 that such singular points are always in $\mathcal{J}_{A,B,C,D}$.)

It is worth emphasizing that the Julia set is non-empty for every choice of parameters; see, for example, Lemma 5.3. With more effort one can show that for every choice of parameters (A, B, C, D) the group $\mathcal{G}_{A,B,C,D}$ contains elements f having positive entropy and exhibiting dynamics quite similar to that of Hénon maps; see [39, 9, 13]. The Julia set associated with iteration of each such individual mapping is a subset of $\mathcal{J}_{A,B,C,D}$, including the collection of all saddle-type periodic points of each such mapping. However, the Fatou set can be empty for some parameter values; indeed this happens for the Picard Parameters (Theorem D(i), below).

1.6. **Main results.** Classical results from the holomorphic dynamics of rational maps of the Riemann sphere assert that there is a dense orbit in the Julia set (topological transitivity) and that repelling periodic points are dense in the Julia set. We search for analogous statements for the action of $\mathcal{G}_{A,B,C,D}$ on $S_{A,B,C,D}$.

Theorem A. *For any parameters (A, B, C, D) there is a point $p \in \mathcal{J}_{A,B,C,D}$ such that*

$$\overline{\mathcal{G}(p)} = \mathcal{J}_{A,B,C,D},$$

i.e., there is a dense orbit of $\mathcal{G}$ in $\mathcal{J}_{A,B,C,D}$.

In the setting of group actions, the natural analog of having a dense set of repelling periodic points consists of looking for a dense set $\mathcal{J}^*_{A,B,C,D} \subset \mathcal{J}_{A,B,C,D}$ such that each point $p \in \mathcal{J}^*_{A,B,C,D}$ has an element of its stabilizer whose derivative at p is hyperbolic. In Theorem D, below, we will see that this does not hold for the Picard Parameters $(0, 0, 0, 4)$. We leave it as an open question to characterize for which parameters (A, B, C, D), if any, the set $\mathcal{J}^*_{A,B,C,D}$ is dense in $\mathcal{J}_{A,B,C,D}$.

A more modest statement is obtained by replacing "hyperbolic derivative" by derivative conjugate to a "shear map". This is the content of Theorem B below.

Theorem B. *For any choice of parameters (A, B, C, D) there is a dense set $\mathcal{J}^{\#}_{A,B,C,D} \subset \mathcal{J}_{A,B,C,D}$ such that for every $p \in \mathcal{J}^{\#}_{A,B,C,D}$ there exists $\gamma \in \mathcal{G}$ such that $\gamma(p) = p$ and*

$$D\gamma(p) \qquad \text{is conjugate to} \qquad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Often parametric families of rational maps of the Riemann sphere have some mappings with connected Julia set and other mappings with disconnected Julia set. In our context we have a slightly surprising general topological property of Julia sets, namely:

Theorem C. *For any parameters A, B, C, D the Julia set $\mathcal{J}_{A,B,C,D}$ is connected.*

Let us now transition from results that hold for all parameters to results that only hold for certain parameters.

Theorem D. *For the Picard Parameters $(A, B, C, D) = (0, 0, 0, 4)$ we have:*

(i) *$\mathcal{J}_{0,0,0,4} = S_{0,0,0,4}$ and consequently $\mathcal{F}_{0,0,0,4} = \emptyset$,*
(ii) *The action of $\mathcal{G}_{0,0,0,4}$ is locally discrete on any open subset of $S_{0,0,0,4}$, and*

(iii) *The closure of the set of points* $\mathcal{J}^*_{0,0,0,4}$ *that have hyperbolic stabilizers is contained in* $S_{0,0,0,4} \cap [-2,2]^3$ *and hence is a proper subset of* $\mathcal{J}_{0,0,0,4} = S_{0,0,0,4}$.

In contrast to the Picard parameters, for which the corresponding Fatou set is empty, Teichmüller theory can be applied to show that the Fatou set is non-empty for certain values of the parameters, including the Markov parameters $(A, B, C, D) = (0, 0, 0, 0)$. A more powerful approach stems from issues related to the Bowditch Conjecture and the Bowditch BQ Conditions, as introduced by Bowditch [7] and Tan, Wong, and Zhang [63] and later studied by Maloni, Palesi, and Tan [45], Hu, Tan, and Zhang [34], and several others. See Section 3.2 for more details. Their methods can be adapted to prove the following theorem:

Theorem E. *The Fatou set* $\mathcal{F}_{A,B,C,D}$ *is non-empty for any* (A, B, C, D) *in an open neighborhood in* $\mathbb{C}^4$ *of*

(1) *any punctured torus parameters* $(0, 0, 0, D)$ *for* $D \neq 4$*, and*
(2) *any Dubrovin-Mazzocco Parameter* $(A(a), B(a), C(a), D(a))$*, where* $a \in (-2, 2)$.

We will denote the subset of the Fatou set obtained in the proof of Theorem E by $V_{\text{BQ}} \equiv V_{\text{BQ}}(A, B, C, D)$ and call it the *Bowditch set* because any point $p \in V_{\text{BQ}}$ satisfies the BQ Conditions. (See Section 3.2 for the precise statement of these conditions.) As a matter of fact, the action of $\mathcal{G}$ on V_{BQ} is properly discontinuous, as was shown in [63] and [45]. (We will also show this in the proof of Theorem F, below.) The Bowditch set seems to be of considerable interest and, in particular, the following rephrasing of Statement (1) in Theorem E seems to be new:

Corollary to Theorem E. *Consider the punctured torus parameters* $(0, 0, 0, D)$ *where* $D \in \mathbb{C}$. *The Bowditch set* $V_{\text{BQ}}(0, 0, 0, D)$ *is non-empty if and only if* $D \neq 4$.

However, let us note that we recently discovered that Statement (2) of Theorem E is an immediate consequence of a stronger statement [45, Theorem 5.3]. In general, a complete classification of all $(A, B, C, D) \in \mathbb{C}^4$ for which $V_{\text{BQ}}(A, B, C, D)$ is non-empty seems to be a delicate question.

Because the Fatou set can be non-empty it is interesting to determine how "large" the Julia set is, especially if one wishes to apply Theorems A and B. For this reason we study the interplay of the Fatou/Julia dichotomy with the locally non-discrete/discrete dichotomy. We will first discuss the locally non-discrete/discrete dichotomy and then relate it to the Fatou/Julia dichotomy.

In Section 8, we will show the existence of a very large set of parameters (A, B, C, D) for which $\mathcal{G}_{A,B,C,D}$ is locally non-discrete on some open subset of the surface $S_{A,B,C,D}$. Later in Section 12 we obtain as a consequence the following theorem about the coexistence of both locally discrete and locally non-discrete dynamics:

Theorem F. *There is an open neighborhood* $\mathcal{P} \subset \mathbb{C}^4$ *of the Markoff Parameters* $(0, 0, 0, 0)$ *and of each of the Dubrovin-Mazzocco Parameters* $(A(a), B(a), C(a), D(a))$*, where* $a \in (-2, 2)$*, with the following property.*

For any $(A, B, C, D) \in \mathcal{P}$ *there are disjoint non-empty opens sets* $U, V_{\text{BQ}} \subset S_{A,B,C,D}$ *such that:*

(1) *The action of* $\mathcal{G}_{A,B,C,D}$ *is locally non-discrete on* U*; i.e.* $U \subset \mathcal{N}_{A,B,C,D}$.
(2) *The action of* $\mathcal{G}_{A,B,C,D}$ *is locally discrete on any open neighborhood of any point from* V_{BQ}*, i.e.* $V_{\text{BQ}} \subset \mathcal{D}_{A,B,C,D}$*. Indeed, the action of* $\mathcal{G}_{A,B,C,D}$ *on* V_{BQ} *is properly discontinuous.*

There are non-commuting pairs of element of $\mathcal{G}_{A,B,C,D}$ *both of which are arbitrarily close to the identity on* U.

Corollary to Theorem F. *For every choice of parameters* $(A, B, C, D) \in \mathcal{P}$ *there there is a set*

$$\mathscr{B}_{A,B,C,D} \subset \partial\mathcal{N}_{A,B,C,D} = \partial\mathcal{D}_{A,B,C,D}$$

that has topological dimension equal to three and is invariant under $\mathcal{G}_{A,B,C,D}$.

Remark. The invariant set $\mathscr{B}_{A,B,C,D} \subset S_{A,B,C,D}$ of topological dimension 3 "persists" over the open subset of parameters $\mathcal{P} \subset \mathbb{C}^4$. The existence of persistent invariant sets of topological dimension 3 for the action of a large group ($\mathcal{G}$ is free on two generators) hints at a fractal nature for $\mathscr{B}_{A,B,C,D}$. In fact, in the smooth category, by combining general position arguments with Baire's theorem, it can be shown that two "generic" diffeomorphisms cannot simultaneously preserve a smooth submanifold. Whereas the same methods are not immediately available in the holomorphic setting, the same conclusion seems likely to still hold true.

With the notation of Theorem F, we expect that for each $(A, B, C, D) \in \mathcal{P}$ we have $U \subset \mathcal{J}_{A,B,C,D}$. However, at present, we can only prove this under one further (weak) assumption, namely:

(P) any fixed point of any $\gamma \in \mathcal{G}_{A,B,C,D} \setminus \{\mathrm{id}\}$ is in $\mathcal{J}_{A,B,C,D}$.

We prove in Proposition 10.9 that there is a countable union of real-algebraic hypersurfaces $\mathcal{H} \subset \mathbb{C}^4$ such that Hypothesis (P) holds if $(A, B, C, D) \in \mathbb{C}^4 \setminus \mathcal{H}$.

Theorem G. *Let $\mathcal{P} \subset \mathbb{C}^4$ be the open neighborhood of the Markoff Parameters and of the Dubrovin-Mazzocco parameters, $a \in (-2, 2)$, given in Theorem F and let $\mathcal{H} \subset \mathbb{C}^4$ be the countable union of real-algebraic hypersurfaces provided by Proposition 10.9. For any $(A, B, C, D) \in \mathcal{P} \setminus \mathcal{H}$ we have:*

$$U \subset \mathcal{J}_{A,B,C,D} \qquad \text{and} \qquad V_{\mathrm{BQ}} \subset \mathcal{F}_{A,B,C,D}.$$

Here, U and V_{BQ} are the (non-empty) open subsets of $S_{A,B,C,D}$ from the statement of Theorem F.

Corollary to Theorem G. *For any $(A, B, C, D) \in \mathcal{P} \setminus \mathcal{H}$ there exists $p \in \mathcal{J}_{A,B,C,D}$ whose orbit closure has interior but is not all of the surface $S_{A,B,C,D}$.*

Remark 1.2. Any $\gamma \in \mathcal{G}_{A,B,C,D}$ preserves an invariant holomorphic 2-form Ω whose equation is given in Section 4.5. This forces that the eigenvalues of $D\gamma$ at any fixed point $p \in S_{A,B,C,D}$ are resonant, $\lambda_1 = 1/\lambda_2$, making it rather unlikely that such a fixed point is in the Fatou set.

Indeed, the possibility that Hypothesis (P), above, fails for any parameters (A, B, C, D) is a known challenge in holomorphic dynamics. It is explained at length in the paper [47] by McMullen, see especially the remark on p. 220 of that paper.

1.7. **Strategy for Proof of Theorem G.** Let us briefly describe the strategy for proving Theorem G, which is arguably the most elaborate result in our paper. It is straightforward to prove that any Fatou component V is Kobayashi hyperbolic. In particular, if $\mathcal{G}$ is locally non-discrete on any open subset of V then it is locally non-discrete on all of V. The idea is then to show that a region U where $\mathcal{G}$ induces a "complicated enough" locally non-discrete dynamics is not compatible with the structure of a (Kobayashi hyperbolic) Fatou set. This region must hence be contained in $\mathcal{J}_{A,B,C,D}$ and this yields Julia sets with non-empty interior.

In order to rule out the possibility that this region U intersects an unbounded Fatou component, the following theorem will be needed. Note that even though we typically consider s_x as an automorphism of a given surface $S_{A,B,C,D}$, Equation (1) actually defines a polynomial automorphism of $\mathbb{C}^3$ which depends only on the parameters A, B, and C. The same holds for s_y and s_z and we will denote the group of automorphisms of $\mathbb{C}^3$ generated by these three mappings with $\mathcal{G}^{\pm}_{A,B,C}$. The index two subgroup generated by g_x, g_y, and g_z (considered as automorphisms of $\mathbb{C}^3$) will be denoted by $\mathcal{G}_{A,B,C}$.

Theorem H. *Suppose that for some parameters A, B, C there is a point $p \in \mathbb{C}^3$ and $\epsilon > 0$ such that for any two vertices $v_i \neq v_j \in \mathcal{V}_\infty$, $i \neq j$, there is a hyperbolic element $\gamma_{i,j} \in \mathcal{G}_{A,B,C}$ satisfying:*

(A) $\mathrm{Ind}(\gamma_{i,j}) = v_i$ *and* $\mathrm{Attr}(\gamma_{i,j}) = v_j$, *and*

(B) $\sup_{z \in B_\epsilon(p)} \|\gamma_{i,j}(z) - z\| < K(\epsilon)$.

Then, for any D, we have that $B_{\epsilon/2}(p) \cap S_{A,B,C,D}$ is disjoint from any unbounded Fatou components of $\mathcal{G}_{A,B,C,D}$. Here, $K(\epsilon) > 0$ denotes the constant given in Proposition 8.1.

We refer the reader to Section 9 for the definition of $\mathcal{V}_\infty$ and to Proposition 9.3 for the definition of hyperbolic element γ along with the corresponding points $\text{Ind}(\gamma)$ and $\text{Attr}(\gamma)$. Hypothesis (A) requires that the six elements $\gamma_{i,j}$ have sufficiently rich "combinatorial behavior" at infinity while Hypothesis (B) requires that these six elements are sufficiently close to the identity on the ball $\mathbb{B}_\epsilon(p)$. Note that the conditions of Theorem H are explicit and easy to check. In particular, for any $(A, B, C, D) \in \mathcal{P}$ they imply that U is disjoint from any unbounded Fatou component.

The idea of the proof of Theorem H is that if $\mathbb{B}_\epsilon(p)$ were in an unbounded Fatou component V then we use local non-discreteness to produce a sequence of elements converging uniformly on compact subsets of V to the identity and we use the prescribed combinatorial behavior at infinity to show that this same sequence of elements sends compact subsets of V uniformly to infinity.

Having ruled out the possibility that an unbounded Fatou component intersects U we then use the following theorem to prove that no bounded Fatou component intersects U.

Theorem K. *Suppose that $(A, B, C, D) \in \mathbb{C}^4 \setminus \mathcal{H}$ and that V is a bounded Fatou component for $\mathcal{G}_{A,B,C,D}$. Then the stabilizer $\mathcal{G}_V$ of V is cyclic.*

It is a standard result that the group $\text{Aut}(V)$ of holomorphic automorphisms of a Kobayashi hyperbolic manifold V is a real Lie group. To prove Theorem K we use that $\mathcal{G}_{A,B,C,D}$ preserves a volume form (see Section 4.5) to show that the closure $G = \overline{\mathcal{G}_V}$ of $\mathcal{G}_V$ is a compact Lie group. Checking that any element of $\mathcal{G}_V$ has infinite order we conclude that G has positive dimension. Supposing that $\mathcal{G}_V$ is non-cyclic we can conclude that it is non-Abelian and moreover that there are non-commuting elements arbitrarily close to the identity. This gives that the connected component of the identity, G_0, is non-Abelian. Since G_0 is compact, it must therefore have real-dimension 3. The assumption that no element of $\mathcal{G}$ has a fixed point in V gives that G_0 acts freely on V and thus that V/G is a manifold of real dimension 1. This allows us to derive a contradiction to the fact that the Julia set is connected (Theorem C).

Note that the works of Bedford-Smillie [3] and Bedford-Kim [2] about Fatou components for volume preserving Hénon maps and rational surface automorphisms serve as a kind of prototype for how Lie Groups are used in the proof of Theorem K.

1.8. **Open problems.** Several interesting open problems arose while writing this paper. They are organized into a short companion paper "Questions about the dynamics on a natural family of affine cubic surfaces" [56].

1.9. **Structure of the paper.** In Section 2 we describe several of the previous works that are related to this paper. We present in Section 3 a discussion of the motivations for a dynamical study of $\mathcal{G}_{A,B,C,D}$, with emphasis on the connections to dynamics on character varieties and to the Painlevé 6 differential equation. In Section 4 we present several basic properties of the group $\mathcal{G} \equiv \mathcal{G}_{A,B,C,D}$ and of the surface $S \equiv S_{A,B,C,D}$ that will be used throughout the paper. In Section 5 we study properties of the "parabolic" elements of $\mathcal{G}$ and use them and Montel's Theorem to prove Theorems A, B, and C. Section 6 is devoted to a careful study of the Picard parameters $(A, B, C, D) = (0, 0, 0, 4)$ and a proof of Theorem D. In Section 7 we prove Theorem E about existence of the unbounded Fatou components, following the techniques from [7, 63, 45, 34]. In Section 8 we produce examples of locally non-discrete dynamics and that will be needed for Theorem F. In Section 9 we collect several important properties of how $\mathcal{G}$ acts near infinity that will be needed later in the paper, including a proof of Proposition 10.9, which plays a key role in Theorem G. Section 10 is devoted to a proof of Theorem H about unbounded Fatou components and Theorem K about bounded Fatou components. We finish the proofs of our theorems with Section 12 where we prove Theorems F and G.

Remark: Sections 2 and 3 are largely about placing this paper in a broader context, highlighting further applications and connections with previous works. These sections can be skimmed over on a first reading and then returned to later as additional information becomes needed.

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2. Related works.

The previous works arising from dynamics on character varieties can be traced back to Markoff's Theorem and to works on the Markoff Surface, see [7] and [59]. To the best of our knowledge, deeper investigations of these dynamics follow two main trends:

(i) Global dynamics on the 2-dimensional real (singular) surfaces for real parameters A, B, C, D, as initiated by Goldman [30] (see also [29]).
(ii) Study of domains in the complex surface $S_{A,B,C,D}$ on which the group $\mathcal{G}_{A,B,C,D}$ acts properly discontinuously, as initiated by Bowditch [7] and later studied extensively by several authors. We refer the reader especially to the works by Tan, Wong, and Zhang [63], Maloni, Palesi, and Tan [45], and Hu, Tan, and Zhang [34].

Based on motivations from the monodromy of the Painlevé 6 differential equation, the previous dynamical results follow three main trends:

(i) Finite orbits under $\mathcal{G}_{A,B,C,D}$ correspond to algebraic solutions of Painlevé 6. They were classified by Dubrovin and Mazzocco [22] and by Lisovyy and Tykhyy [43]. See also [13, Section 4] for a classification of bounded orbits.
(ii) Study of individual mappings from $\mathcal{G}$ and $\mathcal{G}^{\pm}$ displaying rather interesting dynamics by Iwasaki and Uehara [39], including uniformly hyperbolic ones by Cantat [9]. These mappings share many features in common with the complex Hénon mappings.
(iii) Proof by Cantat and Loray [13] that except for the case Picard parameters ($A = B = C = 0$, $D = 4$), the action of $\mathcal{G}$ on $S_{A,B,C,D}$ preserves neither an (multi) affine structure nor a (multi) holomorphic foliation. This corresponds to the Malgrange irreducibility of Painlevé 6, as explained in their paper.

To the best of our knowledge, the previous dynamical works related to the aperiodic Schrödinger equation primarily involve delicate issues about the iteration of a single real mapping $\gamma \in \mathcal{G}_{A,B,C,D}$ on the real slice of $S_{A,B,C,D}$. There is a huge literature on the subject and we refer the reader to the papers by Casdagli and Roberts [18, 57] and references therein for an introduction. For more contemporary works, see, for example, the paper of Damanik, Gorodetski, and Yessen [20] and the paper of Yessen [65].

The fact that mapping class group actions on character varieties can be formulated as a complex dynamical systems allows one to see the present work as joining a recent trend of papers where methods of holomorphic dynamics were used to investigate certain natural dynamical systems, see [9], [15], and [16].

2.1. **Relationship to the recent work of S. Cantat and R. Dujardin.** A recent work by S. Cantat and R. Dujardin [12] studies the dynamics of subgroups of the automorphism group of (compact) real and complex projective surfaces (or, more generally, Kähler surfaces). They obtain deep results on the structure of stationary measures for these subgroups, including criteria to determine when they must be invariant by the group in question (stiffness) as well as detailed descriptions of the resulting invariant measures. Whereas their results clearly have serious implications on the corresponding dynamics, the situation seems to be genuinely more subtle in the case of groups of birational maps. Relevant examples include the case of the actions of $\mathcal{G}$ and $\mathcal{G}^{\pm}$ since, as shown in the present paper, for certain values of the parameters these actions display both non-empty Fatou set and Julia sets with non-empty interior. This phenomenon indicates that the problem of invariance of stationary measures, and of their subsequent description, will hardly allow for a reasonably "compact" classification. Also, the nature of the Fatou components constructed in our Theorem E shows that even the convergence of stochastic processes will no longer be automatic which, in turn, seriously limits the applications of stationary measures in topological problems such as description of non-compact invariant closed sets.

2.2. **Relationship to previous works using the locally non-discrete/discrete dichotomy.** The notion of locally non-discrete group (or even pseudogroup) first appeared in [54] after previous works by A. Shcherbakov, I. Nakai, and E. Ghys, see [62, 50, 28]. The notion was further elaborated in [44]. The main common tool of [54] and [44] is the construction of certain vector fields obtained as a sort of limit of certain dynamics in the group/pseudogroup in question which allow for a detailed analysis of the corresponding group dynamics. In turn, the method put forward in those papers ensures the existence of the desired vector fields for locally non-discrete (pseudo-) groups provided there is a local expansion in the dynamics: typically, we would like some element in group to have a fixed point where all its eigenvalues are of modulus greater than 1. By way of contrast, elements of $\mathcal{G}$ all preserve the same volume form (cf. Section 4.5) and therefore never have repelling fixed points. So we must pursue different arguments.

It should be pointed out that the structure of *locally discrete* groups of diffeomorphisms of the circle is rather well described, see for example [1], [17]. In particular, it follows that the class formed by *locally non-discrete* groups is extremely large and, as a matter of fact, much can be said about the dynamics associated with the latter groups, see for example [24] and the references therein. However, in the case of the circle, the relatively simple nature of the topological dynamics of groups acting on (real) 1-dimensional manifolds leads to the fact that coexistence of local discreteness and local non-discreteness is impossible. More precisely, in $\mathrm{Diff}^{\omega}(S^1)$, if a non-Abelian group $G \subset \mathrm{Diff}^{\omega}(S^1)$ is locally non-discrete on an interval $I \subset S^1$, then every point $p \in S^1$ has a neighborhood where G is locally non-discrete, see [24]. This dramatically contrasts with our Theorem F which asserts that the same group action $\mathcal{G}_{A,B,C,D}$ can simultaneously have open sets U and V on which it acts locally non-discretely and locally discretely, respectively. When combined with the complicated behavior of the pointwise dynamics of $\mathcal{G}_{A,B,C,D}$, in particular the existence of invariant sets with topological dimension 3 (Corollary of Theorem F), we find very rich dynamics in the system that we study.

Let us also mention that V. Kleptsyn and his collaborators have found examples of locally discrete subgroups of $\mathrm{Diff}^{\omega}(S^1)$ that are not conjugate to Fuchsian groups, up to finite covering [1]. In our context, the Picard parameters provide an analogous non-trivial example of a group that is "purely locally discrete", i.e., there is no open set $U \subset S_{A,B,C,D}$, $U \neq \emptyset$, where the group acts in a locally non-discrete way.

3. Dynamics on Character Varieties and the Painlevé 6 equation.

The most compelling motivations for studying the dynamical systems considered here comes from dynamics on character varieties and from the sixth Painlevé equation: the two motivations being closely related as will soon be seen.

3.1. **Motivations from Dynamics on Character Varieties.** We begin with character varieties as the dynamical system discussed in this paper is equivalent to the action of the mapping class group of a surface on the space of $\mathrm{SL}(2,\mathbb{C})$-representations of its fundamental group, up to conjugation. This contains, in particular, the standard action of mapping class groups on Teichmüller spaces and hence can be viewed as belonging to the setting of "higher Teichmüller theory" as well; see, e.g., [64].

The dynamics we are interested in can be cast in the broader framework of (natural) dynamics on character varieties as was first pointed out in W. Goldman's papers [30] and [29]. Let us briefly explain how character varieties arise in this context. Consider, for example, the case of a punctured torus. Its fundamental group Π is isomorphic to the free group F_2 on two generators. The space of representations of Π in $\mathrm{SL}(2,\mathbb{C})$ is clearly identified with $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ which, in turn, is acted upon by $\mathrm{SL}(2,\mathbb{C})$ via simultaneous conjugation. The corresponding character variety is, by definition, the (categorical) quotient of the latter action. Next, note that automorphism group of Π also acts on the space of representations by pre-composition and this action descends to the character varieties. However, on the character variety, the action of inner automorphism of Π becomes trivial so that the action $\mathrm{Aut}(\Pi)$ factors through an action of the group $\mathrm{Out}\,(\Pi)$ consisting of outer automorphisms of Π.

Whereas the previous construction essentially makes sense for representations of Π in any group G, the fact that we are dealing with $G = \mathrm{SL}(2,\mathbb{C})$ can further be exploited as follows. As pointed out, the character variety in question is identified with a pair of matrices in $\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})$ up to simultaneous conjugation. A classical result of Fricke, however, shows that the latter space can be parameterized by $\mathbb{C}^3$. We therefore inherit an action of $\mathrm{Out}\,(\Pi)$ on $\mathbb{C}^3$ which, in addition, coincides with the action on $\mathbb{C}^3$ of the group $\mathcal{G}^{\pm}$ obtained by setting $A = B = C = 0$. Incidentally, the latter group coincides with the group of polynomial automorphisms of $\mathbb{C}^3$ preserving the (Markoff) polynomial $x^2+y^2+z^2-xyz-2$, cf. [33] and see [52] for a general result on the polynomial nature of similar actions. Finally, note that an analogous construction applies to the quadruply-punctured sphere. In the latter case, the corresponding action of $\mathrm{Out}\,(\Pi)$ recovers the action of $\mathcal{G}$ and $\mathcal{G}^{\pm}$ with all the parameters A, B, C, and D, see [4] for a detailed account. Other than [4], more or less comprehensive versions of the previous discussion appear in a number of papers including [30], [29], [9], [13], [33].

3.2. **Properly Discontinuous Dynamics, quasi-Fuchsian representations, and Bowditch BQ Conditions.** We now elaborate how the dynamical context explained in Section 3.1 leads to the existence of parameters with non-empty Fatou sets.

Consider the general case of the 4-holed sphere $\Sigma_{0,4}$ and note that its fundamental group can be identified with the free group on three generators $\Pi \cong \mathrm{F}_3$. Furthermore, since the orbits of the action of $\mathrm{SL}(2,\mathbb{C})$ on itself by conjugation are also of dimension 3, we see that the space of representations from Π to $\mathrm{SL}(2,\mathbb{C})$ (up to conjugation) has 6 complex dimensions. A far more accurate description is possible: the space of representations can be identified with the *quartic hypersurface of* $\mathbb{C}^7$ given by

$$\mathbb{S} = \{(a_1,a_2,a_3,a_4,x,y,z) \in \mathbb{C}^7 \,:\, x^2+y^2+z^2+xyz = Ax+By+Cz+D\}\,,$$

where A, B, C, D are as follows:

$$A = a_1a_4 + a_2a_3\,,\ \ B = a_2a_4 + a_1a_3\,,\ \ C = a_3a_4 + a_1a_2\,, \tag{4}$$

and

$$D = 4 - [a_1a_2a_3a_4 + a_1^2 + a_2^2 + a_3^2 + a_4^2]\,. \tag{5}$$

In particular, by fixing the variables $a_1,\ldots,a_4$, we obtain the surface $S_{A,B,C,D}$. The parameters $a_1,\ldots,a_4$ are identified with the traces of the matrices in $\mathrm{SL}(2,\mathbb{C})$ arising from the loops around the holes of $\Sigma_{0,4}$.

Quasi-Fuchsian representations: Consider the space of quasi-Fuchsian representations inside $\mathbb{S}$. The Bers Simultaneous Uniformization theorem implies that the space of quasi-Fuchsian representations $\mathrm{Rep}_{qf}(\Sigma_{0,4})$ can be identified with the product of two copies of the Teichmüller space

Teich $(\Sigma_{0,4})$ of the 4-holed sphere $\Sigma_{0,4}$, with geodesic boundaries, i.e.

$$\mathrm{Rep}_{qf}(\Sigma_{0,4}) = \mathrm{Teich}\,(\Sigma_{0,4}) \times \mathrm{Teich}\,(\Sigma_{0,4})\,. \tag{6}$$

In turn, recalling that the group of outer automorphisms of the fundamental group of $\Sigma_{0,4}$ is isomorphic to the mapping class group of $\Sigma_{0,4}$, the latter acts on the space $\mathrm{Rep}_{qf}(\Sigma_{0,4}) = \mathrm{Teich}\,(\Sigma_{0,4}) \times \mathrm{Teich}\,(\Sigma_{0,4})$ diagonally with respect to its standard action on Teich $(\Sigma_{0,4})$. In particular, the action of the mapping class group on $\mathrm{Rep}_{qf}(\Sigma_{0,4})$ is *properly discontinuous*.

Now, note that the 4-holed sphere consists of two pair of pants joined by the "waist". Hence the real dimension of Teich $(\Sigma_{0,4})$ is 6 so that the real dimension of $\mathrm{Rep}_{qf}(\Sigma_{0,4})$ is 12 and therefore $\mathrm{Rep}_{qf}(\Sigma_{0,4})$ has has non-empty interior in $\mathbb{S}$. Finally, if the parameters $a_1, \ldots, a_4$ are chosen so that the corresponding surface $S_{A,B,C,D}$ intersects $\mathrm{Rep}_{qf}(\Sigma_{0,4})$ and this intersection contains an open set $U \subset S_{A,B,C,D}$, then the preceding shows that $\mathcal{G}$ acts properly discontinuously on U and hence that U is contained in the Fatou set of the action of $\mathcal{G}$ on $S_{A,B,C,D}$.

Note that it is easy to find parameters $a_1, \ldots, a_4$ so that the resulting surface $S_{A,B,C,D}$ does not intersect $\mathrm{Rep}_{qf}(\Sigma_{0,4})$. Recalling that $a_1, \ldots, a_4$ are the traces of matrices in $\mathrm{SL}(2,\mathbb{C})$ arising from loops around the holes, it is enough to force one of these matrices to be an elliptic element conjugate to an irrational rotation. An alternative argument leading to open sets of parameters $a_1, \ldots, a_4$ for which $S_{A,B,C,D}$ is disjoint from $\mathrm{Rep}_{qf}(\Sigma_{0,4})$ can be formulated by using the well-known Jorgensen inequality.

Bowditch Conjecture and BQ Conditions: Bowditch conjectures in [7] that a point $(x, y, z) \in S_{0,0,0,0}$ is in $\mathrm{Rep}_{qf}(\Sigma_{0,4})$ if and only if the following two simple *BQ conditions* hold:

(1) None of the coordinates of $\gamma(x, y, z)$ is in $[-2, 2]$ for any $\gamma \in \mathcal{G}$, and
(2) $\gamma(x, y, z) \in \left(\mathbb{C} \setminus \overline{\mathbb{D}_2}\right)^3$ for all but finitely many $\gamma \in \mathcal{G}$.

It is easy to prove that if $(x, y, z) \in \mathrm{Rep}_{qf}(\Sigma_{0,4})$ then (1) and (2) hold, but the converse is known as the *Bowditch Conjecture*. See [61, 41, 60] for recent related works and more details.

Conditions (1) and (2) were studied for the punctured torus parameters $(0, 0, 0, D)$ by Tan, Wong, and Zhang [63] and for arbitrary parameters (A, B, C, D) by Maloni, Palesi, and Tan [45]. (The "radius" 2 in Condition (2) needs to be adjusted when $(A, B, C) \neq (0, 0, 0)$, as explained in [45]). Let $\mathrm{Rep}_{BQ}(\Sigma_{0,4}) \subset \mathbb{S}$ denote the set of points for which the BQ Conditions hold. It is proved in [63] and [45] that for any (A, B, C, D) the set

$$V_{\mathrm{BQ}}(A, B, C, D) := \mathrm{Rep}_{BQ}(\Sigma_{0,4}) \cap S_{A,B,C,D}$$

is open and that $\mathcal{G}_{A,B,C,D}$ acts properly discontinuously on it. In particular, $V_{\mathrm{BQ}} \equiv V_{\mathrm{BQ}}(A, B, C, D)$ forms part of the Fatou set for $\mathcal{G}_{A,B,C,D}$.

Notice that V_{BQ} can be non-empty for parameters $a_1, \ldots, a_4$ beyond those for which $S_{A,B,C,D}$ intersects $\mathrm{Rep}_{qf}(\Sigma_{0,4})$. Several such examples are presented in [63] and also [45, Theorem 5.3].

The proof of Theorem E provided in this paper will be an adaptation of ideas from the papers [63, 45] and also the later paper by Hu, Tan, and Zhang [34].

3.3. **Motivations from the Painlevé equation P6.** From a different perspective, the complex dynamical system studied in this paper describes the transverse dynamics of the celebrated Painlevé 6 equation. In particular, the splitting of dynamics into two regions with contrasting dynamical behavior - a region called *Fatou* where the dynamics is simple (e.g. properly discontinuous) and another region named *Julia* where the dynamics is chaotic - also accounts for the somehow "double nature" of Painlevé 6: in the vast literature on the subject, it is possible to find a thread where this equation is viewed as part of integrable system and another one where it is regarded as a complicated dynamical systems. The simultaneous existence of large Fatou and Julia sets (both with non-empty interiors cf. Theorem G) justifies and conciliates both perspectives. In particular, the results obtained in this paper - and especially those involving the *Julia set* - are in line with general programs aimed at the dynamical study of Painlevé equations, cf. [37] and [53].

To clarify this connection, let us begin by recalling that in the most standard notation, the sixth Painlevé equation takes on the form

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right)\left(\frac{dy}{dx}\right)^2-\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right)\frac{dy}{dx}+ \\ &\quad +\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha+\beta\frac{x}{y^2}+\gamma\frac{x-1}{(y-1)^2}+\delta\frac{x(x-1)}{(y-x)^2}\right), \end{aligned} \tag{7}$$

where α, β, γ, and δ are complex parameters. K. Iwasaki in [36] provided a useful alternative way to represent the parameters involved in this equation by denoting them as κ_1, κ_2, κ_3, and κ_4 where the following formulas hold:

$$\alpha = \kappa_4^2/2\,,\quad \beta = -\kappa_1^2/2 \quad \gamma = \kappa_2^2/2\,,\quad \text{and} \quad \delta = (1-\kappa_3^2)/2\,. \tag{8}$$

Now, as a non-autonomous differential equation of second order, P6 can equivalently be viewed as a vector field $Z_{\rm VI}$ on $\mathbb{C}^3$ having the form

$$Z_{\rm VI} = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + \mathcal{H}_{\alpha,\beta,\gamma,\delta}(x,y,z)\frac{\partial}{\partial z}, \tag{9}$$

where the function $\mathcal{H}_{\alpha,\beta,\gamma,\delta}(x,y,z)$ is obtained from the right side of (7) by substituting z for dy/dx. In particular, the variable x can naturally be identified with "time" in the standard form (7).

Let $\mathcal{D}$ denote the foliation defined on $\mathbb{C}^3$ defined by the local orbits of $Z_{\rm VI}$. The foliation $\mathcal{D}$ is holomorphic since it can be defined by a *polynomial vector field* obtained from $Z_{\rm VI}$ by multiplying it by the denominator appearing in the rational expression for the function $\mathcal{H}_{\alpha,\beta,\gamma,\delta}(x,y,z)$. It is immediate to check that the foliation $\mathcal{D}$ on $\mathbb{C}^3$ is transverse to the fibers of the standard fibration (projection) π_x of $\mathbb{C}^3$ to the x-axis, away from the *invariant fibers* sitting over $\{x=0\}$ and $\{x=1\}$. In [51], Okamoto obtained a much nicer birational model for the foliation $\mathcal{D}$. By compactifying the fibers of π_x in a suitable Hirzebruch surface, performing a number of well chosen blow up maps, and removing a certain resulting divisor the following setting for the foliation $\mathcal{D}$ (see for example [51], [38]):

- A complex (open) manifold N of dimension 3 fibering over $\mathbb{C}\setminus\{0,1\}$ with an open surface denoted by F as typical fiber. Moreover the projection map $\mathsf{p}: M \to \mathbb{C}\setminus\{0,1\}$ arises as the transform of the initial projection $\pi_x : \mathbb{C}^3 \to \mathbb{C}$ through the corresponding sequence of blow-up maps.
- N is equipped with the corresponding transform (still denoted by $\mathcal{D}$) of the extended foliation $\mathcal{D}$ of $\mathbb{C}\times \mathrm{F}_\epsilon$ by the corresponding blow-up maps.
- The foliation $\mathcal{D}$ is transverse to the fibers of p and the base $\mathbb{C}\setminus\{0,1\}$ can still be naturally identified with the "time" in (7).
- The restriction of p to a leaf L of $\mathcal{D}$ yields a covering map from L to $\mathbb{C}\setminus\{0,1\}$.

Owing to the fourth condition, paths contained in $\mathbb{C}\setminus\{0,1\}$ can be lifted in the leaves of $\mathcal{D}$ so as to yield a homomorphism ρ from the fundamental group of $\mathbb{C}\setminus\{0,1\}$ to the group of holomorphic diffeomorphisms $\mathrm{Diff}\,(F)$ of F. In other words, $\rho(\pi_1(\mathbb{C}\setminus\{0,1\})\subset \mathrm{Diff}\,(F)$ is the holonomy group of $\mathcal{D}$ whose action on F encodes the transverse dynamics of the initial Painlevé 6 equation.

The Riemann-Hilbert map conjugates the monodromy action of $\rho(\pi_1(\mathbb{C}\setminus\{0,1\}))$ on F at parameters $(\kappa_1,\ldots,\kappa_4)$ with the action of $\mathcal{G}_{A,B,C,D}$ on $S_{A,B,C,D}$. More precisely there are two mappings

$$\mathfrak{rh}:\ \mathbb{C}^4\to\mathbb{C}^4 \qquad \text{and} \qquad \mathrm{RH}_{(\kappa_1,\kappa_2,\kappa_3,\kappa_4)} : F \to S_{A,B,C,D},$$

where $(A,B,C,D) = \mathfrak{rh}(\kappa_1,\kappa_2,\kappa_3,\kappa_4)$. They have the property that for any choice of parameters $(\kappa_1,\kappa_2,\kappa_3,\kappa_4)$ the mapping $\mathrm{RH}_{(\kappa_1,\kappa_2,\kappa_3,\kappa_4)}$ conjugates the monodromy action of $\rho(\pi_1(\mathbb{C}\setminus\{0,1\}))$ on F to the action of $\mathcal{G}_{A,B,C,D}$ on $S_{A,B,C,D}$.

The mapping $\mathfrak{rh}$ is relatively simple. If Iwasaki parameters $\kappa_1, \ldots, \kappa_4$ are considered for P6, then we define

$$a_i = 2\cos(\pi\kappa_i)\,,$$

for $i = 1, \ldots, 4$. Next, to the 4-tuple of complex numbers $(a_1, \ldots, a_4)$, let us assign the 4-tuple (A, B, C, D) by means of Formulas (4) and (5). Applying these changes of parameters defines $(A, B, C, D) = \mathfrak{rh}(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$.

As mentioned, it is well known that the mapping $\mathrm{RH}_{(\kappa_1,\kappa_2,\kappa_3,\kappa_4)}$ provides a holomorphic conjugation between the dynamics of $\rho(\pi_1(\mathbb{C} \setminus \{0, 1\})$ on F and the dynamics of $\mathcal{G}$ on $S_{A,B,C,D}$, see [35, Theorem 8.4] and also to [22] for further details. Now, considering the holonomy representation $\rho(\pi_1(\mathbb{C} \setminus \{0, 1\})) \subset \mathrm{Diff}\,(F)$, it is well known that the mapping $\mathrm{RH}_{(\kappa_1,\kappa_2,\kappa_3,\kappa_4)}$ provides a holomorphic conjugation between the dynamics of $\rho(\pi_1(\mathbb{C}\setminus\{0, 1\})$ on F and the dynamics of $\mathcal{G}$ on $S_{A,B,C,D}$, see [35, Theorem 8.4] and also to [22] for further details.

Whereas the a mapping $\mathrm{RH}_{(\kappa_1,\kappa_2,\kappa_3,\kappa_4)}$ can hardly be computed due to its highly transcendental nature, it still provides a holomorphic conjugation between the dynamical systems in question. In particular, at least on compact parts of F, conjugation invariant properties of the dynamics of $\mathcal{G}$ can immediately be translated into dynamical properties of Painlevé 6, and conversely. This applies in particular to our results involving the dynamics of $\mathcal{G}$ in its *Julia set*.

3.4. **Consequences of our results to Painlevé 6.** Taken together, our theorems provide plenty of rather explicit examples of parameters (A, B, C, D) along with open sets $U \subset \mathcal{J}_{A,B,C,D}$ in which the action of $\mathcal{G}_{A,B,C,D}$ has dense orbits. In connection with these examples, we would like to point out a computational issue that deserves further attention. These have to do with the fact that the group action $\mathcal{G}$ is conjugate by the Riemann-Hilbert map to the monodromy of the actual Painlevé 6 equation. In our examples, it is not too hard to estimate the "size" of the open sets U provided by Theorem G: for example, we can provide some explicit $r > 0$ and a point $p \in U$ such that the ball of radius r about p is certainly contained in U. Yet, it for applications, it would be very interesting to have similar estimates for the size/place of the image of U by the Riemann-Hilbert map, so as to provide accurate numerics for an open set in which the actual Painlevé 6 equation must have dense orbits. The difficulty of this problem largely stems from well known difficulties in computing the Riemann-Hilbert map.

4. Basic properties of $S_{A,B,C,D}$ and $\mathcal{G}_{A,B,C,D}$

4.1. **Projective compactification and triangle at infinity.** We begin by considering the natural compactification of $\mathbb{C}^3$ in $\mathbb{CP}^3$ so that $\mathbb{CP}^3 = \mathbb{C}^3 \cup \Pi_\infty$, where $\Pi_\infty \simeq \mathbb{CP}^2$ is the plane at infinity in $\mathbb{CP}^3$. Next, denote by $\overline{S}_{A,B,C,D}$ the closure of $S_{A,B,C,D}$ in $\mathbb{CP}^3$. By using the standard affine atlas for $\mathbb{CP}^3$, it is straightforward to check that $\overline{S}_{A,B,C,D} \cap \Pi_\infty$ consists of three projective lines forming a triangle Δ_∞ in Π_∞. Indeed, if (u, v, w) are affine coordinates for $\mathbb{CP}^3$ satisfying $(1/u, v/u, w/u) = (x, y, z)$, then the surface $S_{A,B,C,D}$ is determined in (u, v, w)-coordinates by the equation

$$u + uv^2 + uw^2 + vw = Au^2 + Bu^2v + Cu^2w + Du^3\,.$$

In particular, it follows that $\overline{S}_{A,B,C,D} \cap \Pi_\infty$ locally coincides with the axes $\{u = v = 0\}$ and $\{u = w = 0\}$. An analogous use of the remaining coordinates shows that the third side of the mentioned triangle coincides with the projective line of $\Pi_\infty \simeq \mathbb{CP}^2$ which is missed by the domain of the affine coordinates $(v, w) \simeq (u = 0, v, w)$. A slightly less immediate computation with the above equation for $\overline{S}_{A,B,C,D}$ also shows that $\overline{S}_{A,B,C,D}$ is smooth on a neighborhood of Π_∞.

4.2. **Relation to the family of all cubic surfaces.** We now explain how the families of surfaces $S_{A,B,C,D}$ (and their projective compactifications $\overline{S}_{A,B,C,D}$) relate to the family of all cubic surfaces. This subsection is largely based on the work of Goldman and Toledo [31]. Suitable general background on cubic surfaces is given in [58, 8, 46], and the papers quoted therein.

We start with the fact that two minimal projective cubic surfaces are birationally equivalent if and only if they are projectively equivalent; see [46, p. 184]. Therefore, to decide if a given cubic surface appears in the family $S_{A,B,C,D}$ (respectively $\overline{S}_{A,B,C,D}$) for any values of (A,B,C,D), we can restrict ourselves to affine (respectively projective) equivalence.

A plane meeting a cubic in three lines is called a *tritangent plane*. If these three lines are in general position, then we say that we have a *generic tritangent plane*. As explained in Section 4.1, for any choice of parameters (A,B,C,D), the plane at infinity Π_∞ is a generic tritangent plane for $\overline{S}_{A,B,C,D}$. It is shown in [31] that a projective cubic surface belongs to the family $\overline{S}_{A,B,C,D}$ if and only if it admits a generic tritangent plane. Every smooth project cubic surface admits a generic tritangent plane and therefore appears in $\overline{S}_{A,B,C,D}$ for a suitable choice of parameters. However, there exist singular cubic surfaces that do not admit a generic tritangent plan, and therefore are not represented in the family $\overline{S}_{A,B,C,D}$. Alternatively, the singularities of the cubic surfaces in $\overline{S}_{A,B,C,D}$ are classified in [31] and the possible types form a proper subset of the types of singular points occurring for general cubic surfaces.

A generic projective cubic surface $\mathcal{S}$ has 45 tritangent planes and all of them are generic. It follows from the argument in [31] that each choice of a tritangent plane of $\mathcal{S}$ leads to a projective transformation of $\mathbb{CP}^3$ sending that tritangent plane to Π_∞ and sending $\mathcal{S}$ to a member of the family $\overline{S}_{A,B,C,D}$. One can then make any choice of $\epsilon_i \in \{-1,1\}$ for $i = 1,2,3$ such that $\epsilon_1\epsilon_2\epsilon_3 = 1$ and apply the affine map $(x,y,z) \mapsto (\epsilon_1 x, \epsilon_2 y, \epsilon_3 z)$ to identify $\mathcal{S}$ with four different choices of surface $\overline{S}_{A,B,C,D}$. We have therefore identified $\mathcal{S}$ with elements of the family $\overline{S}_{A,B,C,D}$ by applied $4\times 45 = 180$ different projective transformations. Since the automorphism group of a generic projective cubic is trivial we conclude that $\mathcal{S}$ is identified a surface $\overline{S}_{A,B,C,D}$ for 180 different choices of parameters (A,B,C,D).

Note that if the generic cubic $\mathcal{S}$ is identified with $\overline{S}_{A,B,C,D}$ and also with $\overline{S}_{A',B',C',D'}$ by using two different choices of tritangent planes of $\mathcal{S}$, then the corresponding actions of $\mathcal{G}_{A,B,C,D}$ and $\mathcal{G}_{A',B',C',D'}$ are not typically conjugate because they are defined with respect to affine coordinates corresponding to different choices of "plane at infinity".

4.3. **Singular points of $S_{A,B,C,D}$.** As explained in Section 4.1, for any parameters (A,B,C,D) we have that $\overline{S}_{A,B,C,D}$ is smooth on a neighborhood of Π_∞. Therefore $\overline{S}_{A,B,C,D}$ is singular if and only if the affine surface $S_{A,B,C,D}$ is so (and the corresponding singular sets always coincide). In particular, the singular set of $\overline{S}_{A,B,C,D}$ is a compact subvariety of $\mathbb{C}^3$ and it is therefore finite.

The surface $S_{A,B,C,D}$ is singular if and only if at least one of the following conditions is satisfied (see, e.g. [4, 36]):

- We have $a_i = \pm 2$ for at least one $i \in \{1,2,3,4\}$.
- The coefficients $a_1,\ldots,a_4$ satisfy the relation

$$[2(a_1^2+a_2^2+a_3^2+a_4^2) - a_1a_2a_3a_4 - 16]^2 - (4-a_1^2)(4-a_2^2)(4-a_3^2)(4-a_4^2) = 0\,.$$

(The parameters a_1, a_2, a_3 and a_4 were introduced in Section 3.2 and their relationship to A, B, C, and D was described there.)

Since $\mathcal{G}$ must preserve $\operatorname{Sing}(S_{A,B,C,D})$, it that $\mathcal{G}$ has a finite orbit whenever $\operatorname{Sing}(S_{A,B,C,D}) \neq \emptyset$. The existence of a finite orbit for $\mathcal{G}$ is naturally yields some insights in the dynamics of $\mathcal{G}$, as will be seen in the course of this work.

Remark 4.1. Every singular point p of $S_{A,B,C,D}$ lies in the Julia set $\mathcal{J}_{A,B,C,D}$. Indeed, if U is a neighborhood of p, [13, Theorem C] implies the existence of a point $q \in U$ and of a sequence $\gamma_n \in \mathcal{G}$ such that $\gamma_n(q)$ diverges to infinity. Meanwhile, since every element of $\mathcal{G}$ permutes the singular set of $S_{A,B,C,D}$, we have that $\gamma_n(p)$ remains bounded.

4.4. **Algebraic properties of $\mathcal{G}^{\pm}_{A,B,C,D}$ and $\mathcal{G}_{A,B,C,D}$.** The group of automorphisms of $\mathbb{C}^3$ generated by s_x, s_y, and s_z is isomorphic to the free product $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. In particular, the group of automorphisms generated by g_x, g_y, and g_z is free on two generators. However, a stronger statement holds: for every choice of the parameters A, B, C, and D, the group $\mathcal{G}^{\pm}$ is isomorphic to the indicated free product and the group $\mathcal{G}$ is free on two generators, *when viewed as a group of automorphisms of* $S_{A,B,C,D}$.

The non-existence of additional relations between the maps s_x, s_y, and s_z even when restricted to a particular surface $S_{A,B,C,D}$ is a consequence of El-Huiti's theorem in [23] – albeit not an immediate one. For details the reader can check [9] and [13].

4.5. **Invariant volume form.** Considering s_x, s_y, and s_z as self-mappings of $\mathbb{C}^3$, a simple calculation yields:

$$s_x^*(dx \wedge dy \wedge dz) = s_y^*(dx \wedge dy \wedge dz) = s_z^*(dx \wedge dy \wedge dz) = -dx \wedge dy \wedge dz.$$

Therefore elements of the index two subgroup $\mathcal{G}_{A,B,C} = \langle g_x, g_y, g_z \rangle < \mathrm{Aut}(\mathbb{C}^3)$ preserve $dx \wedge dy \wedge dz$ and also the associated Euclidean volume form $dx \wedge d\bar{x} \wedge dy \wedge d\bar{y} \wedge dz \wedge d\bar{z}$ on $\mathbb{C}^3$.

The smooth part of $S_{A,B,C,D}$ comes equipped with a holomorphic volume form

$$\Omega = \frac{dx \wedge dy}{2z + xy - C} = \frac{dy \wedge dz}{2x + yz - A} = \frac{dz \wedge dx}{2y + zx - B} \tag{10}$$

which is obtained by contracting the form $dx \wedge dy \wedge dz$ with the gradient of the polynomial function on $\mathbb{C}^3$ that defines $S_{A,B,C,D}$. We again have $s_x^* \Omega = -\Omega$ and similarly for s_y and s_z. Therefore, the elements of $\mathcal{G}_{A,B,C,D} < \mathrm{Aut}(S_{A,B,C,D})$ preserve Ω and hence also preserve the associated real volume form $\Omega \wedge \overline{\Omega}$. It assigns infinite volume to the whole surface $S_{A,B,C,D}$ but finite volume to a sufficiently small neighborhood of each singular point of $S_{A,B,C,D}$.

Let p be a smooth point of $S_{A,B,C,D}$. An immediate consequence of the existence of the invariant volume form is that if p is fixed for $\gamma \in \mathcal{G}_{A,B,C,D}$ then $\det(D\gamma(p)) = 1$. In particular, if p is a hyperbolic fixed point for γ, then it must be of saddle type.

4.6. **Pencils of rational curves.** Recall that $\pi_x : \mathbb{C}^3 \to \mathbb{C}$ sends $(x, y, z) \in \mathbb{C}^3$ to $x \in \mathbb{C}$. Given $x_0 \in \mathbb{C}$, let

$$\Pi_{x=x_0} = \{(x, y, z) \in \mathbb{C}^3 \, : \, \pi_x(x, y, z) = x_0\} \qquad \text{and} \qquad S_{x=x_0} = S_{A,B,C,D} \cap \Pi_{x=x_0}.$$

Let $\overline{\Pi}_{x=x_0}$ be the closure of $\Pi_{x=x_0}$ in $\mathbb{CP}^3$ and let $\overline{S}_{x=x_0}$ denote the closure of $S_{x=x_0}$ in $\overline{S}_{A,B,C,D}$. Since $\overline{S}_{x=x_0}$ has degree two in $\overline{\Pi}_{x=x_0} \cong \mathbb{CP}^2$, it is uniformized by the Riemann Sphere provided that $\overline{S_{x_0}}$ is smooth. The statement remains valid in the case where $\overline{S}_{x=x_0}$ is singular (i.e. a union of two lines with a single simple intersection) up to passing from $\overline{S}_{x=x_0}$ to its normalization.

Denoting by π_y and π_z the projections of $\mathbb{C}^3$ to $\mathbb{C}$ respectively defined by $\pi_y(x, y, z) = y$ and $\pi_z(x, y, z) = z$, the fibers $\Pi_{y=y_0}, S_{y=y_0}, \Pi_{z=z_0}$, and $S_{z=z_0}$ are analogously defined.

Clearly the collection of rational curves obtained from $\overline{S}_{x=x_0}$, $x_0 \in \mathbb{C}$, defines a *rational pencil* in $\overline{S}_{A,B,C,D}$. By performing finitely many blow-ups, this pencil becomes a singular rational fibration $\mathcal{D}_x$, with connected fibers, over a suitable Riemann surface $\widetilde{\overline{S}}_{A,B,C,D}$. Naturally, there are analogous pencils $\mathcal{D}_y$ and $\mathcal{D}_z$ defined on $\overline{S}_{A,B,C,D}$ with the help of the collection of rational curves $\overline{S}_{y=y_0}$ and $\overline{S}_{z=z_0}$ contained in $\overline{S}_{A,B,C,D}$.

Let us close this section with an elementary lemma, whose simple proof will be omitted.

Lemma 4.2. *For all but finitely many values of $x_0 \in \mathbb{C}$, the rational curve $\overline{S}_{x=x_0}$ is smooth and intersects the plane at infinity Π_∞ of $\mathbb{CP}^3$ in two distinct points. Analogous statements hold for the rational curves $\overline{S}_{y=y_0}$ and $\overline{S}_{z=z_0}$.*

For the pencil $\mathcal{D}_x$ on $\overline{S}_{A,B,C,D}$, we fix a (minimal) blow-up procedure turning $\mathcal{D}_x$ into a (singular) rational fibration $\widetilde{\mathcal{D}}_x$. Clearly there are only finitely many values of $x_0 \in \mathbb{C}$ corresponding to singular fibers of this fibration. We then define a finite set $\mathcal{B}_x^+ \subset \mathbb{C}$ by saying that $x_0 \in \mathcal{B}_x$ if at least one of the following conditions fails to hold:

- The rational curve $\overline{S}_{x=x_0}$ is smooth and intersects the plane at infinity at two distinct points.
- The fibration $\widetilde{\mathcal{D}}_x$ is regular on a neighborhood of the fiber sitting over x_0.

We also define $\mathcal{B}_x \subseteq \mathcal{B}_x^+$ to be the set of points at which the first of the two conditions above fails to hold. The corresponding sets for the coordinates y and z will similarly be denoted by $\mathcal{B}_y^+$, $\mathcal{B}_y$ and by $\mathcal{B}_z^+$, $\mathcal{B}_z$.

5. Dynamics of parabolic maps and proof of Theorems A, B, and C

We begin by studying the generators $g_x = s_z \circ s_y, g_z = s_x \circ s_z$, and $g_z = s_y \circ s_x$ of $\mathcal{G}$. Here these mappings are explicitly written as

$$g_x\,(x,y,z) = \left(x,\, -y - xz + B,\, xy + (x^2-1)z + C - Bx\right),$$
$$g_y\,(x,y,z) = \left((y^2-1)x + yz + A - Cy,\, y,\, -yx - z + C\right), \quad \text{and}$$
$$g_z\,(x,y,z) = \left(-x - yz + A,\, zx + (z^2-1)y + B - Az,\, z\right).$$

The map g_x preserves the coordinate x and hence each affine plane $\Pi_{x=x_0} \subset \mathbb{C}^3$. Since, in addition, g_x clearly preserves the (affine) surface $S_{A,B,C,D}$, it follows that g_x individually preserves each one of the curves $S_{x=x_0}$ (and hence also the rational curves $\overline{S}_{x=x_0} \subset \overline{S}_{A,B,C,D}$). Similar conclusions hold for the maps g_y and g_z.

We can now complement the discussion in Section 4 revolving around Lemma 4.2. Owing to these statements, we know that the rational curve $\overline{S}_{x=x_0}$ is smooth and intersects the divisor at infinity Π_∞ of $\mathbb{CP}^3$ in two distinct points for all but finitely many values of $x_0 \in \mathbb{C}$. It is then natural to consider the automorphism of $\overline{S}_{x=x_0}$ induced by g_x. The following proposition can be found in [13] (see in particular Proposition 4.1 in the paper in question). The proof is elementary, in the spirit of most of the discussion in the previous section.

Proposition 5.1. *Let $x_0 \in \mathbb{C}$ be such that the rational curve $\overline{S}_{x=x_0}$ is smooth and intersects the divisor at infinity Π_∞ in two distinct points. Then the restriction $\overline{g_{x_0}}$ of g_x to $\overline{S}_{x=x_0}$ is a Möbius transformation whose two fixed points are at infinity. Furthermore, we have:*

- *If $x_0 \in (-2, 2)$ then $\overline{g_{x_0}}$ is elliptic. It is periodic if and only if $x_0 = \pm 2\cos(\theta\pi)$ with θ rational.*
- *If $x_0 \in \mathbb{C} \setminus [-2, 2]$ then $\overline{g_{x_0}}$ is loxodromic.*

The analogous statements hold for restrictions of g_y to the fibers $\overline{S}_{y=y_0}$ and of g_z to the fibers $\overline{S}_{z=z_0}$.

Remark 5.2. The case $x_0 = \pm 2$ deserves an additional comment for the sake of clarity. First note that the affine singular points of the curves $\overline{S}_{x=x_0}$ are contained in the set of points where the intersection of $S_{A,B,C,D}$ and $\Pi_{x=x_0}$ is not transverse. They are obtained by solving for where the second and third components of the gradient of the defining polynomial for $S_{A,B,C,D}$ both vanish. A short calculation shows that they lie in the affine curve given by

$$x_0 \longrightarrow \left(x_0, \frac{Cx_0 - 2B}{x_0^2 - 4}, \frac{Bx_0 - 2C}{x_0^2 - 4}\right).$$

We see that this curve intersects the plane at infinity for $x_0 = \pm 2$. In particular, the affine curve $S_{x=2}$ (resp. $S_{x=-2}$) is always smooth. On the other hand, this curve intersects the plane $\Pi_\infty \subset \mathbb{CP}^3$ at a single point (with multiplicity 2) which, depending on the coefficients, may or may not be a singular point of $\overline{S}_{x=2}$ (resp. $\overline{S}_{x=-2}$). The restriction $\overline{g_2}$ of g_x to $\overline{S}_{x=2}$ is as follows:

(1) When $\overline{S}_{x=2}$ is smooth, then $\overline{g_2}$ is a parabolic map whose single fixed point coincides with the intersection of $\overline{S}_{x=2}$ with Π_∞.

(2) Otherwise $\overline{S}_{x=2}$ consists of the union of two projective lines intersecting each other at a point in Π_∞. Each line is then preserved by $\overline{g_2}$ and, in each of these lines, $\overline{g_2}$ induces a parabolic map whose fixed point coincides with their intersection.

The analogous statement holds for $\overline{g_{-2}}$ and $\overline{S}_{x=-2}$.

Proposition 5.1 yields the following lemma:

Lemma 5.3. *If $x_0 \in (-2,2) \setminus \mathcal{B}_x$ then $S_{x=x_0} \subset \mathcal{J}(g_x)$. Analogous statements hold for the g_y and g_z mappings.*

Proof. Suppose that U is an open neighborhood of a point $p \in S_{x=x_0}$. Because $x_0 \in (-2,2) \setminus \mathcal{B}_x$, Proposition 5.1 ensures that g_x restricted to the fiber $S_{x=x_0}$ is elliptic with both fixed points lying in Π_∞. In particular, the iterates $g_x^n(p)$ remain bounded. Meanwhile, since U is open in S there exists a point $q \in U$ with $x_1 = \pi_x(q) \notin [-2,2] \cup \mathcal{B}_x$. According to Proposition 5.1, the restriction of g_x to the fiber $\overline{S}_{x=x_1}$ is hyperbolic with both fixed points at infinity. Therefore the orbit $g_x^n(q)$ tends to infinity. Hence U cannot be contained in the Fatou set of g_x since it contains points with both bounded and unbounded orbits. The lemma follows. □

We will need the following elementary lemma, whose simple proof is omitted.

Lemma 5.4. *Fix any $x_0 \in \mathbb{C}$. Then, for all but finitely many choices of y_0 the fibers $S_{x=x_0}$ and $S_{y=y_0}$ intersect transversally. When the fibers intersect transversally, they do so at two distinct points.*

Now fix a point $x_0 \in \mathbb{C} \setminus \mathcal{B}_x^+$. For sufficiently small $\epsilon > 0$, consider the "tube" $T^\epsilon_{x=x_0}$ defined by

$$T^\epsilon_{x=x_0} = \{(x,y,z) \in S \,:\, |x - x_0| < \epsilon\} \tag{11}$$

Clearly $T^\epsilon_{x=x_0}$ is filled (foliated) by the curves $S_{x=x_1}$ where x_1 satisfies $|x_1 - x_0| < \epsilon$. Let $y_0 \in \mathbb{C}$ be such that the curve $S_{y=y_0}$ intersects the curves $S_{x=x_1} \subset T^\epsilon_{x=x_0}$ transversally.

Lemma 5.5. *With the preceding notation and up to choosing $\epsilon > 0$ sufficiently small, the open set $T^\epsilon_{x=x_0} \setminus S_{y=y_0}$ is Kobayashi hyperbolic.*

Proof. Since $\mathcal{B}_x^+$ is finite and $x_0 \in \mathbb{C} \setminus \mathcal{B}_x^+$, there is $\epsilon > 0$ such that $\mathbb{D}_\epsilon = \{x_1 \in \mathbb{C}\,;\ |x_1 - x_0| < \epsilon\}$ is contained in $\mathbb{C} \setminus \mathcal{B}_x^+$. Now, in view of the definition of $\mathcal{B}_x^+$, for every $S_{x=x_1} \subset T^\epsilon_{x=x_0}$, the corresponding rational curve $\overline{S}_{x=x_1}$ intersects Π_∞ transversally and at two distinct points. In other words, $\overline{S}_{x=x_1} \setminus S_{x=x_1}$ consists of two distinct points provided that $S_{x=x_1} \subset T^\epsilon_{x=x_0}$. On the other hand, up to a birational transformation, the projective curves $\overline{S}_{x=x_1}$ define a (regular) holomorphic fibration over the disc $\mathbb{D}_\epsilon$. Because the fibers are rational curves and therefore pairwise isomorphic as Riemann surfaces, the theorem of Fischer and Grauert [26] implies that this fibration is holomorphically trivial, i.e., it is holomorphically equivalent to $\mathbb{D}_\epsilon \times \mathbb{CP}^1$. Since $\overline{S}_{x=x_1} \setminus S_{x=x_1}$ consists of two points, there also follows that

$$T^\epsilon_{x=x_0} = \mathbb{D}_\epsilon \times \mathbb{C} \setminus \{0\} \tag{12}$$

as complex manifolds.

The hypothesis that $S_{y=y_0}$ intersects $S_{x=x_0}$ transversally implies that their intersection consists of exactly two (distinct) points (Lemma 5.4). We can then reduce $\epsilon > 0$, if necessary, so that $S_{y=y_0}$ intersects each $S_{x=x_1}$ from $T^\epsilon_{x=x_0}$ transversally in two points, each of them varying holomorphically with x_1. These points will be referred to as the two branches of $S_{y=y_0}$ in $T^\epsilon_{x=x_0}$. Since each branch is the graph of a holomorphic function on x_1, we can pick either one of them and construct a further holomorphic diffeomorphism to make it correspond to the point $1 \in \mathbb{C} \setminus \{0\}$. By means of this construction, $T^\epsilon_{x=x_0} \setminus S_{y=y_0}$ becomes identified with an open set in $\mathbb{D}_\epsilon \times \mathbb{C} \setminus \{0,1\}$. The lemma follows since the latter manifold is Kobayashi hyperbolic as the product of two hyperbolic Riemann surfaces. □

Lemma 5.6. *Let $x_0 \in (-2, 2) \setminus \mathcal{B}_x^+$ and let y_0 be any point chosen so that $S_{x=x_0}$ and $S_{y=y_0}$ intersect transversally. For any open $U \subset S$ with $U \cap S_{x=x_0} \neq \emptyset$ there is an iterate n such that $g_x^n(U) \cap S_{y=y_0} \neq \emptyset$.*

Analogous statements hold when x and y are replaced with any two distinct variables from $\{x, y, z\}$.

Proof. Owing to Lemma 5.5, we fix a tube $T_{x=x_0}^{\epsilon}$ as in (11) so that $T_{x=x_0}^{\epsilon} \setminus S_{y=y_0}$ is Kobayashi hyperbolic. Let then U be a non-empty open set of $S_{A,B,C,D}$ intersecting $S_{x=x_0}$. Up to trimming U, we can assume that $U \subset T_{x=x_0}^{\epsilon}$. Since g_x preserves the $S_{x=\mathrm{const}}$ fibration and fixes the points in $\overline{S}_{x=\mathrm{const}} \cap \Pi_\infty$ (g_x has no poles), it follows that $g_x^n(U)$ remains in $T_{x=x_0}^{\epsilon}$ for every $n \in \mathbb{Z}$. Now assume for a contradiction that $g_x^n(U) \subset T_{x=x_0}^{\epsilon} \setminus S_{y=y_0}$ for every n. Since $T_{x=x_0}^{\epsilon} \setminus S_{y=y_0}$ is Kobayashi hyperbolic, this implies that $\{g_x^n\}$ forms a normal family on U. This is, however, impossible since U intersects $S_{x=x_0}$ and $S_{x=x_0} \subset \mathcal{J}(g_x)$ (Lemma 5.3). Thus there must exist n such that $g_x^n(U) \cap S_{y=y_0} \neq \emptyset$ and the lemma follows. □

Remark 5.7. The above proof actually shows slightly more than the statement of Lemma 5.6. Once $\epsilon > 0$ is chosen sufficiently small so that $T_{x=x_0}^{\epsilon}$ intersects $S_{y=y_0}$ in two branches (each a graph of a holomorphic function of x) and once U is chosen sufficiently small so that $U \subset T_{x=x_0}^{\epsilon}$ then for each branch of $S_{y=y_0}$ in $T_{x=x_0}^{\epsilon}$ there is an iterate n so that $g_x^n(U)$ intersects the branch in question.

Lemma 5.6 admits a useful quantitative version that can directly be proved, namely:

Lemma 5.8. *Assume that $x_0 = \pm 2\cos(\theta\pi) \in (-2, 2) \setminus \mathcal{B}_x^+$ with θ irrational and let $U \subset S$ be an open set such that $U \cap S_{x=x_0} \neq \emptyset$. Next consider a sequence $\{q_j = (x_j, y_j, z_j)\} \subset S_{A,B,C,D} \setminus S_{x=x_0}$ converging to some $q \in S_{x=x_0}$ and assume, in addition, the existence of $\delta > 0$ such that the argument of x_j lies in an interval of the form $[\delta, \pi - \delta] \cup [\pi + \delta, 2\pi - \delta]$ for every j. Then for every j large enough, there exists $n_j \in \mathbb{Z}$ such that $g_x^{n_j}(q_j) \in U$.*

Analogous statements hold when x is replaced by y or z.

Proof. Recall that g_x preserves every leaf of the foliation $\mathcal{D}_x$ induced by the collection of rational curves $\overline{S}_{x=\mathrm{const}} \subset \overline{S}_{A,B,C,D}$. The assumption on x_0 implies that g_x on the curve $\overline{S}_{x=x_0}$ corresponds to an elliptic element that is conjugate to an irrational rotation. Thus, up to saturating U by the dynamics of g_x, there is no loss of generality in assuming that U is a disc bundle over an annulus $A \subset S_{x=x_0}$, where A is invariant under g_x. Now note that the action g_{x_j} of g_x on $S_{x=x_j}$ is loxodromic since $x_j \notin \mathbb{R}$. Furthermore the multiplier at the attracting fixed point of g_{x_j} has modulus tending to 1 as $q_j \to q$. In particular, for j large enough, the annulus A contains a fundamental domain of g_{x_j} on $S_{x=x_j} \simeq \mathbb{CP}^1$. The lemma follows at once. □

The preceding lemmas are summarized by the proposition below.

Proposition 5.9. *Let $x_0 \in (-2, 2) \setminus \mathcal{B}_x^+$. Given two open sets $U_1, U_2 \subset S$, both of which intersect $S_{x=x_0}$, there is an iterate n such that $g_x^n(U_1) \cap U_2 \neq \emptyset$. Analogous statements hold when x_0 is replaced with y_0 or z_0.*

Proof. As in the proof of Lemma 5.6, we work within the tube $T_{x=x_0}^{\epsilon}$ defined in Equation (12). Since U_2 is open and $U_2 \cap S_{x=x_0} \neq \emptyset$ they have infinitely many points of intersection. We can therefore pick a point $(x_0, y_0, z_0) \in U_2 \cap S_{x=x_0}$ so that $S_{y=y_0}$ intersects $S_{x=x_0}$ transversally (Lemma 5.4). We can make $\epsilon > 0$ smaller, if necessary, so that $S_{y=y_0} \cap T_\epsilon(x_0)$ is expressed as two smooth branches, each of them coinciding with the graph of a holomorphic function of x. Let $S_{y=y_0}(\epsilon)$ denote the branch passing through (x_0, y_0, z_0). Since U_2 is open, it follows that $U_2 \cap S_{y=y_0}(\epsilon)$ is an open neighborhood of (x_0, y_0, z_0) in $S_{y=y_0}(\epsilon)$. This means that we can reduce $\epsilon > 0$ even further, if necessary, so that we can assume the entire branch $S_{y=y_0}(\epsilon)$ is contained in U_2.

With the above setting, the combination of Lemma 5.6 and Remark 5.7 ensures the existence of n such that $g_x^n(U_1) \cap S_{y=y_0}(\epsilon) \neq \emptyset$. Since $S_{y=y_0}(\epsilon) \subset U_2$, the proposition follows. □

Recalling that the sets $\mathcal{B}_x^+$, $\mathcal{B}_y^+$, and $\mathcal{B}_z^+$ are all finite, Lemma 5.4 allows us to choose points $x_0 \neq x_1 \in (-2,2) \setminus \mathcal{B}_x$, $y_0 \neq y_1 \in (-2,2) \setminus \mathcal{B}_y$, $z_0 \neq z_1 \in (-2,2) \setminus \mathcal{B}_z$ and to form the "grid"

$$\mathbb{G} = S_{x=x_0} \cup S_{x=x_1} \cup S_{y=y_0} \cup S_{y=y_1} \cup S_{z=z_0} \cup S_{z=z_1} \tag{13}$$

so that

(i) every pair of curves (fibers) have transverse intersection (possibly empty), and
(ii) each x_0, x_1, y_0, y_1, z_0, and z_1 is of the form $\pm 2\cos(\theta\pi)$ for some irrational θ.

We will also say that any pair of irreducible components of $\mathcal{G}$ with empty intersection in $S_{A,B,C,D}$ are "parallel", e.g. $S_{x=x_0}$ and $S_{x=x_1}$ are parallel.

Proposition 5.10. *Let U be any open set in $S_{A,B,C,D}$ that intersects the grid $\mathbb{G}$. Then, for any irreducible component of the grid (say $S_{z=x_0}$) there exists $\gamma \in \mathcal{G}$ with $\gamma(U)$ intersecting that chosen irreducible component (say $\gamma(U) \cap S_{z=x_0} \neq \emptyset$).*

Proof. It is clearly enough to show the existence of $\gamma \in \mathcal{G}$ with $\gamma(U) \cap S_{z=x_0} \neq \emptyset$. If U already has non-trivial intersection with $S_{x=x_0}$ then there is nothing to prove. Otherwise, there are two cases to be considered:

Case 1: U intersects an irreducible components of $\mathbb{G}$ that is not parallel to $S_{x=x_0}$. Without loss of generality, we can suppose U intersects $S_{y=y_0}$. Lemma 5.6 then implies the existence of an iterate g_y^n of g_y so that $g_y^n(U) \cap S_{x=x_0} \neq \emptyset$.

Case 2: U intersects the component $S_{x=x_1}$ that is parallel to $S_{x=x_0}$. In this case, we first apply Lemma 5.6 to find an iterate g_x^n of g_x such that $g_x^n(U) \cap S_{y=y_0} \neq \emptyset$. Hence the problem is reduced to the situation treated in Case 1 so that it suffices to proceed accordingly. □

Corollary 5.11. *Let U_1 and U_2 be any two open sets in $S_{A,B,C,D}$ both of which intersect the grid $\mathbb{G}$. Then, there exists $\gamma \in \mathcal{G}$ with $\gamma(U_1) \cap U_2 \neq \emptyset$.*

Proof. Using Propositition 5.10 we can find some $\gamma_1 \in \mathcal{G}$ so that $\gamma_1(U_1)$ and U_2 both intersect the same irreducible component of $\mathbb{G}$. The result then follows immediately from Proposition 5.9. □

Remark 5.12. Concerning Corollary 5.11, note that the non-empty intersection $\gamma(U_1) \cap U_2$ may be disjoint from $\mathbb{G}$ (and hence it might be disjoint from $\mathcal{J}_{A,B,C,D}$ as well).

Proposition 5.13 below is the last ingredient in the proof of Theorem A. Notwithstanding its very elementary nature, this proposition is likely to find further applications in the study of the dynamics associated with the group $\mathcal{G}$.

Proposition 5.13. *For any open set U intersecting the Julia set $\mathcal{J}_{A,B,C,D}$ of $\mathcal{G}$ non-trivially, there exists some element $\gamma \in \mathcal{G}$ such that $\gamma(U) \cap \mathbb{G} \neq \emptyset$.*

Proof. Assume aiming at a contradiction that $\gamma(U)$ is disjoint from $\mathbb{G}$ for every $\gamma \in \mathcal{G}$. Then, for every $\gamma \in \mathcal{G}$ we have that

$$\iota \circ \gamma(U) \subset \big(\mathbb{C} \setminus \{x_0, x_1\}\big) \times \big(\mathbb{C} \setminus \{y_0, y_1\}\big) \times \big(\mathbb{C} \setminus \{z_0, z_1\}\big),$$

where $\iota : S_{A,B,C,D} \hookrightarrow \mathbb{C}^3$ denotes the inclusion. Applying Montel's Theorem to each coordinate, this implies that the whole group $\mathcal{G}$ forms a normal family on U. This contradicts the assumption that $U \cap \mathcal{J}_{A,B,C,D} \neq \emptyset$ and establishes the statement. □

Proof of Theorem A. The proof is based on Baire's argument. First note that $\mathcal{J}_{A,B,C,D}$ has the Baire property since it is a complete metric space as a closed subset of a manifold. The topology in $\mathcal{J}_{A,B,C,D}$ is the one inherited from the topology of $S_{A,B,C,D}$ and hence it is second countable, i.e., there is a countable basis $\{V_k\}_{k=1}^{\infty}$ for the topology of $\mathcal{J}_{A,B,C,D}$. By definition the open sets $V_k \subset \mathcal{J}_{A,B,C,D}$ are given by

$$V_k = U_k \cap \mathcal{J}_{A,B,C,D}\,, \tag{14}$$

where U_k is an open set of $S_{A,B,C,D}$. To prove the theorem, it suffices to show that the restriction to $\mathcal{J}_{A,B,C,D}$ of the action of $\mathcal{G}$ is topologically transitive in the basis $\{V_k\}_{k=1}^{\infty}$. Namely, given k_1 and k_2, we need to prove the existence of $\gamma \in \mathcal{G}$ such that $\gamma(V_{k_1}) \cap V_{k_2} \neq \emptyset$. In fact, assuming the existence of these elements γ, for every $n \in \mathbb{N}$, consider the set

$$\bigcup_{\gamma \in \mathcal{G}} \gamma^{-1}(V_n)$$

formed by all points in $\mathcal{J}_{A,B,C,D}$ whose orbit intersects V_n. This set is clearly open since V_n is so. It is also dense in $\mathcal{J}_{A,B,C,D}$ since it intersects non-trivially every set V_k defining a basis for the topology of $\mathcal{J}_{A,B,C,D}$. Taking then the intersection over n

$$\bigcap_{n=1}^{\infty} \bigcup_{\gamma \in \mathcal{G}} \gamma^{-1}(V_n)$$

we obtain a G_δ-dense subset of $\mathcal{J}_{A,B,C,D}$. By definition, the $\mathcal{G}$-orbit of any point in this intersection visits all the open sets V_k so that these points have dense orbits in $\mathcal{J}_{A,B,C,D}$.

It remains to show that for any two open sets V_{k_1} and V_{k_2} from our basis there exists $\gamma \in \mathcal{G}$ satisfying $\gamma(V_{k_1}) \cap V_{k_2} \neq \emptyset$.

We start by working with the corresponding open sets U_{k_1} and U_{k_2} of S. We first use Proposition 5.13 to find $\gamma_1, \gamma_2 \in \mathcal{G}$ such that $\gamma_1(U_{k_1})$ and $\gamma_2(U_{k_2})$ each hit the grid $\mathbb{G}$. We can then use Proposition 5.10 to find $\gamma_3 \in \mathcal{G}$ such that $\gamma_3 \circ \gamma_1(U_{k_1})$ and $\gamma_2(U_{k_2})$ intersect the same irreducible component of $\mathbb{G}$. Without loss of generality, we suppose it is $S_{x=x_0}$; i.e. that it is the first of the six irreducible components of $\mathbb{G}$ listed in (13).

Since $S_{y=y_0}$ is transverse to $S_{x=x_0}$, we can choose a sequence of points $\{q_j\}_{j=1}^{\infty} \subset S_{y=y_0}$ converging to $q \in S_{x=x_0} \cap S_{y=y_0}$ that satisfies the hypotheses of Lemma 5.8. Moreover, by Lemma 5.3, $S_{y=y_0} \subset \mathcal{J}_{A,B,C,D}$ so each element of the sequence is in $\mathcal{J}_{A,B,C,D}$. We therefore find a point $q_N \in \mathcal{J}_{A,B,C,D}$ and $\gamma_4, \gamma_5 \in \mathcal{G}$ with $\gamma_4(q_N) \in \gamma_3 \circ \gamma_1(U_{k_1})$ and $\gamma_5(q_N) \in \gamma_2(U_{k_2})$. In other words,

$$q_N \in \gamma_4^{-1} \circ \gamma_3 \circ \gamma_1(U_{k_1}) \quad \bigcap \quad \gamma_5^{-1} \circ \gamma_2(U_{k_2}) \quad \bigcap \quad \mathcal{J}_{A,B,C,D}.$$

Since $\mathcal{J}_{A,B,C,D}$ is invariant under $\mathcal{G}$, this proves that

$$\gamma_4^{-1} \circ \gamma_3 \circ \gamma_1(V_{k_1}) \cap \gamma_5^{-1} \circ \gamma_2(V_{k_2}) \neq \emptyset.$$

We conclude that $\gamma(V_{k_1}) \cap V_{k_2} \neq \emptyset$ with $\gamma = \gamma_2^{-1} \circ \gamma_5 \circ \gamma_4^{-1} \circ \gamma_3 \circ \gamma_1$. □

Remark 5.14. After finding $\gamma_1, \gamma_2 \in \mathcal{G}$ such that $\gamma_1(U_{k_1})$ and $\gamma_2(U_{k_2})$ each hit the grid $\mathbb{G}$, it is tempting to use Corollary 5.11 to find γ_3 with

$$\gamma_3(\gamma_1(U_{k_1})) \cap \gamma_2(U_{k_2}) \neq \emptyset. \tag{15}$$

However, this does not necessarily prove that $\gamma_3(\gamma_1(V_{k_1})) \cap \gamma_2(V_{k_2}) \neq \emptyset$ because the intersection (15) need not be in $\mathbb{G}$ and hence it potentially might not contain any points of $\mathcal{J}_{A,B,C,D}$; see Remark 5.12. This is why we use the "quantitative" Lemma 5.8 instead.

Proof of Theorem B. Let us define a new "grid" $\mathbb{G}'$ using Equation (13), but this time we will use

$$x_0 = y_0 = z_0 = 0 \qquad \text{and} \qquad x_1 = y_1 = z_1 = \sqrt{2}.$$

These values are chosen so that

$$g_x^2|_{S_{x=x_0}} = \text{id}, \quad g_y^2|_{S_{y=y_0}} = \text{id}, \quad g_z^2|_{S_{z=z_0}} = \text{id}, \qquad g_x^4|_{S_{x=x_1}} = \text{id}, \quad g_y^4|_{S_{y=y_1}} = \text{id}, \quad \text{and} \quad g_z^4|_{S_{z=z_1}} = \text{id}.$$

As in Proposition 5.13, if U is any open set that intersects $\mathcal{J}_{A,B,C,D}$ non-trivially then there is an element $\gamma \in \mathcal{G}$ with $\gamma(U)$ intersecting $\mathbb{G}'$. In fact, the proof of Proposition 5.13 does not use the choices that $x_0, x_1 \notin \mathcal{B}_x$, $y_0, y_1 \notin \mathcal{B}_y$, or $z_0, z_1 \notin \mathcal{B}_z$ that were made in the construction of our original grid $\mathbb{G}$ so that it applies equally well to $\mathbb{G}'$.

Conjugating by γ, if necessary, it then suffices to prove that shear fixed points are dense in our newly chosen grid $\mathbb{G}'$. We will prove it for $S_{x=x_0}$ and $S_{x=x_1}$ and leave the completely analogous proofs for $S_{y=y_0}, S_{z=z_0}, S_{y=y_1}$ and $S_{z=z_1}$ to the reader.

Every point of $S_{x=x_0}$ is a fixed point for g_x^2. We will show that all but finitely many of them are shear fixed points. Clearly this assertion is, in turn, equivalent to showing that the derivative $D(g_x)^2$ has two dimensional generalized eigenspace associated to eigenvalue 1 but only one eigenvector associated to eigenvalue 1. Considering g_x^2 as a mapping from $\mathbb{C}^3 \to \mathbb{C}^3$ we have

$$D(g_x^2)|_{x=0} = \begin{bmatrix} 1 & 0 & 0 \\ 2\,z - C & 1 & 0 \\ B - 2\,y & 0 & 1 \end{bmatrix}.$$

If $z \neq C/2$ or $y \neq B/2$ then this matrix has generalized eigenspace of dimension 3 associated to the eigenvalue 1 but only two eigenvectors, namely $\boldsymbol{e}_2 = [0,1,0]$ and $\boldsymbol{e}_3 = [0,0,1]$. Therefore, it suffices to prove that

$$T_p S_{A,B,C,D} \neq \mathrm{span}(\boldsymbol{e}_2, \boldsymbol{e}_3) \tag{16}$$

for every $p \in S_{x=x_0}$ bar some finite set. Taking the gradient of the defining equation for $S_{A,B,C,D}$ yields that

$$T_p S_{A,B,C,D} = \ker \left[\ yz - A + 2\,x \quad zx - B + 2\,y \quad xy - C + 2\,z\ \right].$$

Restricted to $x = x_0 = 0$ we can only have $\boldsymbol{e}_2 \in T_p S_{A,B,C,D}$ if $y = B/2$ and we can only have $\boldsymbol{e}_3 \in T_p S_{A,B,C,D}$ if $z = C/2$. Combined with $x = 0$ each of these conditions amounts to at most two points of $S_{x=x_0}$. Therefore, all but at most finitely many points of $S_{x=x_0}$ are shear fixed points of g_x^2.

The situation for $S_{x=x_1}$ is essentially the same, except that one must work with g_x^4. We leave the details to the reader. □

Proof of Theorem C. Let p_1 and p_2 be arbitrary points in $\mathcal{J}_{A,B,C,D}$. We will show that for any neighborhoods U_1 of p_1 and U_2 of p_2, there is a path in $\mathcal{J}_{A,B,C,D}$ from U_1 to U_2. Theorem C will immediately follow.

The grid $\mathbb{G}$ given in (13) is path connected and, by virtue of Lemma 5.3, we have $\mathbb{G} \subset \mathcal{J}_{A,B,C,D}$. Therefore, it suffices to find a path from U_1 to $\mathbb{G}$ and a path from U_2 to $\mathbb{G}$. As the situation is symmetric, it suffices to consider U_1.

Since $p_1 \in U_1 \cap \mathcal{J}_{A,B,C,D}$, Proposition 5.13 gives some $\gamma \in \mathcal{G}$ such that $\gamma(U_1) \cap \mathbb{G} \neq \emptyset$. Let C be an irreducible component of $\mathbb{G}$ with $\gamma(U_1) \cap C \neq \emptyset$. Since we have chosen the irreducible components of $\mathbb{G}$ to be smooth, C is biholomorphic to $\mathbb{C} \setminus \{0\}$.

Consider now the Riemann surface $\gamma^{-1}(C)$ contained in $S_{A,B,C,D} \subset \mathbb{C}^3$. Since γ is a holomorphic diffeomorphism of $S_{A,B,C,D}$, it follows that $\gamma^{-1}(C)$ is again biholomorphic to $\mathbb{C} \setminus \{0\}$ and hence it is uniformized by $\mathbb{C}$ and contained in $\mathbb{C}^3$. We claim that $\gamma^{-1}(C)$ intersects the grid $\mathbb{G}$. Indeed, if we had $\gamma^{-1}(C) \cap \mathbb{G} = \emptyset$, the uniformization map from $\mathbb{C}$ to $\gamma^{-1}(C)$ would yield a (non-constant) holomorphic map from $\mathbb{C}$ to $\mathbb{C}^3$ each of whose coordinates omits two values in $\mathbb{C}$. Picard's Theorem would then imply that this map must be constant and this is impossible.

Finally, note that $\gamma^{-1}(C) \subset \mathcal{J}_{A,B,C,D}$ since $\mathcal{J}_{A,B,C,D}$ is invariant by $\mathcal{G}$ and $C \subset \mathbb{G} \subset \mathcal{J}_{A,B,C,D}$. Furthermore, by construction, $\gamma^{-1}(C)$ also intersects U_1. Since $\gamma^{-1}(C)$ is path connected, we can therefore find a path contained in $\gamma^{-1}(C) \subset \mathcal{J}_{A,B,C,D}$ going from U_1 to $\mathbb{G}$. The proof of Theorem C is complete. □

6. Picard parameters and proof of Theorem D

The parameters $(A,B,C,D) = (0,0,0,4)$ are quite special for at least two reasons:

- The surface $S_{(0,0,0,4)}$ has the maximal number of singularities among all cubic surfaces and for this reason it is called the *Cayley Cubic*. They are at the four points

$$\{(-2,-2,-2),\, (-2,2,2),\, (2,-2,2),\, (2,2,-2)\}. \tag{17}$$

- It was proved by Cantat-Loray [13, Theorem 5.4] that $\mathcal{G}$ has an invariant affine structure on $S_{A,B,C,D}$ if and only if $(A,B,C,D) = (0,0,0,4)$.

An *affine structure* on a complex surface consists of a collection of coordinate charts whose transition functions are (restrictions of) affine mappings of $\mathbb{C}^2$. One says that a group G *preserves an affine structure* if the expression of each element of G in the preferred collection of charts (associated to the specified affine structure) consists again of affine mappings. Existence of the invariant affine structure dates back to work of Picard on the Painlevé 6 equation corresponding to the parameters $(A,B,C,D) = (0,0,0,4)$. It is for this reason that the parameters $(A,B,C,D) = (0,0,0,4)$ are called the *Picard Parameters*.

From our point of view, this case is also very interesting as it will soon be clear. More importantly, however, the information collected in the course of this discussion will enable us to prove Proposition 10.9 in Section 10. Albeit a somewhat technical statement, Proposition 10.9 plays an important role in the proofs of Theorems G and K.

Throughout this section we will typically drop the parameters from our notation, writing $S \equiv S_{0,0,0,4}$, $\mathcal{G} \equiv \mathcal{G}_{0,0,0,4}$, $\mathcal{J} \equiv \mathcal{J}_{0,0,0,4}$, and so on. The singular locus of S will be denoted by $S_{\rm sing}$.

Proposition 6.1. *For the Picard parameters, $\mathcal{G}$ acts locally discretely on any open $U \subset S$.*

We will use the existence of a semi-conjugacy between the action of $\mathcal{G}$ and the group action of monomial mappings on $\mathbb{C}^* \times \mathbb{C}^*$, which we describe now (see for example [13, Section 1.5]). Consider the following two matrix groups

(18)

$$\widetilde{\Gamma}_2 := \{M \in {\rm SL}(2,\mathbb{Z}) \,:\, M \equiv {\rm Id} \bmod 2\} \qquad \text{and} \qquad \Gamma_2 := \{[M] \in {\rm PSL}(2,\mathbb{Z}) \,:\, M \equiv {\rm Id} \bmod 2\}.$$

The square brackets around M in the definition of Γ_2 denote that we take the equivalence class modulo multiplication by $\pm{\rm Id}$. Note that Γ_2 is the famous congruence subgroup which has well-known generating set consisting of $[M_x]$ and $[M_y]$ where

$$M_x := \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad M_y := \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}. \tag{19}$$

See, for example, [19, Section 16.3] or Exercises 6 and 7 from [66, Chapter 13]. Meanwhile, $\widetilde{\Gamma}_2$ is generated by M_x, M_y, and $-{\rm Id}$.

There is a group isomorphism from Γ_2 to $\mathcal{G}$ induced by sending $[M_x]$ to g_x and $[M_y]$ to g_y. More generally, we denote the image of any $[M] \in \Gamma_2$ under this isomorphism by $f_{[M]} \in \mathcal{G}$. This isomorphism can be seen directly, but it also fits nicely within the context of dynamics on character varieties; see [13, Section 2.3] for more details.

Associated with a matrix $M = \{m_{ij}\} \in \widetilde{\Gamma}_2$ is a monomial mapping $\eta_M : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^* \times \mathbb{C}^*$ given by

$$\eta_M\,(u,v) = (u^{m_{11}} v^{m_{12}}, u^{m_{21}} v^{m_{22}})\,.$$

Also let $\Phi : \mathbb{C}^* \times \mathbb{C}^* \to S$ be defined by

$$\Phi(u,v) = (-u - 1/u, -v - 1/v, -u/v - v/u)\,.$$

It turns out that Φ is a degree two orbifold cover. Furthermore, a straightforward verification shows that that the critical points of Φ are precisely the four points $(u, v) = (\pm 1, \pm 1)$ while the corresponding critical values are the four points of S_{sing}.

Proposition 6.2. *Φ semi-conjugates the action of $\widetilde{\Gamma}_2$ on $\mathbb{C}^* \times \mathbb{C}^*$ to the action of $\mathcal{G}$ on S. More specifically, given $M \in \widetilde{\Gamma}_2$ and $(u, v) \in \mathbb{C}^* \times \mathbb{C}^*$, we have*

$$\Phi \circ \eta_M(u, v) = f_{[M]} \circ \Phi(u, v). \tag{20}$$

Proof. One can directly check Formula (20) for the three generators of $\widetilde{\Gamma}_2$. □

Proof of Proposition 6.1. Suppose that there is an open $U \subset S$ and a sequence $[M_n] \in \Gamma_2 \setminus \{[\text{Id}]\}$ such that $f_{[M_n]}|_U$ converges locally uniformly to the identity on U. Making U smaller, if necessary, we can assume that U is evenly covered by Φ and that V is one of the two connected components of $\Phi^{-1}(U)$. Then, up to appropriately choosing the matrix M_n representing $[M_n]$, the monomial maps η_{M_n} converge locally uniformly to the identity on V. However, this is impossible because $\widetilde{\Gamma}_2$ is a discrete subgroup of $SL(2, \mathbb{Z})$. Since the action is question is linear, from the discrete character of $\widetilde{\Gamma}_2$ it follows that the induced action on pairs $(\log |u|, \log |v|)$ is locally discrete as well. □

The semiconjugacy from Proposition 6.2 also allows us to completely determine the Julia set associated with Picard parameters. Namely, we have:

Proposition 6.3. *The Julia set $\mathcal{J}_{0,0,0,4}$ is the whole surface S.*

The proof of Proposition 6.3 will, however, require the following lemma:

Lemma 6.4. *The set formed by the union over all hyperbolic elements of $\widetilde{\Gamma}_2$ of the corresponding eigendirections is dense in $\mathbb{P}^1(\mathbb{R})$.*

Proof. Conjugating by elements of $\widetilde{\Gamma}_2$, the problem reduces to considering directions between $(1, 0)^T$ and $(1, 1)^T$. Let p be any prime number and let $1 \le q \le p - 1$ be any even number. Then, there exist positive integers a and b such that $bp - aq = 1$. Since q is even, b is odd. If a is odd then we can replace a and b by $p + a$ and $q + b$, respectively. This allows us to assume that

$$\begin{pmatrix} p & a \\ q & b \end{pmatrix} \in \widetilde{\Gamma}_2.$$

Choosing p sufficiently large, it is a consequence of the Perron-Frobenius Theorem that this matrix has an eigenvector whose direction is arbitrarily close to $(p, q)^T$. Finally, by appropriately choosing q, every vector $(v_1, v_2)^T$ between $(1, 0)^T$ and $(1, 1)^T$ can be approximated. The lemma follows. □

Proof of Proposition 6.3. It suffices to prove that the Julia set J for the monomial action on $\mathbb{C}^* \times \mathbb{C}^*$ is all of $\mathbb{C}^* \times \mathbb{C}^*$. Indeed, suppose there is an open $U \subset S$ contained in the Fatou set for the action of $\mathcal{G}$ on S. Making U smaller, if necessary, we can assume that U is evenly covered by Φ and that V is one of the two connected components of $\Phi^{-1}(U)$. Because of the semi-conjugacy Φ, the assumption on U would imply that V is in the Fatou set for the monomial action of $\widetilde{\Gamma}_2$ on $\mathbb{C}^* \times \mathbb{C}^*$.

The unit torus $\mathbb{T}^2 \subset \mathbb{C}^* \times \mathbb{C}^*$ is invariant under the monomial action of $\widetilde{\Gamma}_2$ with the hyperbolic elements of $\widetilde{\Gamma}_2$ corresponding to Anosov mappings $\eta_M : \mathbb{T}^2 \to \mathbb{T}^2$. Therefore, $\mathbb{T}^2 \subset J$. Moreover, for each hyperbolic $M \in \widetilde{\Gamma}_2$ the hyperbolic set $\mathbb{T}^2$ has stable and unstable manifolds under η_M, namely:

$$\mathcal{W}^s_M(\mathbb{T}^2) = \{(u, v) \in \mathbb{C}^* \times \mathbb{C}^* \, : \, (\log |u|, \log |v|)^T \text{ is a stable eigenvector for } M\}, \text{ and}$$
$$\mathcal{W}^u_M(\mathbb{T}^2) = \{(u, v) \in \mathbb{C}^* \times \mathbb{C}^* \, : \, (\log |u|, \log |v|)^T \text{ is an unstable eigenvector for } M\}.$$

These invariant manifolds are of real-dimension three. Note that $\mathcal{W}^s_M(\mathbb{T}^2)$ is in the Julia set for the monomial mapping η_M associated to M and $\mathcal{W}^u_M(\mathbb{T}^2)$ is in the Julia set for η_M^{-1}. It follows from

Lemma 6.4 that the union of these stable and unstable manifolds is dense. Since J is closed, it must be all of $\mathbb{C}^* \times \mathbb{C}^*$. $\square$

In the remainder of this section, we will focus on points stabilized by non-trivial elements of $\mathcal{G}$. As mentioned, the discussion below will allow us to prove Proposition 10.9.

Let $S(\mathbb{R}) = S \cap \mathbb{R}^3$ denote the real slice of S. It is well known that $S(\mathbb{R}) \setminus S_{\rm sing}$ consists of one bounded component and three unbounded components (see [4]). Let $S(\mathbb{R})_0$ denote the closure of the bounded component of $S(\mathbb{R}) \setminus S_{\rm sing}$. It is straightforward to check that

$$S(\mathbb{R})_0 = S \cap [-2, 2]^3 = \Phi(\mathbb{T}^2).$$

For every $M \in \widetilde{\Gamma}_2$ and any fixed point $p \in \mathbb{C}^* \times \mathbb{C}^*$ of η_M, there exist suitable local coordinates (complexified angular coordinates) in which $D\eta_M(p) = M$. In particular, the eigenvalues of $D\eta_M(p)$ and of M coincide. Moreover, by noticing that $(\eta_M)^k = \eta_{M^k}$ for every integer $k > 1$, it follows that the analogous statement holds for periodic points as well.

Note also that when $M \in \widetilde{\Gamma}_2$ is parabolic, η_M may have fixed points away from of the real torus $\mathbb{T}^2 \subset \mathbb{C}^* \times \mathbb{C}^*$. However, by the preceding discussion, the derivative $D\eta_m(p)$ at such a fixed point will always have eigenvalues equal to ± 1. When $M \in \widetilde{\Gamma}_2$ is hyperbolic, any fixed point p of η_M is on $\mathbb{T}^2$ and the fixed point is a saddle, with one of the eigenvalues of $D\eta_M(p)$ having absolute value less than one and the other having absolute value greater than one.

Lemma 6.5. *Let $M \in \widetilde{\Gamma}_2$ and let $f_{[M]} : S \to S$ be the associated element of $\mathcal{G}$. If p is a smooth point of S and a fixed point of $f_{[M]}$ then the eigenvalues of $Df(p)$ have the same absolute values as the eigenvalues of M.*

Proof. The critical values of Ψ are precisely the singular point of S, which are permuted by elements of $\mathcal{G}$. Therefore, $\Psi^{-1}(p) = \{q_1, q_2\}$ and they will either each be a fixed point for η_M or they will form a period two cycle for η_M. In either case $(Df_{[M]}(p))^2$ and $D\eta_M(q_2)D\eta_M(q_1)$ will be conjugate matrices and hence have the same eigenvalues. Meanwhile $D\eta_M(q_2)D\eta_M(q_1)$ and M^2 have the same eigenvalues, so the result follows. $\square$

Proposition 6.6. *For the Picard parameters, whenever M is a hyperbolic matrix, every fixed point of the corresponding mapping $f_{[M]}$ not lying in $S_{\rm sing}$ must be a hyperbolic saddle. In addition, these fixed points are all located on $S(\mathbb{R})_0 = \Phi(\mathbb{T}^2)$.*

In particular, there is no dense subset $\mathcal{J}^*_{0,0,0,4} \subset \mathcal{J}_{0,0,0,4}$ consisting of points with hyperbolic stabilizers.

Proof. It follows from Lemma 6.5 and from the discussion in the paragraph before this lemma that a hyperbolic fixed point of an arbitrary element in $\mathcal{G}$, in fact, must be a fixed point of some mapping $f_{[M]}$, where M is hyperbolic. Every such fixed point is therefore a hyperbolic saddle and is contained in $S(\mathbb{R})_0$. Clearly $S(\mathbb{R})_0$ is a proper subset of S which, in turn, coincides with $\mathcal{J}$ in view of Proposition 6.3. The proposition follows. $\square$

Proof of Theorem D. It follows directly from the combination of Propositions 6.1, 6.3, and 6.6. $\square$

Recall that, by construction, every mapping $f_{[M]} : S \to S$ is the restriction of a polynomial diffeomorphism of $\mathbb{C}^3$ which will be denoted by $F_{[M]} : \mathbb{C}^3 \to \mathbb{C}^3$. These maps $F_{[M]}$ leave invariant all the surfaces of the form $S_{0,0,0,D}$, with $D \in \mathbb{C}$. From this it follows that if $p \in S_{0,0,0,4}$ is a fixed point of $f_{[M]}$ then two of the eigenvalues of the 3×3 matrix $DF_{[M]}(p)$ are the same as those of $Df_{[M]}(p)$ and the third eigenvalue is 1, provided that p is a regular point of $S_{0,0,0,D}$. Owing to Lemma 6.5, we conclude that two of the eigenvalues of $DF_{[M]}(p)$ have the same absolute values as the eigenvalues of M and the remaining eigenvalue is 1.

The following proposition describes the eigenvalues of $DF_{[M]}(p)$ at singular points p of S. The fact that the eigenvalues of M are squared is essentially the same phenomenon that occurs for the classical one-dimensional Chebeyshev map, and we are grateful to Michał Misiurewicz for explaining it to us.

Proposition 6.7. *Let (A, B, C, D) be the Picard Parameters $(0, 0, 0, 4)$. For any $M \in \widetilde{\Gamma}_2$ let $f_{[M]} : S \to S$ be the corresponding element of $\mathcal{G}$ and let $F_{[M]} : \mathbb{C}^3 \to \mathbb{C}^3$ be its extension to $\mathbb{C}^3$. For any $p \in S_{\rm sing}$ two of the eigenvalues of $DF_{[M]}(p)$ are the squares of the eigenvalues of M and the remaining eigenvalue is* 1*.*

Proof. The proof relies upon the semiconjugacy Ψ from Proposition 6.2. We apply it to points $(u, v) = (e^{i\theta}, e^{i\phi}) \in \mathbb{T}^2$ and abuse notation slightly by writing

$$(x, y, z) = \Psi(\theta, \phi) = (-2\cos\theta, -2\cos\phi, -2\cos(\theta - \phi)).$$

The points $(0, 0), (0, \pi), (\pi, 0)$, and (π, π) map by Ψ to the singular points in (17) in the respective order that they are listed there.

Here we focus on the singular point $p = (-2, -2, -2) = \Psi(0, 0)$. The minor adaptations required for the other singular points essentially amount to some sign modifications in the equations below and thus can safely be left to the reader.

Since $M \in \widetilde{\Gamma}_2$ we have $\det(M) = 1$. Using this, the characteristic polynomial for M^2 is

$$P_{M^2}(x) = x^2 - (m_{11}^2 + m_{22}^2 + 2m_{12}m_{21})\ x + 1,$$

where m_{jk} denotes jk-th entry of M.

Let $N = DF_{[M]}(p)$ and recall from Section 4.5 that any element of $\mathcal{G}_{A,B,C} = \langle g_x, g_y, g_z \rangle < \mathrm{Aut}(\mathbb{C}^3)$ preserves the Euclidean volume form on $\mathbb{C}^3$. This implies that $\det(N) = 1$. Moreover, $F_{[M]}(S_D) = S_D$ for every $D \in \mathbb{C}$, implying that $Q \circ F_{[M]}(x, y, z) = Q(x, y, z)$ for the polynomial $Q(x, y, z) = x^2 + y^2 + z^2 + xyz$. This gives that one of the eigenvalues of N equals 1.

Hence,the characteristic polynomial of N is

$$P_N(x) = x^3 - (n_{11} + n_{22} + n_{33})\ x^2 + (n_{11} + n_{22} + n_{33})\ x - 1\,,$$

where n_{jk} denotes jk-th entry of N. In the sequel we will show that

$$n_{11} + n_{22} + n_{33} = m_{11}^2 + m_{22}^2 + 2m_{12}m_{21} + 1\,. \tag{21}$$

This will imply that $P_N(x) = P_{M^2}(x)(x - 1)$ therefore completing the proof of Proposition 6.7.

To begin, consider the x-coordinate of the semi-conjugacy (20):

$$-2\cos(m_{11}\theta + m_{12}\phi) = F_{[M],1}(-2\cos\theta, -2\cos\phi, -2\cos(\theta - \phi)),$$

where we have added the subscript 1 to denote the first coordinate of $F_{[M]}$. Setting $\phi = 0$ and taking the partial derivative with respect to θ yields

$$2\sin(m_{11}\theta)m_{11} = \frac{\partial F_{[M],1}}{\partial x}(-2\cos\theta, -2, -2\cos\theta)2\sin\theta + \frac{\partial F_{[M],1}}{\partial z}(-2\cos\theta, -2, -2\cos\theta)2\sin\theta.$$

Next, for $\theta \neq 0$, we divide both sides of the above equation by 2θ so as to obtain

$$\frac{\sin(m_{11}\theta)}{m_{11}\theta}m_{11}^2 = \frac{\partial F_{[M],1}}{\partial x}(-2\cos\theta, -2, -2\cos\theta)\frac{\sin\theta}{\theta} + \frac{\partial F_{[M],1}}{\partial z}(-2\cos\theta, -2, -2\cos\theta)\frac{\sin\theta}{\theta}\,.$$

Now it suffices to take the limit as θ goes to 0 to conclude that $m_{11}^2 = n_{11} + n_{13}$.

Similarly, setting $\theta = 0$ and doing the analogous computation involving partial derivatives with respect to ϕ yields $m_{12}^2 = n_{12} + n_{13}$. Finally, the analogous computation with $\phi = \theta$ lead to $(m_{11} + m_{12})^2 = n_{11} + n_{12}$. The three previous equations can be solved for n_{11} to find

$$n_{11} = m_{11}^2 + m_{11}m_{12}. \tag{22}$$

The same computations with the second and third coordinate of the semi-conjugacy (20) yield

$$n_{22} = m_{22}^2 + m_{22}m_{21}, \qquad \text{and} \qquad n_{33} = (m_{11} - m_{21})(m_{22} - m_{12}). \tag{23}$$

Combined with the fact that $\det(M) = 1$, Equations (22) and (23) imply that the condition expressed by (21) holds. The proof of the proposition is completed. □

Corollary 6.8. *Let (A, B, C, D) be the Picard Parameters $(0, 0, 0, 4)$. Assume that $M \in \widetilde{\Gamma}_2$ is hyperbolic. Denote by $f_{[M]} : S \to S$ the element of $\mathcal{G}$ associated with M and let $F_{[M]} : \mathbb{C}^3 \to \mathbb{C}^3$ be the corresponding extension of $f_{[M]}$ to $\mathbb{C}^3$. If $p \in S$ is a fixed point of $F_{[M]}$ then $DF_{[M]}(p)$ has one eigenvalue of modulus less than one, one eigenvalue equal to one, and one eigenvalue of modulus greater than one.*

7. Existence of Fatou Components and Proof of Theorem E

Recall from Section 1.7 that for fixed choice of (A, B, C) the equations for s_x, s_y, and s_z can be interpreted as polynomial automorphisms of $\mathbb{C}^3$. We denote the group of automorphisms of $\mathbb{C}^3$ generated by s_x, s_y, and s_z by $\mathcal{G}^{\pm}_{A,B,C}$. We can consider the Fatou set of this action on $\mathbb{C}^3$ and denote it by $\mathcal{F}^{\pm}_{A,B,C}$. As in Section 1.5, convergence to infinity is again allowed in our definition of normal families.

The first step toward proving Theorem E will be Proposition 7.1, below, where for any choice of parameters (A, B, C) will provide a point $p_0 \in \mathbb{C}^3$ and $\epsilon > 0$ so that the ball $B_\epsilon(p_0)$ of radius ϵ around p_0 is contained in $\mathcal{F}_{A,B,C}$. Then to obtain 4-tuples of parameters $(A, B, C, D) \in \mathbb{C}^4$ for which the action of $\mathcal{G}_{A,B,C,D}$ on $S_{A,B,C,D}$ has non-empty Fatou set $\mathcal{F}_{A,B,C,D}$, it will be enough to select D so that $S_{A,B,C,D} \cap B_\epsilon(p_0) \neq \emptyset$; see Corollary 7.2.

Proposition 7.1. *For any choice of parameters $(A, B, C) \in \mathbb{C}^3$, let $r = \max\{|A|, |B|, |C|\}$. Next, given $R > 2 + \sqrt{r}$, let*

$$\epsilon = \min\{R - (2 + \sqrt{r}), R + 1 - \sqrt{4\,R + r + 1}\} > 0. \tag{24}$$

If $|u| = R$ and $p_0 = (u, u, u) \in \mathbb{C}^3$ then the open ball $B_\epsilon(p_0)$ is contained in the Fatou set $\mathcal{F}^{\pm}_{A,B,C}$ for the action of $\mathcal{G}^{\pm}_{A,B,C}$ on $\mathbb{C}^3$. In particular $\mathcal{F}^{\pm}_{A,B,C} \neq \emptyset$.

The idea for the proof of Proposition 7.1 comes from the papers of Bowditch [7], Tan, Wong, and Zhang [63], Maloni, Palesi, and Tan [45], and Hu, Tan, and Zhang [34]. See Section 3.2 for more details.

Proof. Let $p = (x, y, z) \in B_\epsilon(p_0)$ and note that this implies that each coordinate of p has modulus larger than $2 + \sqrt{r}$. We will show that for any integer $k \geq 1$, and any reduced word $w_k w_{k-1} \dots w_1$ in the mappings s_x, s_y, and s_z that each coordinate of $w_k w_{k-1} \dots w_1(p)$ has modulus at least as large as the corresponding coordinate for $w_{k-1} \dots w_1(p)$. In particular, this will imply that

$$w_k w_{k-1} \dots w_1 \left(B_\epsilon(p_0)\right) \subset \left(\mathbb{C} \setminus \overline{\mathbb{D}_{2+\sqrt{r}}(0)}\right)^3$$

for any such word of any length $k \geq 1$. Applying Montel's Theorem to each coordinate implies that the action of $\mathcal{G}^{\pm}$ is normal on $B_\epsilon(p_0)$ so that the statement follows.

We first check that our claim holds for $k = 1$. Consider the involution s_x and the points p and $s_x(p)$. Clearly the coordinates y and z of these two points coincide. To show that the modulus of the x coordinate of $s_x(p)$ is strictly larger than the modulus of the x coordinate of p, note that

$$|\pi_x(s_x(p))| = |-yz - x + A| > (R - \epsilon)^2 - (R + \epsilon) - r \geq R + \epsilon > |\pi_x(p)|\,, \tag{25}$$

where the second inequality follows from the assumption that $\epsilon \leq R + 1 - \sqrt{4\,R + r + 1}$. Indeed, this condition can be reformulated as $R - \epsilon + 1 \geq \sqrt{4\,R + r + 1}$ which, by taking squares on both sides, leads right away to the inequality in question. Naturally, analogous estimates hold with respect to the coordinates y or z when s_x is replaced by s_y and s_z. Therefore, we have shown that for every point $p \in B_\epsilon(p_0)$, applying s_x, s_y, or s_z to p strictly increases the modulus of one of the coordinates while leaving the other two coordinates unchanged.

We now prove the claim for arbitrary $k \geq 2$ by means of contradiction. We therefore assume that $k \geq 2$ is the smallest index for which there is a reduced word $w_k w_{k-1} w_{k-2} \ldots w_1$ such that some coordinate of $w_{k-1} w_{k-2} \ldots w_1(p)$ has has modulus strictly larger than the corresponding coordinate of $w_k w_{k-1} w_{k-2} \ldots w_1(p)$. Note also that taking k to be minimal implies that each coordinate of $w_{k-1} \ldots w_1(p)$ has modulus greater than or equal to the minimal modulus of a coordinate of p which, in turn, exceeds $2 + \sqrt{r} = 2 + \sqrt{\max\{|A|, |B|, |C|\}}$. Let

$$(x, y, z) = w_{k-2} \ldots w_1(p), \qquad (x', y', z') = w_{k-1} \ldots w_1(p), \quad \text{and} \quad (x'', y'', z'') = w_k \ldots w_1(p).$$

(If $k = 2$ then we interpret $w_{k-2} \ldots w_1$ as the identity mapping.) In particular, the preceding ensures that $\min\{|x'|, |y'|, |z'|\} \geq 2 + \sqrt{r}$. On the other hand, since the word w is reduced and all generators s_x, s_y, s_z are involutions, we must have $w_{k-1} \neq w_k$. Without loss of generality, we can then suppose $w_{k-1} = s_x$ and $w_k = s_y$. This yields

$$(x', y', z') = (-x - yz + A, y, z), \qquad \text{and} \qquad (x'', y'', z'') = (x', -y' - x'z' + B, z').$$

Our assumption on k implies $|x| \leq |x'|$ and $|y'| > |y''|$. Therefore,

$$\begin{aligned} &2|x'| \geq |x' + x| = |-yz + A| = |-y'z' + A| \qquad \text{and} \\ &2|y'| > |y' + y''| = |-x'z' + B|. \end{aligned} \tag{26}$$

We now split the discussion in two cases. Assume first that $|x'| \geq |y'|$. Then, the second inequality from (26) gives

$$|x'z'| - |B| \leq 2|y'| \leq 2|x'|.$$

In turn, moving $|B|$ to the right side, dividing by $|x'|$, and recalling that $|x'| \geq 2 + \sqrt{r}$ leads to

$$|z'| \leq 2 + \frac{|B|}{|x'|} < 2 + \frac{r}{2 + \sqrt{r}} < 2 + \sqrt{r}.$$

This is impossible since $\min\{|x'|, |y'|, |z'|\} \geq 2 + \sqrt{r}$. If we consider now the case where $|y'| > |x'|$, we just need to use the first inequality from (26) to similarly show that

$$|z'| < 2 + \frac{|A|}{|y'|} < 2 + \sqrt{r}\,.$$

Thus, in any event, we obtain a contradiction that proves our initial claim.

In summary, for any $p \in B_\epsilon(p_0)$ we have shown that for any integer $k \geq 1$, and any reduced word $w_k w_{k-1} \ldots w_1$ in the mappings s_x, s_y, and s_z that each coordinate of $w_k w_{k-1} \ldots w_1(p)$ has modulus at least as large as the corresponding coordinate for $w_{k-1} \ldots w_1(p)$. As already pointed out, this implies that the ball $B_\epsilon(p)$ must therefore lie in the Fatou set of the action of $\mathcal{G}^{\pm}_{A,B,C}$ on $\mathbb{C}^3$. □

Corollary 7.2. *For any parameters $(A_0, B_0, C_0) \in \mathbb{C}^3$ suppose that $p_0 = (u, u, u)$ with $|u| > 2 + \sqrt{r}$, where r is given as in Proposition 7.1. Let D_0 be chosen so that $p_0 \in S_{A_0,B_0,C_0,D_0}$.*

Then there is some $\delta > 0$ such that for all parameters $(A, B, C, D) \in \mathbb{B}_\delta(A_0, B_0, C_0, D_0) \subset \mathbb{C}^4$ the Fatou set $\mathcal{F}_{A,B,C,D}$ for the action of $\mathcal{G}_{A,B,C,D}$ on $S_{A,B,C,D}$ is non-empty.

Proof. The condition on $\epsilon > 0$ given in (24) depends continuously on $r = \max\{|A|, |B|, |C|\}$. Therefore there exists $\delta_0 > 0$ and $\epsilon_0 > 0$ such that if $(A, B, C) \in \mathbb{B}_{\delta_0}(A_0, B_0, C_0) \subset \mathbb{C}^3$ then $B_{\epsilon_0}(p_0) \subset \mathcal{F}^{\pm}_{A,B,C} \subset \mathbb{C}^3$. On the other hand, the point (x, u, u) lies in the surface $S_{A,B,C,D}$ where

$$D = x^2 + 2u^2 + xu^2 - Ax - Bu - Cu.$$

This polynomial is monic and non-constant in x so that its roots vary continuously with (A, B, C, D). Since it has a root at $x = u$ when $(A, B, C, D) = (A_0, B_0, C_0, D_0)$ we can find some $0 < \delta < \delta_1$ such that if $(A, B, C, D) \in \mathbb{B}_\delta(A_0, B_0, C_0, D_0)$ then

$$(x, u, u) \in B_{\epsilon_0}(p_0) \cap S_{A,B,C,D} \subset \mathcal{F}^{\pm}_{A,B,C} \cap S_{A,B,C,D} \subset \mathcal{F}_{A,B,C,D}.$$

□

The proof of Theorem E will be a quick application of Corollary 7.2 combined with the following elementary lemma whose proof we leave to the reader.

Lemma 7.3. *For every $D \in \mathbb{C} \setminus \{4\}$, the polynomial $q(u) = u^3 + 3u^2 = D$ has a solution with modulus strictly larger than* 2.

Proof of Theorem E. Consider first the Punctured Torus Parameters $A = B = C = 0$. The condition for a point $p_0 = (u, u, u)$ to lie in $S_{0,0,0,D_0}$ is $u^3 + 3u^2 = D_0$. If $D_0 \neq 4$, Lemma 7.3 ensures that there is a point $p_0 = (u, u, u) \in S_{0,0,0,D_0}$ with $|u| > 2$. Corollary 7.2 ensures the existence of $\delta > 0$ such that for all $(A, B, C, D) \in \mathbb{B}_\delta((0, 0, 0, D_0))$ the Fatou set $\mathcal{F}_{A,B,C,D}$ is non-empty. This establishes the first part of Theorem E.

Now consider the Dubrovin-Mazzocco parameters $A(a) = B(a) = C(a) = 2a + 4$, and $D(a) = -(a^2 + 8a + 8)$ for $a \in (-2, 2)$. Let us denote the surface $S_{A,B,C,D}$ at these parameters by S_a. The condition for (u, u, u) to belong to S_a is given by

$$q_a(u) = u^3 - 3\,(2\,a + 4)\,u + 3\,u^2 + a^2 + 8\,a + 8 = 0\,.$$

A direct calculation shows that if you substitute $u = -(2 + \sqrt{2\,a + 4}) = -(2 + \sqrt{r})$ into $q_a(u)$ the result is positive. Hence, there is a real $u_0 < -(2 + \sqrt{r})$ satisfying $q(u_0) = 0$. Hence, again Corollary 7.2 implies that for any $a \in (-2, 2)$ there exists $\delta > 0$ such that for all $(A, B, C, D) \in \mathbb{B}_\delta((A(a), B(a), C(a), D(a)))$ the Fatou set $\mathcal{F}_{A,B,C,D}$ is non-empty. The proof of Theorem E is complete. □

8. Locally non-discrete dynamics in $\mathcal{G}_{A,B,C,D}$.

Let M be a (possibly open) connected complex manifold and consider a group G of holomorphic diffeomorphisms of M. The group G is said to be *locally non-discrete* on an open $U \subset M$ if there is a sequence of maps $\{f_n\}_{n=0}^{\infty} \in G$ satisfying the following conditions (see for example [55]):

(1) For every n, f_n is different from the identity.
(2) The sequence of maps f_n converges uniformly to the identity on compact subsets of U.

If no such sequence of maps $\{f_n\}_{n=0}^{\infty} \in G$ exists then G is said to be *locally discrete on* U. The reader will note that a group G may be locally non-discrete on one open set U and locally discrete on a disjoint open set V.

We remind the reader from Section 1.4 that we denote the open subset of $S_{A,B,C,D}$ on which $\mathcal{G}_{A,B,C,D}$ is locally non-discrete (the "locally non-discrete locus") by $\mathcal{N}_{A,B,C,D}$ and its complement (the "locally discrete locus") by $\mathcal{D}_{A,B,C,D}$.

Let us return for a while to a more general context. It is convenient to begin our discussion with Proposition 8.1 below. This proposition constitutes a simple general result of which specific variants appear in [55, p. 9-10] and in [44, Sec. 3] while the main idea dates back to Ghys [28]. Both Proposition 8.1 and Lemma 8.2, below, are stated in the wider context of pseudogroups G of holomorphic maps of open sets of M to M. In this context, the definition of G being locally non-discrete on U becomes:

(0) The open set U is contained in the domain of definition of f_n (as an element of the pseudogroup G), for every n.
(1) For every n, the restriction of f_n to U is different from the identity.
(2) The sequence of maps f_n converges uniformly to the identity on compact subsets of U.

Condition (0) is required for Conditions (1) and (2) to make sense and Condition (1) has been modified because of the possibility that U not be connected.

Consider a ball $B_\epsilon(0) \subset \mathbb{C}^n$ of radius $\epsilon > 0$ around the origin of $\mathbb{C}^n$. Assume we are given local holomorphic diffeomorphisms $F_1, F_2 : B_\epsilon(0) \to \mathbb{C}^n$ and denote by G the pseudogroup of maps from $B_\epsilon(0)$ to $\mathbb{C}^n$ generated by F_1, F_2. Naturally the inverses of F_1, F_2 are respectively denoted by F_1^{-1} and F_2^{-1}. In what follows we can assume without loss of generality that the domain of definition of F_1^{-1} and F_2^{-1} as elements of G is non-empty. Let us then define a sequence $S(n)$ of sets of elements in G by letting $S(0) = \{F_1, F_1^{-1}, F_2, F_2^{-1}\}$. The sets $S(n)$ are now inductively defined by stating that $S(n+1)$ is constituted by all elements of the form $[\gamma_i, \gamma_j] = \gamma_i \circ \gamma_j \circ \gamma_i^{-1} \circ \gamma_j^{-1}$ with $\gamma_i, \gamma_j \in S(n)$. Note that the construction of these elements is so far purely formal in the sense that the domain of definition (contained in $B_\epsilon(0)$) of diffeomorphisms in $S(n)$ viewed as elements of G may be empty. Nonetheless, we have:

Proposition 8.1. *Given $\epsilon > 0$, there is $K = K(\epsilon) > 0$ such that, if*

$$\left\{ \sup_{z \in B_\epsilon(0)} \|F_1(z) - z\| \,,\ \sup_{z \in B_\epsilon(0)} \|F_2(z) - z\| \right\} < K \,,$$

then the following hold:

1. *For every n and every $\gamma \in S(n)$, the domain of definition of γ as element in G contains the ball $B_{\epsilon/2}(0) \subset \mathbb{C}^n$ of radius $\epsilon/2$ around the origin.*
2. *Furthermore, if γ belongs to $S(n)$ then we have $\sup_{p \in B_{\epsilon/2}(0)} \|\gamma(p) - p\| \le \frac{K}{2^n}$.*

Note that, in general, the above proposition falls short of implying that the pseudogroup G is locally non-discrete since the sets $S(n)$ may degenerate so as to only contain the identity map.

As mentioned, variants of Proposition 8.1 can be found in the literature and, to the best of our knowledge, the idea goes back to Ghys in [28]. Otherwise, its proof is nearly identical to that from [55, p. 9-10], so we provide only a sketch.

Proof of Proposition 8.1: The proof is based on the following estimate from [44, Lemma 3.0]. Let $B_\epsilon(0) \subset \mathbb{C}^n$ be an open ball and suppose $f_1, f_2 : B_\epsilon(0) \to \mathbb{C}^n$ are holomorphic local diffeomorphisms. If

$$\max \left\{ \sup_{z \in B_\epsilon} \left\| f_1^{\pm 1}(z) - z \right\|, \sup_{z \in B_\epsilon} \left\| f_2^{\pm 1}(z) - z \right\| \right\} \le K \tag{27}$$

for some $K > 0$, then for any $\tau > 0$ satisfying $4K + \tau < \epsilon$ the commutator $[f_1, f_2]$ is defined on the ball of radius $\epsilon - 4K - \tau$ and satisfies

$$\sup_{z \in B_{\epsilon - 4\delta - \tau}} \| [f_1, f_2](z) - z \| \le \frac{2}{\tau} \sup_{z \in B_\epsilon} \| f_1(z) - z \| \cdot \sup_{z \in B_\epsilon} \| f_2(z) - z \| \,. \tag{28}$$

Starting with $S(0)$, we choose $\tau = \tau_0 = K = K_0$ and $\epsilon_1 = \epsilon - 8K$ so that the preceding yields

$$\sup_{z \in B_{\epsilon_1}} \|\gamma(z) - z\| \le \frac{K}{2} \,,$$

for every $\gamma \in S(1)$. Now inductively setting $K_i = K/2^i$, $\tau_i = 4K_i$ it is straightforward to conclude that

$$\sup_{z \in B_{\epsilon_n}} \|\gamma(z) - z\| \le \frac{K}{2^n} \,,$$

for every $\gamma \in S(n)$, where

$$\epsilon_n = \epsilon - 8K - K \sum_{j=1}^{n-1} 2^{3-j}. \tag{29}$$

The proposition then follows by choosing K sufficiently small that $\epsilon_n \ge \epsilon/2$ for all n. □

The following simple lemma complements Proposition 8.1.

Lemma 8.2. *Let* $F_1, F_2 : B_\epsilon(0) \to \mathbb{C}^n$ *be as above with* $F_1(0) = F_2(0) = 0$*. Assume that their derivatives at the origin satisfy*

$$\max\{\|D_0F_1 - \mathrm{Id}\|, \|D_0F_2 - \mathrm{Id}\|\} < \tau, \tag{30}$$

where $\tau > 0$ *is some uniform (universal) to be determined later (the norm used here is the standard norm on linear operators on* $\mathbb{C}^n$*). Then, there is some* $0 < \delta < \epsilon$ *such that* F_1 *and* F_2 *satisfy the hypotheses of Proposition 8.1 on* $B_\delta(0)$*.*

Proof. It follows from the proof of Proposition 8.1 that the relation between ϵ and $K = K(\epsilon)$ given by (29) is linear. Thus for $\epsilon = 1$, $K = K(1)$ becomes a uniform (universal) constant. We then choose $\tau = K/2$ so that τ is also universal.

In view of the above remark, we proceed as follows. Fix $\delta < \epsilon$ and consider the homothety $\Lambda_\delta : \mathbb{C}^n \to \mathbb{C}^n$ sending $(x_1, \ldots, x_n)$ to $(\delta x_1, \ldots, \delta x_n)$. Set $F_{j,\delta} = \Lambda_\delta^{-1} \circ F_j \circ \Lambda_\delta$ and note that F_1, F_2 satisfy the conditions of Proposition 8.1 on the ball of radius δ if and only if we have

$$\max\left\{\sup_{z\in B_1}\|F_{1,\delta}(z) - z\|, \sup_{z\in B_1}\|F_{2,\delta}(z) - z\|\right\} < K = K(1)\,. \tag{31}$$

We will now check that Estimate (31) is always satisfied provided that δ is small enough. Clearly, it suffices to consider the case of $F_{1,\delta}$. Owing to Taylor's formula, we have

$$\sup_{z\in B_1}\|F_{1,\delta}(z) - z\| \le \|D_0F_1(z) - z\| + O(\delta) \le K/2 + O(\delta) < K$$

provided that δ is small enough. The result is proved. □

Let $\mathcal{ND} \subset \mathbb{C}^4$ the set of those parameters (A, B, C, D) giving rise to a group $\mathcal{G}_{A,B,C,D}$ acting locally non-discretely on some non-empty open subset of the surface $S_{A,B,C,D}$. The remainder of this section is devoted to exhibiting several explicit examples of parameters in the interior of $\mathcal{ND}$.

Proposition 8.3. *Suppose that* (A_0, B_0, C_0) *are parameters for which there are two non-commuting elements* $F_1, F_2 \in \mathcal{G}$ *sharing a common fixed point* $p \in \mathbb{C}^3$*. Assume also that the derivatives of* F_1, F_2 *at* p *satisfy inequality (30). Then the following holds:*

1. *There exists* $r > 0$ *such that for all parameters* (A, B, C) *sufficiently close to* (A_0, B_0, C_0) *the group* $\mathcal{G}$ *acting on* $\mathbb{C}^3$ *is locally non-discrete on the open ball* $B_r(p) \subset \mathbb{C}^3$*.*
2. *Let* D_0 *be chosen so that* $p \in S_{A_0,B_0,C_0,D_0}$*. Then for all parameters* (A, B, C, D) *sufficiently close to* (A_0, B_0, C_0, D_0) *the group* $\mathcal{G}$ *acting on* $S_{A,B,C,D}$ *is locally non-discrete on the open set* $S_{A,B,C,D} \cap B_r(p)$*. In other words,* (A_0, B_0, C_0, D_0) *is an interior point of* $\mathcal{ND}$*.*

Recall from the introduction that El-Huiti's theorem implies that $\mathcal{G}$ is the free group on two generators (one can choose any two of the three mappings g_x, g_y, and g_z as generators). Therefore, two elements $\gamma_1, \gamma_2 \in \mathcal{G}$ commute if and only if there exists $a \in \mathcal{G}$ so that γ_1 and γ_2 are both powers of a. This is an immediate consequence of the Nielsen-Schreier Theorem which states that any subgroup of a free group is free.

The proof of Proposition 8.3 will require a simple algebraic lemma:

Lemma 8.4. *Let* F_n *denote the free group on* $n \ge 2$ *symbols and suppose* $a, b \in F_n$ *do not commute. Then the subgroup*

$$H = \langle [a, b], [a^{-1}, b^{-1}] \rangle \le F_n$$

is again free on two or more symbols.

Proof. The Nielsen-Schreier Theorem implies that H is a free group. To see that it has rank two, it suffices check that $[a,b]$ and $[a^{-1},b^{-1}]$ do not commute. This follows because

$$[a,b][a^{-1},b^{-1}][a,b]^{-1}[a^{-1},b^{-1}]^{-1}$$

is a non-trivial reduced word in a, b, a^{-1}, and b^{-1}. Since a and b do not commute this word does not reduce to the identity in F_n. □

Proof of Proposition 8.3. Lemma 8.2 implies the existence of some $\delta > 0$ such that for parameters (A_0, B_0, C_0) the mappings F_1 and F_2 satisfy the hypotheses of Proposition 8.1 on $B_\delta(p)$. Moreover the conditions of Proposition 8.1 are open in the C^0 topology (on $B_\delta(p)$). This implies that for all parameters (A, B, C) sufficiently close to (A_0, B_0, C_0) the mappings F_1 and F_2 continue to satisfy the hypotheses of Proposition 8.1 on $B_\delta(p)$. Thus, for these parameters (A, B, C), the elements in the iterated commutators $S(n)$ converge uniformly to the identity on the ball $B_{\delta/2}(0) \subset \mathbb{C}^3$.

Since F_1 and F_2 do not commute, Lemma 8.4 can inductively be applied to ensure that each set $S(n)$ contains at least two non-commuting elements. Therefore, for every $n \geq 0$, there are elements in $S(n)$ that are different from the identity which, in turn, proves that the pseudogroup generated by F_1 and F_2 is locally non-discrete on $B_r(p)$.

Statement (2) then follows immediately because elements of $\mathcal{G}$ different from the identity cannot coincide with the identity when restricted to any surface $S_{A,B,C,D}$. In other words, for every choice of parameters (A, B, C, D), any non-trivial reduced word in g_x and g_y (or in any two of the three mappings g_x, g_y, and g_z) induces a mapping of $S_{A,B,C,D}$ that is different from the identity. This is a consequence of El-Huiti's theorem in [23]; see Section 4.4. □

Example 1 - Markoff Parameters. It is an easy observation that for the parameters $A = B = C = 0$ the origin $(0,0,0) \in \mathbb{C}^3$ is a common fixed point for g_x, g_y, and g_z and that their derivatives at the origin satisfy $D_0 g_x = \mathrm{diag}(1,-1,-1)$, $D_0 g_y = \mathrm{diag}(-1,1,-1)$, and $D_0 g_z = \mathrm{diag}(-1,-1,1)$.

If we let $h_x = g_x^2, h_y = g_y^2$, and $h_z = g_z^2$, then each of these maps is tangent to the identity at the origin. Notice that when $D = 0$ the surface $S_{0,0,0,D}$ passes through $(0,0,0)$. Therefore, applying Proposition 8.3 to the non-commuting pair of elements h_x and h_y yields:

Lemma 8.5. *There is a neighborhood W of the origin in $\mathbb{C}^4$ such that for all $(A, B, C, D) \in W$, the action of $\mathcal{G}$ is locally non-discrete on an open subset of $S_{A,B,C,D}$ obtained by intersecting $S_{A,B,C,D}$ with a small ball centered at the origin in $\mathbb{C}^3$. In other words, $(0,0,0,0)$ is an interior point of $\mathcal{ND}$.*

Example 2 - Dubrovin-Mazzocco parameters. Recall from the introduction the real 1-parameter family studied by Dubrovin and Mazzocco [22]: In our notations, the Dubrovin-Mazzocco parameters correspond to

$$A(a) = B(a) = C(a) = 2a + 4, \quad \text{and} \quad D(a) = -(a^2 + 8a + 8)$$

for $a \in (-2, 2)$. To simplify notations, let us denote the surface $S_{A,B,C,D}$ for these parameters by S_a and the group $\mathcal{G}_{A,B,C,D}$ by $\mathcal{G}_a$.

Lemma 8.6. *For the Dubrovin-Mazzocco family introduced above the surface S_a contains exactly three singular points p_1, p_2, p_3 given by*

$$p_1 = (x_1, y_1, z_1) = (a, 2, 2), \quad p_2 = (x_2, y_2, z_2) = (2, a, 2), \quad \textit{and} \quad p_3 = (x_3, y_3, z_3) = (2, 2, a). \tag{32}$$

Proof. The singular points correspond to the common fixed points of s_x, s_y, and s_z which, in turn, correspond to solutions in (x, y, z) of the equations

$$-x - yz + 2a + 4 = x, \qquad -y - xz + 2a + 4 = y, \qquad \text{and} \qquad -z - xy + 2a + 4 = z\ . \tag{33}$$

It is immediate to check that the three above singular points satisfy these equations.

Meanwhile, we must check that these three points lie on the surface S_a. By symmetry, it suffices to check for $p_1 = (x_1, y_1, z_1)$ and, in this case, we have

$$\begin{aligned} x_1^2 + y_1^2 + z_1^2 + x_1y_1z_1 - Ax_1 - By_1 - Cz_1 &= a^2 + 4 + 4 + 4a - (2a+4)[a - 2 - 2] \\ &= -(a^2 + 8a + 8) = D \end{aligned} \tag{34}$$

so that $p_1 \in S_a$. Note that there are other common fixed points for s_x, s_y, and s_z but they are located away from S_a. □

Proposition 8.7. *For every $a \in (-2, 2)$ the group $\mathcal{G}_a$ acting on S_a has locally non-discrete dynamics in some neighborhood of each of the singular points p_1, p_2 and p_3.*

Moreover, the Dubrovin-Mazzocco parameters at this fixed value of a yield an interior point of $\mathcal{N}D$. More precisely, given parameters (A, B, C, D) sufficiently close to the Dubrovin-Mazzocco parameters in question, the group $\mathcal{G}$ is locally non-discrete on the intersection of $S_{A,B,C,D}$ with $B_r(p_1) \cup B_r(p_2) \cup B_r(p_3)$ for some $r > 0$.

Note that the parameter $a = -2$, which we do not consider a Dubrovin Mazzocco parameter, corresponds to the Picard Parameters $(A, B, C, D) = (0, 0, 0, 4)$ for which $\mathcal{G}$ is locally discrete on all of the corresponding surface, as was shown in Section 6.

Proof. Because $A(a) = B(a) = C(a)$ the action of $\mathcal{G}_{A,B,C,D}$ on $S_{A,B,C,D}$ is self-conjugate under any permutation of the variables x, y, and z. Therefore it suffices to consider p_1, with the corresponding results for p_2 and p_3 following via these conjugacies. Note that p_1 is fixed by every element in $\mathcal{G}$ (in fact, by every element of the group generated by s_x, s_y, and s_z).

Let us first show that a suitable iterate of $g_x = s_z \circ s_y$ is close to the identity on some suitable neighborhood of p_1. A direct calculation yields that the eigenvalues of $D_{p_1}g_x$ are

$$\lambda_1 = \frac{a^2}{2} - 1 + \frac{\sqrt{a^4 - 4a^2}}{2}, \quad \lambda_2 = \frac{a^2}{2} - 1 - \frac{\sqrt{a^4 - 4a^2}}{2}, \quad \text{and} \quad \lambda_3 = 1\,.$$

For $a \in (-2, 0) \cup (0, 2)$ we see that λ_1 and λ_2 form a complex conjugate pair of eigenvalues, each of which has modulus one. In this case, $D_{p_1}g_x$ is conjugate to a rotation in suitable coordinates. In particular, for any $\tau > 0$ there is k sufficiently large so that

$$\|\left(D_{p_1}g_x\right)^k - \mathrm{Id}\| < \tau.$$

To find a second mapping satisfying the hypotheses of Lemma 8.2, it suffices to consider the map h obtained by conjugating g_x^k by, say, g_y. Since p_1 is fixed by g_y as well, it follows that $D_{p_1}h$ is conjugate to $D_{p_1}g_x^k$ and that the conjugating matrix does not depend on k (it is simply the matrix given by $Dg_y(p_1)$). Therefore, up to making k larger if needed, we can assume that both g_x^k and h satisfy the hypotheses of Lemma 8.2. Since g_x^k and h do not commute in $\mathcal{G}$, the result then follows from Proposition 8.3.

In the special case that $a = 0$ another direct calculation yields that $(D_{p_1}g_x)^2 = \mathrm{Id}$, in which case g_x^2 is tangent to the identity at p_1. Letting $h = g_y g_x^2 g_y^{-1}$ we obtain a second mapping tangent to the identity at p_1. Since, as previously seen, these two maps do not commute, the desired result in this special case follows again from Proposition 8.3. □

Remark 8.8. *The combination of the results obtained in this section and in the previous one substantiate our claim that, in general, the action of $\mathcal{G}_{A,B,C,D}$ is genuinely non-linear in the sense that it cannot preserve a rigid geometric structure. More precisely, whenever the locally non-discrete locus $\mathcal{N}_{A,B,C,D}$ is non-empty but not dense (as in the above examples), the group $\mathcal{G}_{A,B,C,D}$ cannot preserve any rigid geometric structure on $S_{A,B,C,D}$.*

Roughly speaking, a structure on a manifold M is called rigid if there is a uniform k such that the k-jet of a local isometry at a given point determines its C^∞-jet; see [32] *for detail. Examples of these structures include (pseudo-) Riemannian metrics and affine connections. Also, the action of* *a finite dimensional Lie group G on a manifold can be viewed as an action by isometries of some rigid structure provided that there is some k such that the induced action on the k-jet bundle $J^k(M)$ over M is free and proper.*

It follows essentially from the Frobenius theorem that, at least in real analytic category, every germ of isometry can be extended along paths in M. In particular, the standard results on continuous/differentiable dependence on the initial conditions for solutions of differential equations can be applied to these extensions. Consider a sequence of local isometries $\{f_n\}$ of some k-rigid structure fixing a point p and let $c : [0,1] \to M$ be a fixed path with $c(0) = p$. Denoting by $\{f_{n,c}\}$ the extension of f along c, the preceding implies that if the k-jets of f_n at p converge to the identity then so do the k-jets of $f_{n,c}$ at the point $c(1)$. Since holomorphic convergence implies C^∞-convergence, we conclude that if $\mathcal{G}_{A,B,C,D}$ preserves some rigid structure on M then the locally non-discrete locus $\mathcal{N}_{A,B,C,D}$ is either empty or coincides with the whole surface $S_{A,B,C,D}$.

A specific instance of this is the case of the Picard parameters where the action of $\mathcal{G}_{0,0,0,4}$ is known to preserve an affine structure, which is an example of a rigid structure. We have seen in Theorem D(ii) that for these parameters the locally non-discrete locus is empty.

9. Dynamics near infinity

In this section we will collect a few important, if slightly technical, results on the dynamics of the group $\mathcal{G}$ as well as the dynamics of (individual) hyperbolic maps near Δ_∞. The corresponding results, especially Proposition 9.6 and Lemma 9.8, will play major roles in the proofs of Theorems H and K to be supplied in the forthcoming sections.

Let us begin with a general review of the behavior of a given map in $\mathcal{G} = \mathcal{G}_{A,B,C,D}$ near infinity. Up to compactifying $\mathbb{C}^3$ into the projective space, we begin by recalling that the closure of any surface $S_{A,B,C,D}$ intersects the hyperplane at infinity $\Pi_\infty \subset \mathbb{CP}^3$ in a triangle Δ_∞. In homogeneous coordinates, this triangle is given by

$$\Delta_\infty = \{(X : Y : Z : W) \in \mathbb{CP}^3 \quad : \quad W = 0 \quad \text{and} \quad XYZ = 0\}.$$

The vertices of Δ_∞ are denoted by

$$\mathcal{V}_\infty = \big\{v_1 = (1 : 0 : 0 : 0), \qquad v_2 = (0 : 1 : 0 : 0), \qquad \text{and} \qquad v_3 = (0 : 0 : 1 : 0)\big\}.$$

As previously seen in Section 4.1, for every choice of the parameters (A, B, C, D), the surface $S_{A,B,C,D}$ is smooth on a neighborhood of Δ_∞. Furthermore, $S_{A,B,C,D}$ is tangent to the plane at infinity exactly at the vertices in $\mathcal{V}_\infty$ and, hence, it has transverse intersection with the plane at infinity elsewhere in Δ_∞. In the sequel, to abridge notation, the parameters A, B, C, D will be dropped whenever there is no possibility of misunderstanding. Thus, we will sometimes write $\mathcal{G}$ for $\mathcal{G}_{A,B,C,D}$, S for $S_{A,B,C,D}$, and $\overline{S}$ for the closure of S.

Because $\overline{S}$ is tangent to the plane at infinity at the vertices of Δ_∞, near v_1 we can use the affine coordinates $(Y/X, Z/X, W/X)$ on $\mathbb{CP}^3$ and express the surface $\overline{S}$ so that W/X becomes a holomorphic function of $(Y/X, Z/X)$. In other words, S is locally given as the graph of a holomorphic function in the variables $(Y/X, Z/X)$ on a neighborhood of v_1. This local representation will be referred to as the "standard coordinates" on $\overline{S}$ in a neighborhood of v_1. The analogous construction leads to "standard coordinates" on $\overline{S}$ near v_2 and near v_3, respectively.

Inside the plane at infinity, consider a neighborhood N_1 of v_1 where $\overline{S}$ is the graph of some holomorphic function h_1 as indicated above. Neighborhoods N_2 and N_3 respectively of v_2 and v_3 are similarly defined along with the corresponding holomorphic functions h_2 and h_3. The following lemma is rather immediate:

Lemma 9.1. *Consider parameters A_0, B_0, C_0 and D_0 and apply the above construction to the surface $\overline{S}_{A_0,B_0,C_0,D_0}$. Up to reducing the neighborhoods N_i, $i = 1, 2, 3$, there exists a neighborhood $\mathcal{N} \subset \mathbb{C}^4$ of (A_0, B_0, C_0, D_0) such that all of the following holds:*

(1) *For every* $(A,B,C,D) \in \mathcal{N}$*, the surface* $\overline{S}_{A,B,C,D}$ *is (locally) the graph of a function* h_i *defined on* N_i*. Furthermore, for* $i = 1,2,3$*, the functions* h_i *vary continuously with the parameters.*

(2) *For every* $(A,B,C,D) \in \mathcal{N}$*, the intersection of the surface* $\overline{S}_{A,B,C,D}$ *with the plane at infinity over the set* $\Delta_\infty \setminus (N_1 \cup N_2 \cup N_3)$ *is uniformly transverse. Furthermore, the slopes vary continuously with the point and with the parameters.*

□

We will also need some facts about birational extensions of elements from $\mathcal{G}^{\pm}_{A,B,C,D}$ and $\mathcal{G}_{A,B,C,D}$ to $\overline{S}_{A,B,C,D}$.

Remark 9.2. *Doing this birational extension is more subtle than one might initially guess. Any* $f \in \mathcal{G}^{\pm}_{A,B,C,D}$ *is obtained as the restriction of a polynomial automorphism* $F : \mathbb{C}^3 \to \mathbb{C}^3$ *to* $S_{A,B,C,D}$*. However, the birational extension* $F : \mathbb{CP}^3 \dashrightarrow \mathbb{CP}^3$ *typically has entire sides of the triangle at infinity* Δ_∞ *contained in its indeterminacy set. Meanwhile the birational extension* $f : \overline{S}_{A,B,C,D} \dashrightarrow \overline{S}_{A,B,C,D}$ *must have a finite set of indeterminate points because the indeterminacy locus of any rational mapping is always of codimension at least two. Indeed, restricting the mapping* F *to the surface* $\overline{S}_{A,B,C,D}$ *resolves this "excess indeterminacy" for the birational extension* $F : \mathbb{CP}^3 \dashrightarrow \mathbb{CP}^3$*. For this reason, working with the birational extension* $f : \overline{S}_{A,B,C,D} \dashrightarrow \overline{S}_{A,B,C,D}$ *is best done by working within local coordinates on* $\overline{S}_{A,B,C,D}$ *in the domain and codomain of* f*, as opposed to working with* 3*-dimensional coordinates.*

Recalling that $\mathcal{G}^{\pm}$ is generated by the involutions s_x, s_y, and s_z. An element γ in $\mathcal{G}^{\pm}$ is said to be *cyclically reduced* if its reduced spelling as a word in the "letters" s_x, s_y, and s_z has the property that every cyclic permutation of the word has no further possible cancellations. This can be expressed equivalently by saying that the word's reduced spelling in the "letters" s_x, s_y, and s_z is not conjugate in $\mathcal{G}^{\pm}$ to another element in $\mathcal{G}^{\pm}$ having strictly smaller length when spelled in the same "letters".

We also recall that a meromorphic self-map from a complex surface to itself is said to be *algebraically stable* if it does not contract a hypersurface to its indeterminate set, cf. [27, 21]. In the present case where elements of $\mathcal{G}$ act on the surface S, to be algebraically stable amounts to saying that γ does not contradict any of the sides of Δ_∞ to an indeterminacy point which, in turn, necessarily lies in Δ_∞ as well.

With this terminology, the following definition/proposition and remark summarize [9, Prop. 3.2] and [13, Prop. 2.3].

Definition/Proposition 9.3. *For any parameters* A,B,C,D *and any* $\gamma \in \mathcal{G}$ *we have*

(i) γ *is said to be hyperbolic if and only if it conjugate to a cyclically reduced word in* s_x, s_y*, and* s_z *that contains all three mappings.*

(ii) *A hyperbolic map* $\gamma \in \mathcal{G}$ *possesses a single indeterminate point which coincides with a vertex of* Δ_∞ *and will be denoted by* $\mathrm{Ind}(\gamma)$.

(iii) *A hyperbolic map* $\gamma \in \mathcal{G}$ *contracts all of* $\Delta_\infty \setminus \{\mathrm{Ind}(\gamma)\}$ *to a vertex of* Δ_∞ *denoted by* $\mathrm{Attr}(\gamma)$*. The vertices* $\mathrm{Ind}(\gamma)$ *and* $\mathrm{Attr}(\gamma)$ *may or may not coincide. In particular, the map* γ *is algebraically stable if and only if* $\mathrm{Ind}(\gamma) \neq \mathrm{Attr}(\gamma)$.

(iv) *Alternatively,* $\gamma : \overline{S} \dashrightarrow \overline{S}$ *is algebraically stable if and only if it is a cyclically reduced composition of* s_x, s_y*, and* s_z *of length at least two.*

(v) *If* γ *is algebraically stable and hyperbolic then* γ *is holomorphic around* $\mathrm{Attr}(\gamma)$ *and, in fact,* $\mathrm{Attr}(\gamma)$ *is a superattracting fixed point of* γ*. Moreover, the roles of* $\mathrm{Ind}(\gamma)$ *and* $\mathrm{Attr}(\gamma)$ *are interchanged if we pass from* γ *to* γ^{-1}*, i.e.,* $\mathrm{Attr}(\gamma) = \mathrm{Ind}(\gamma^{-1})$ *and* $\mathrm{Ind}(\gamma) = \mathrm{Attr}(\gamma^{-1})$.

(vi) *An element* γ *is said to be parabolic if it is conjugate in* γ *to one of the maps* g_x*,* g_y*, or* g_z*. Every element of* $\mathcal{G}$ *different from the identity is either hyperbolic or parabolic.*

Remark 9.4. *While $\mathcal{G}$ only contains hyperbolic and parabolic elements, the bigger group $\mathcal{G}^{\pm}$ also contains elliptic elements. An element of $\mathcal{G}^{\pm}$ is elliptic if and only if it is conjugate in $\mathcal{G}^{\pm}$ to one of the involutions s_x, s_y, or s_z, if and only if it is periodic. (The characterizations of parabolic and hyperbolic elements of $\mathcal{G}$ carry over to $\mathcal{G}^{\pm}$, with conjugation in $\mathcal{G}$ replaced by conjugation in $\mathcal{G}^{\pm}$.)*

Remark 9.5. *There is an equivalent but more intrinsic way of classifying the elements of $\mathcal{G}^{\pm}_{A,B,C,D}$ as being hyperbolic, parabolic, or elliptic. It will not be needed in this paper, but we describe it here for the benefit of the reader. Let us introduce one additional matrix group:*

$$\Gamma_2^{\pm} := \{M \in \mathrm{PGL}(2,\mathbb{Z}) \,:\, M \equiv \mathrm{Id} \bmod 2\}.$$

which has the classical congruence group Γ_2 as an index two subgroup. It is generated by (the projective equivalence classes of) the following matrices

$$N_x := \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}, \qquad N_y := \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}, \qquad \text{and} \qquad N_z := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

For any parameters (A,B,C,D) there is a group isomorphism from $\Gamma_2^{\pm}$ to $\mathcal{G}^{\pm}_{A,B,C,D}$ induced by sending N_x, N_y, and N_z to the involutions s_x, s_y, and s_z, respectively. It is an extension of the group isomorphism $\Gamma_2 \to \mathcal{G}_{A,B,C,D}$ discussed in the case of the Picard Parameters (Section 6) and it arises naturally from the context of dynamics on character varieties; see [13, Section 2.3].

Given $[N] \in \Gamma_2^{\pm}$ let us denote $f_{[N]} \in \mathcal{G}^{\pm}_{A,B,C,D}$ the associated automorphism. One then has that $f_{[N]}$ is hyperbolic, parabolic, or elliptic if and only if the associated matrix N induces a hyperbolic, parabolic, or elliptic transformation of the hyperbolic plane $\mathbb{H}$.

Moreover, the degrees of the k-th iterate $f^k_{[N]}$ grows at the same rate as the entries of N^k. Therefore, one has yet another equivalent classification that $f_{[N]}$ is hyperbolic, parabolic, or elliptic if and only if the degrees of $f^k_{[N]}$ grow exponentially, grow polynomially, or remain bounded. Therefore, the classification of elements of $\mathcal{G}^{\pm}$ into the classes (hyperbolic, parabolic, elliptic) is compatible with use of this terminology for elements of the Cremona group (see [11, Theorem 4.6]*) and also with the use of this terminology for automorphisms of compact surfaces (see* [10, Section 2.4.4]*).*

Note that the properties described in Definition/Proposition 9.3 are independent of the parameters A, B, C, D. They depend only on the spelling of γ as an element in the abstract group generated by three letters a, b, and c with the relation $abc = \mathrm{id}$ (up to the substitutions $a = g_x$, $b = g_y$, and $c = g_z$). In particular, the points $\mathrm{Ind}(\gamma)$ and $\mathrm{Ind}(\gamma^{-1}) = \mathrm{Attr}(\gamma)$ are independent of the parameters and this plays an important role in the statement of the following proposition.

Proposition 9.6. *Consider an hyperbolic element $\gamma \in \mathcal{G}$ and, for a choice of parameters A_0, B_0, C_0, and D_0, let $\overline{f}_{A_0,B_0,C_0,D_0}$ be the resulting birational map of the (compact) surface $\overline{S}_{A_0,B_0,C_0,D_0}$. Then there exist a neighborhood U_Δ of $\Delta_\infty \subset \mathbb{CP}^3$ and a neighborhood $\mathcal{U}_0 \subset \mathbb{C}^4$ of $(A_0,B_0,C_0,D_0) \in \mathbb{C}^4$ such that the following holds:*

- *For every $(A,B,C,D) \in \mathcal{U}_0$, the intersection $U_\Delta \cap \overline{S}_{A,B,C,D}$ is a neighborhood of Δ_∞ in $\overline{S}_{A,B,C,D}$.*
- *For every $(A,B,C,D) \in \mathcal{U}_0$, the map $\overline{f}_{A,B,C,D}$ has no periodic point in $(U_\Delta \cap \overline{S}_{A,B,C,D}) \setminus \Delta_\infty$.*

Proof. It suffices to prove the proposition for an algebraically stable hyperbolic γ since, every hyperbolic element in $\mathcal{G}$ is conjugate in $\mathcal{G}$ to an algebraically stable one and all elements in $\mathcal{G}$ preserve infinity.

Let then $\overline{f}_{A_0,B_0,C_0,D_0}$ be an algebraically stable hyperbolic map as indicated above. The statement amounts to showing the existence of a neighborhood $V_0 \subset \overline{S}_{A_0,B_0,C_0,D_0}$ of $\Delta_\infty \subset \overline{S}_{A_0,B_0,C_0,D_0}$ satisfying the two conditions below:

(1) $\overline{f}_{A_0,B_0,C_0,D_0}$ has no periodic point in $V_0 \setminus \Delta_\infty$.

(2) The neighborhood V_0 can be chosen to vary continuously with the parameters A, B, C, and D.

To prove the first assertion, we consider the points $\mathrm{Attr}(\overline{f}_{A_0,B_0,C_0,D_0})$ and $\mathrm{Ind}(\overline{f}_{A_0,B_0,C_0,D_0})$ in Δ_∞. As mentioned, these points are distinct and do not depend on the choice of the parameters. To abridge notation, we then set $P = \mathrm{Attr}(\overline{f}_{A_0,B_0,C_0,D_0})$ and $Q = \mathrm{Ind}(\overline{f}_{A_0,B_0,C_0,D_0})$. We also recall that P is a super-attracting fixed point of $\overline{f}_{A_0,B_0,C_0,D_0}$ and, in fact, a super-attracting fixed point of $\overline{f}_{A,B,C,D}$ for every choice of the parameters A, B, C, and D. Similarly, Q is a super-attracting fixed point of $\overline{f}^{-1}_{A,B,C,D}$ for any $(A,B,C,D) \in \mathbb{C}^4$. Now let $U \subset \overline{S}_{A_0,B_0,C_0,D_0}$ be a small neighborhood of Q which is contained in the basin of attraction of Q for $\overline{f}^{-1}_{A_0,B_0,C_0,D_0}$. Since $\overline{f}_{A_0,B_0,C_0,D_0}$ sends $\Delta_\infty \setminus \{Q\}$ to P, there exists a neighborhood V of $\Delta_\infty \setminus U$ which is in the basin of attraction of P, with respect to $\overline{f}_{A_0,B_0,C_0,D_0}$. Therefore $V \cup U$ contains no periodic point of $\overline{f}_{A_0,B_0,C_0,D_0}$: in fact, every point in $(V \cup U) \setminus \{P, Q\}$ converges to P under iteration by $\overline{f}_{A_0,B_0,C_0,D_0}$ or to Q under iteration by $\overline{f}^{-1}_{A_0,B_0,C_0,D_0}$.

It remains to prove that the above neighborhood can be assumed to vary continuously with the parameters. For this, we first consider the (local) description of the surfaces $\overline{S}_{A,B,C,D}$ provided by Lemma 9.1. The lemma in question shows that the surface $\overline{S}_{A,B,C,D}$ converges towards $\overline{S}_{A_0,B_0,C_0,D_0}$ as $(A,B,C,D) \to (A_0,B_0,C_0,D_0)$ on a neighborhood of $\Delta_\infty \subset \mathbb{CP}^3$. In fact, on a neighborhood of the points P and Q, this follows from the local structure of these surfaces as graphs of holomorphic functions. Conversely, away from these neighborhoods of P and Q, the statement follows from the (uniform) transverse intersection of these surfaces and the plane at infinity of $\mathbb{CP}^3$. In particular the maps $\overline{f}_{A,B,C,D}$ converge uniformly to $\overline{f}_{A_0,B_0,C_0,D_0}$ on a neighborhood of Δ_∞. Thus, the statement is reduced to showing the existence of $\epsilon > 0$ such that for all parameters (A,B,C,D) sufficiently close to (A_0,B_0,C_0,D_0), the ball of radius ϵ around P still is contained in the basin of attraction of P with respect to $\overline{f}_{A,B,C,D}$. Here, the same argument applies to Q and $\overline{f}^{-1}_{A,B,C,D}$ and the notion of "ball" is relative to some auxiliary metric, for example, the Euclidean metric in the "standard coordinates" of Lemma 9.1.

The last claim can be proved as follows. The germ of $\overline{f}_{A,B,C,D}$ is a rigid, reducible, super-attracting germ in the sense of [25] (cf. [9]). It actually falls in the "class 6" of the classification provided in [25] and hence it is conjugate to a monomial map, owing to results of Dloussky and Favre. Since the monomial map does not depend on the parameters, the claim follows from checking directly in the argument of [25] that the conjugating map depends continuously on the initial map. □

As a consequence of Proposition 9.6, we also obtain the following result (cf. Lemma 16 from [39]).

Corollary 9.7. *For every choice of the parameters $(A,B,C,D) \in \mathbb{C}^4$ and every hyperbolic element in $\mathcal{G}$, all the fixed points of the resulting (hyperbolic) map $f_{A,B,C,D}$ are isolated.*

Proof. As a composition of the algebraic maps g_x, g_y, and g_z, $f_{A,B,C,D}$ is itself an algebraic map from $\mathbb{C}^3$ to $\mathbb{C}^3$. Thus the set of its fixed points is an algebraic set of $\mathbb{C}^3$ so that its intersection with $S_{A,B,C,D}$ is an algebraic subset of $S_{A,B,C,D}$. Since this set is compact (Lemma 9.6) is must be finite. □

The remainder of this section will be devoted to the proof of Lemma 9.8 below which is crucial for understanding the structure of unbounded Fatou components as will be seen later on.

We resume the terminology used at the beginning of the section. Thus $(X : Y : Z : W)$ are homogeneous coordinates on $\mathbb{CP}^3$. For any A, B, C, D, the surface $\overline{S} = \overline{S}_{A,B,C,D}$ is tangent to the hyperplane at infinity at each of the vertices v_1, v_2, and v_3 of Δ_∞. Again, near, say v_1, we can use the affine coordinates $(Y/X, Z/X, W/X)$ on $\mathbb{CP}^3$ and express the surface so that W/X is a holomorphic function of $(Y/X, Z/X)$. Similar descriptions apply to the other two vertices, and the resulting coordinates were called "standard coordinates" around the vertices in question.

Next, let $\mathcal{U}_\infty(i)$ be an open neighborhood of v_i in $\overline{S}$ so that $\mathcal{U}_\infty(i)$ is contained in the graph of $h_i : N_i \to \mathbb{C}$, $i = 1, 2, 3$. Without loss of generality, we assume that the three neighborhoods $\mathcal{U}_\infty(i)$, $i = 1, 2, 3$, are pairwise disjoint. If $p \in \mathcal{U}_\infty(i)$, we denote by $\operatorname{dist}(p, v_i)$ the Euclidean distance between p and v_i in the standard coordinate chart on $\mathcal{U}_\infty(i)$. Let

$$\mathcal{U}_\infty = \mathcal{U}_\infty(1) \cup \mathcal{U}_\infty(2) \cup \mathcal{U}_\infty(3).$$

For any $p \in \mathcal{U}_\infty$, we define $\operatorname{dist}(p, \mathcal{V}_\infty)$ to equal $\operatorname{dist}(p, v_i)$ where $\mathcal{U}_\infty(i)$ is the unique one of the three neighborhoods containing p.

Let $S(0)$ be a set consisting of six hyperbolic elements $\gamma_{i,j} \in \mathcal{G}$, $i, j \in \{1, 2, 3\}$, $i \neq j$. As in Proposition 8.1, for every natural number n we will consider the inductively defined sets of iterated commutators $S(n)$. For every n the set $S(n+1)$ contains every possible commutator of any two distinct elements of $S(n)$.

Lemma 9.8. *Given parameters A, B, C, D, assume there are six hyperbolic elements $\gamma_{i,j} \in \mathcal{G}$ as above such that for every pair $i \neq j \in \{1, 2, 3\}$ we have $\operatorname{Ind}(\gamma_{i,j}) = v_i$ and $\operatorname{Attr}(\gamma_{i,j}) = v_j$. Let $S(0)$ be the set consisting of the six elements $\gamma_{i,j}$ and let $\{S(n)\}$ be the corresponding sequence of inductively defined subsets of $\mathcal{G}$. Then, up to reducing the neighborhood $\mathcal{U}_\infty \subset \overline{S}_{A,B,C,D}$, for any point $q \in \mathcal{U}_\infty$, there exists a constant $0 < \lambda < 1$ and a sequence $\{\eta_n\}_{n=0}^\infty \subset \mathcal{G}$ satisfying the two conditions below:*

(i) *$\eta_n \in S(n)$ for every $n \geq 0$, and*
(ii) *$\operatorname{dist}(\eta_n(q), \mathcal{V}_\infty) \leq \lambda^{4^n}$ for every $n \geq 0$.*

Proof. Note that we might have asked the elements $\gamma_{i,j}$ of the set $S(0)$ to satisfy $\gamma_{i,j} = \gamma_{j,i}^{-1}$ though this is not necessary. Also, in view of Definition/Proposition 9.3, all elements $\gamma_{i,j}$ are algebraically stable and $\gamma_{i,j}$ is holomorphic around $\operatorname{Attr}(\gamma_{i,j}) = v_j$.

The proposition will be proved by induction. In fact, we will prove a stronger statement. Namely, there exists λ, $0 < \lambda < 1$, such that for each integer $n \geq 0$, we have:

If n is even then for every pair of distinct $i, j \in \{1, 2, 3\}$ there exists some $\gamma_{i,j}^{(n)} \in S(n)$ such that

(E1) $\gamma_{i,j}^{(n)}$ is holomorphic on $\mathcal{U}_\infty \setminus \mathcal{U}_\infty(i)$, satisfies $\gamma_{i,j}^{(n)}(\mathcal{U}_\infty \setminus \mathcal{U}_\infty(i)) \subset \mathcal{U}_\infty(j)$, and for every $q \in \mathcal{U}_\infty \setminus \mathcal{U}_\infty(i)$ we have $\operatorname{dist}(\gamma_{i,j}^{(n)}(q), v_j) \leq \lambda^{4^n} \operatorname{dist}(q, \mathcal{V}_\infty)$.
(E2) $(\gamma_{i,j}^{(n)})^{-1}$ is holomorphic on $\mathcal{U}_\infty \setminus \mathcal{U}_\infty(j)$, satisfies $(\gamma_{i,j}^{(n)})^{-1}(\mathcal{U}_\infty \setminus \mathcal{U}_\infty(j)) \subset \mathcal{U}_\infty(i)$, and for every $q \in \mathcal{U}_\infty \setminus \mathcal{U}_\infty(j)$ we have $\operatorname{dist}\left((\gamma_{i,j}^{(n)})^{-1}(q), v_i\right) \leq \lambda^{4^n} \operatorname{dist}(q, \mathcal{V}_\infty)$.

If n is odd then for each $i \in \{1, 2, 3\}$ there exists some $\tau_i^{(n)} \in S(n)$ such that

(O) $\tau_i^{(n)}$ and $(\tau_i^{(n)})^{-1}$ are holomorphic on $\mathcal{U}_\infty \setminus \mathcal{U}_\infty(i)$, they satisfy $(\tau_i^{(n)})^{\pm 1}(\mathcal{U}_\infty \setminus \mathcal{U}_\infty(i)) \subset \mathcal{U}_\infty(i)$, and for any $q \in \mathcal{U}_\infty \setminus \mathcal{U}_\infty(i)$ we have $\operatorname{dist}\left((\tau_i^{(n)})^{\pm 1}(q), v_i\right) \leq \lambda^{4^n} \operatorname{dist}(q, \mathcal{V}_\infty)$.

The base of the induction is $n = 0$, in which case for each pair $i \neq j$ we let $\gamma_{i,j}^{(0)} = \gamma_{i,j}$ be the corresponding element of $S(0)$. Fix then a pair $i \neq j$ and let $k \in \{1, 2, 3\}$ be such that $k \neq i$ and $k \neq j$. Consider standard local coordinates (u_1, u_2) around v_k and let (w_1, w_2) stand for standard local coordinates in a neighborhood of v_j. By hypothesis we have $\operatorname{Ind}(\gamma_{i,j}) = v_i$ and $\operatorname{Attr}(\gamma_{i,j}) = v_j$. Therefore, item (iii) of Definition/Proposition 9.3 gives that $\gamma_{i,j}(\Delta_\infty \setminus \{v_i\}) = v_j$. Hence, if $\gamma_{i,j}$ is expressed in local coordinates under the form $(w_1, w_2) = \gamma_{i,j}(u_1, u_2)$, both coordinates of $\gamma_{i,j}(u_1, u_2)$ will vanish along both axes $\{u_1 = 0\}$ and $\{u_2 = 0\}$. Similarly, if we express $\gamma_{i,j}$ from the (w_1, w_2) coordinates to themselves, then both coordinates of $\gamma_{i,j}(w_1, w_2)$ vanish along both axes $\{w_1 = 0\}$

and $\{w_2 = 0\}$. This implies that for any $0 < \lambda < 1$, we can choose the neighborhoods $\mathcal{U}_\infty(k)$ and $\mathcal{U}_\infty(j)$ sufficiently small so as to ensure that for any $q \in \mathcal{U}_\infty(k) \cup \mathcal{U}_\infty(j)$ the estimate

$$\mathrm{dist}(\gamma_{i,j}(q), v_j) \leq \lambda\ \mathrm{dist}(q, \mathcal{V}_\infty) \tag{35}$$

holds. After choosing a sufficiently small neighborhood $\mathcal{U}_\infty(i)$ of v_i and perhaps making $\mathcal{U}_\infty(k)$ smaller, the same reasoning applies to show that for any $q \in \mathcal{U}_\infty(k) \cup \mathcal{U}_\infty(i)$ we have

$$\mathrm{dist}\left(\gamma_{i,j}^{-1}(q), v_i\right) \leq \lambda\ \mathrm{dist}(q, \mathcal{V}_\infty). \tag{36}$$

Repeating for all six distinct pairs $i \neq j$ we obtain sufficiently small neighborhoods $\mathcal{U}_\infty(1), \mathcal{U}_\infty(2)$, and $\mathcal{U}_\infty(3)$ such that (35) and (36) hold. If we then let $\mathcal{U}_\infty(1), \mathcal{U}_\infty(2)$, and $\mathcal{U}_\infty(3)$ be round balls of equal sufficiently small radius in the standard local coordinates, both estimates (35) and (36) will continue to hold. In addition, for all six distinct pairs $i \neq j$ we have $\gamma_{i,j}(\mathcal{U}_\infty \setminus \mathcal{U}_\infty(i)) \subset \mathcal{U}_\infty(j)$ and $\gamma_{i,j}^{-1}(\mathcal{U}_\infty \setminus \mathcal{U}_\infty(j)) \subset \mathcal{U}_\infty(i)$. Therefore we can assume that (E1) and (E2) hold when $n = 0$.

For the remainder of the proof we keep the neighborhood $\mathcal{U}_\infty = \mathcal{U}_\infty(1) \cup \mathcal{U}_\infty(2) \cup \mathcal{U}_\infty(3)$ fixed and inductively prove that (E1) and (E2) hold on $\mathcal{U}_\infty$ for every even n and that (O) holds on $\mathcal{U}_\infty$ for every odd n.

Suppose now that n is even and that the collection of six elements $\gamma_{i,j}^{(n)} \in S(n)$ exist and satisfy (E1) and (E2). For each $i \in \{1, 2, 3\}$, we will prove the existence of an element $\tau_i^{(n+1)} \in S(n+1)$ satisfying (O).

Fix then $i \in \{1, 2, 3\}$ and let $j, k \in \{1, 2, 3\} \setminus \{i\}$ be the other two indices. We define

$$\tau_i^{(n+1)} = \left[\gamma_{i,j}^{(n)}, \gamma_{i,k}^{(n)}\right] = \left(\gamma_{i,j}^{(n)}\right)^{-1} \left(\gamma_{i,k}^{(n)}\right)^{-1} \gamma_{i,j}^{(n)}\, \gamma_{i,k}^{(n)}.$$

Using (E1) and (E2) we can see that the above composition is holomorphic on all of $\mathcal{V}_\infty \setminus \mathcal{V}_\infty(i)$ and that it maps $\mathcal{V}_\infty \setminus \mathcal{V}_\infty(i)$ into $\mathcal{V}_\infty(i)$. Moreover, for any $q \in \mathcal{V}_\infty \setminus \mathcal{V}_\infty(i)$ we have that

$$\gamma_{i,k}^{(n)}(q) \in \mathcal{V}_\infty(k), \quad \gamma_{i,j}^{(n)}\, \gamma_{i,k}^{(n)}(q) \in \mathcal{V}_\infty(j), \quad \text{and} \quad \left(\gamma_{i,k}^{(n)}\right)^{-1} \gamma_{i,j}^{(n)}\, \gamma_{i,k}^{(n)}(q) \in \mathcal{V}_\infty(i).$$

Again using (E1) and (E2) we have the each of the four mappings in the commutator used to define $\tau_i^{(n+1)}$ contracts distance to $\mathcal{V}_\infty$ by a factor of λ^{4^n} and hence

$$\mathrm{dist}\left(\tau_i^{(n+1)}(q), v_i\right) = \mathrm{dist}\left(\left(\gamma_{i,j}^{(n)}\right)^{-1} \left(\gamma_{i,k}^{(n)}\right)^{-1} \gamma_{i,j}^{(n)} \gamma_{i,k}^{(n)}(q), v_i\right) \leq \lambda^{4^{n+1}} \mathrm{dist}(q, \mathcal{V}_\infty).$$

Notice that

$$(\tau_i^{(n+1)})^{-1} = \left[\gamma_{i,k}^{(n)}, \gamma_{i,j}^{(n)}\right] = \left(\gamma_{i,k}^{(n)}\right)^{-1} \left(\gamma_{i,j}^{(n)}\right)^{-1} \gamma_{i,k}^{(n)}\, \gamma_{i,j}^{(n)}.$$

Therefore the same proof as in the previous paragraph applies to $(\tau_i^{(n+1)})^{-1}$ after switching j and k. We conclude that (O) holds for $n + 1$.

Suppose now that n is odd and that the collection of three elements $\tau_i^{(n)} \in S(n)$ exist and satisfy (O). We will prove that all six elements $\gamma_{i,j}^{(n+1)} \in S(n+1)$ satisfying (E1) and (E2) exist.

For any distinct $i, j \in \{1, 2, 3\}$ let $k \in \{1, 2, 3\} \setminus \{i, j\}$ be the remaining element. Now define

$$\gamma_{i,j}^{(n+1)} = \left[\tau_j^{(n)}, \tau_i^{(n)}\right] = \left(\tau_j^{(n)}\right)^{-1} \left(\tau_i^{(n)}\right)^{-1} \tau_j^{(n)} \tau_i^{(n)}.$$

Using (O) we can see that the above composition of mappings is holomorphic on $\mathcal{U}_\infty \setminus \mathcal{U}_\infty(i)$ and that it maps $\mathcal{U}_\infty \setminus \mathcal{U}_\infty(i)$ into $\mathcal{U}_\infty(j)$. Again using (O), each of these four mappings contracts distance to $\mathcal{V}_\infty$ by a factor of λ^{4^n} and hence

$$\operatorname{dist}\left(\gamma_{i,j}^{(n+1)}(q), v_j\right) = \operatorname{dist}\left(\left(\tau_j^{(n)}\right)^{-1}\left(\tau_i^{(n)}\right)^{-1}\tau_j^{(n)}\tau_i^{(n)}(q), v_j\right) \le \lambda^{4^{n+1}}\operatorname{dist}(q, \mathcal{V}_\infty).$$

We conclude that (E1) holds for $n+1$.

To see that (E2) holds for $n+1$, note that

$$(\gamma_{i,j}^{(n+1)})^{-1} = \left[\tau_i^{(n)}, \tau_j^{(n)}\right] = \left(\tau_i^{(n)}\right)^{-1}\left(\tau_j^{(n)}\right)^{-1}\tau_i^{(n)}\tau_j^{(n)}.$$

Therefore the same proof as in the previous paragraph applies to $(\gamma_{i,j}^{(n+1)})^{-1}$ after switching i and j. We conclude that (E2) holds for $n+1$.

Therefore, we conclude that statements (E1) and (E2) hold for every even $n \ge 0$ and that (O) holds for every odd $n \ge 0$. □

10. General properties of Fatou components and good set of parameters $\mathbb{C}^4_{\rm good}$.

Recall that the Fatou set $\mathcal{F}_{A,B,C,D}$ for the action of $\mathcal{G}_{A,B,C,D}$ on $S_{A,B,C,D}$ is the set of points p admitting a neighborhood on which the restrictions of all elements in $\mathcal{G}$ form a normal family. Recall also that, by way of definition, this normal family may contain sequences of maps converging to infinity, cf. Section 1.5. In particular, $\mathcal{F}_{A,B,C,D}$ is an open (possibly empty) set and, according to Remark 4.1, none of the possible singular points of $S_{A,B,C,D}$ lies in $\mathcal{F}_{A,B,C,D}$.

A *Fatou component* $V \subset S_{A,B,C,D}$ is a connected component of $\mathcal{F}_{A,B,C,D}$. Since the Fatou set is invariant, one can consider the stabilizer $\mathcal{G}_V \le \mathcal{G}$ of V, which consists of those elements of $\mathcal{G}$ that map V to V. The purpose of this section is to establish several general properties of Fatou components. By combining these properties with the previous material, the proofs of Theorems H and K will quickly be derived in the next section.

We begin with the following dichotomy which plays an important role in the proof of Theorem F.

Proposition 10.1. *Let V be a Fatou component for $\mathcal{G}_{A,B,C,D}$. Then either:*

- (i) *there is a sequence of mappings $\gamma_n \in \mathcal{G}_{A,B,C,D} \setminus \{\mathrm{id}\}$ that converge uniformly on compact subsets of V to the identity, or*
- (ii) *the action of $\mathcal{G}_{A,B,C,D}$ on V is properly discontinuous.*

Proof. First suppose that V intersects the locally non-discrete locus $\mathcal{N}$ non-trivially. Then there exists a non-empty open set $W \subset V$ and a sequence $\{f_j\} \subset \mathcal{G}$, $f_j \ne \mathrm{id}$ for all $j \in \mathbb{N}$, such that the restrictions of the elements f_j to W converge uniformly to the identity. To show that Claim (i) holds in this case, it suffices to show that $\{f_{j_k}\}$ actually converges to the identity on compact subsets of V. This can be checked as follows. Consider a (connected, relatively compact) set $U \subset V$ containing W. Owing to the fact that $\{f_j\}$ is a normal family on V, we can assume without loss of generality that $\{f_j\}$ converges uniformly on U. However, by assumption, the limit map coincides with the identity on W and hence must coincide with the identity on all of U as required.

Now suppose that V is entirely contained in the locally-discrete locus $\mathcal{D}$. In this case we will prove that $\mathcal{G}$ acts properly discontinuously on V. Let $K \subset V$ be a compact set and, aiming at a contradiction, assume the existence of an infinite sequence $\{f_j\}$ of pairwise distinct elements in $\mathcal{G}$ such that $K \cap f_j(K) \ne \emptyset$ for all j. Thus, since K is compact, we can find a subsequence j_k and points p and q in K such that the sequence $\{y_k = f_{j_k}(p)\}_{k\in\mathbb{N}}$ converges to q. Up to enlarging K, we can assume without loss of generality that q lies in the interior of K. Setting $l_k = j_{k+1} - j_k$, it follows that $g_k = f^{l_k}$ sends y_k to y_{k+1} and both points converge towards q as $k \to \infty$.

On the other hand, since V is contained in the Fatou set, normality implies that the derivatives of the elements g_k are uniformly bounded in K. Similarly, the derivatives of their inverses g_k^{-1} are also bounded on $g_k(K)$. Using the uniform bound on the derivatives of g_k, we conclude that g_k sends some fixed neighborhood U_q of q to a bounded subset of K for k large enough. Again, normality implies that a subsequence $\{g_{k(i)}\}_i$ of $\{g_k\}$ converges uniformly on (compact subsets of) U_q to a non-constant map $g_\infty : U_q \to K$. However, since there is also convergence of derivatives (Cauchy formula), the existence of uniform bounds on the derivatives of g_k and g_k^{-1} implies that limit map g_∞ is locally invertible. Up to reducing the size of U_q we can suppose that g_∞ is actually invertible. It follows that the sequence of maps $h_i = g_{k(i+1)}^{-1} \circ g_{k(i)}$ converges uniformly to the identity on compact subsets of U_q. This contradicts the assumption that V is entirely contained in the locally-discrete locus $\mathcal{D}$. We therefore conclude that Claim (ii) holds. □

Proposition 10.2. *Any Fatou component V of $\mathcal{G}_{A,B,C,D}$ is Kobayashi hyperbolic.*

Proof. Since V is an open subset of $S_{A,B,C,D}$ and V does not contain any singular points of $S_{A,B,C,D}$, V is itself a complex (open) manifold. Recall the "grid"

$$\mathbb{G} = S_{x=x_0} \cup S_{x=x_1} \cup S_{y=y_0} \cup S_{y=y_1} \cup S_{z=z_0} \cup S_{z=z_1},$$

defined in (13), which is a subset of the Julia set $\mathcal{J}_{A,B,C,D}$. Hence V does not intersect $\mathcal{G}$ and we can therefore consider the inclusion

$$\iota : V \to \big(\mathbb{C} \setminus \{x_0, x_1\}\big) \times \big(\mathbb{C} \setminus \{y_0, y_1\}\big) \times \big(\mathbb{C} \setminus \{z_0, z_1\}\big).$$

Clearly the image of ι is contained in a product of hyperbolic Riemann surfaces which is naturally a Kobayashi hyperbolic domain. The fact that holomorphic mappings do not increase the Kobayashi psuedometric then implies that V is also Kobayashi hyperbolic as well; see [40, Proposition 3.2.2]. □

Let then V be a given component of $\mathcal{F}_{A,B,C,D}$ and let $\mathrm{Aut}(V)$ denote its group of holomorphic automorphisms. By building on the general theory of topological transformation groups of Gleason, Montgomery, and Zippin [48], Cartan was able to show that the automorphism group of a bounded domain in $\mathbb{C}^n$ is a finite-dimensional real Lie group. In turn, Kobayashi [40] was able to extend Cartan's theorem to general (Kobayashi) hyperbolic manifolds. Owing to Proposition 10.2, Corollary 10.3 below summarizes these results in the case of a Fatou component.

Recall that the action $\varphi : G \times M \to M$ of a group G on a manifold M is said to be *proper* if the preimage by φ of any compact set of M is again compact in $G \times M$.

Corollary 10.3. *Let V be a Fatou component of $\mathcal{G}_{A,B,C,D}$. Then,*

(1) $\mathrm{Aut}(V)$ *is a real Lie Group of finite dimension in the topology of uniform convergence on compact sets.*
(2) $\mathrm{Aut}(V)$ *acts properly on V.*
(3) *For any $p \in V$ the stabilizer* $\mathrm{Aut}(V)_p = \{f \in \mathrm{Aut}(V) : f(p) = p\}$ *is compact, since the action of* $\mathrm{Aut}(V)$ *on V is proper.*

For more details, see [40, Theorems 5.4.1 and 5.4.2].

Let us now consider the stabilizer $\mathcal{G}_V \leq \mathcal{G}$ of the hyperbolic component V.

Proposition 10.4. *Suppose that $\mathcal{G}_{A,B,C,D}$ is locally non-discrete on an connected open $U \subset \mathcal{F}_{A,B,C,D}$ and let V be the Fatou component containing U. Recalling that $\mathcal{G}_V$ stands for the stabilizer of V in $\mathcal{G}$, the following holds:*

(1) *The closure $G = \overline{\mathcal{G}_V}$ of $\mathcal{G}_V$ in* $\mathrm{Aut}(V)$ *is a real Lie Group of dimension at least* 1.
(2) *For every point $p \in V$, the stabilizer G_p of p in G is such that its local action around p is conjugate to the local (linear) action of a closed subgroup of* $\mathrm{SU}(2)$ *on a neighborhood of* $(0,0) \in \mathbb{C}^2$.

Proof. As a closed subgroup of a Lie Group, G is itself a real Lie Group. Moreover, since $\mathcal{G}$ is locally non-discrete on the open set $U \subset V$, modulo reducing U, there are elements $\{\gamma_n\}_{n=1}^\infty \subset \mathcal{G}$ that converge uniformly to the identity on U. In particular, for n large enough, we have $\gamma_n(V) \cap V \neq \emptyset$ and thus $\gamma_n(V) = V$ since $\mathcal{F}$ is invariant under $\mathcal{G}$. Hence, up to dropping finitely many elements in the sequence in question, we can assume without loss of generality that $\{\gamma_n\}_{n=1}^{\infty} \subset \mathcal{G}_V$ for every $n \in \mathbb{N}$. Next we have:

Claim. The sequence $\{\gamma_n\}_{n=1}^{\infty}$ actually converges to the identity uniformly on compact subsets of V, i.e. as elements of $\mathrm{Aut}(V)$.

Proof of the claim. Consider a relatively compact open set $U' \subset V$ with $U \subset U'$. The claim amounts to checking that $\{\gamma_n\}_{n=1}^{\infty}$ converges uniformly to the identity in U'. If this were not the case then, up to passing to a subsequence, there would exist $\varepsilon > 0$ such that

$$\sup_{x \in U'} \|\gamma_n(x) - x\| \geq \varepsilon > 0\,.$$

Since $\{\gamma_n\}_{n=1}^{\infty}$ is contained in a normal family on V, we can extract a limit map γ_∞ defined on U' and thus satisfying $\sup_{x \in U'} \|\gamma_\infty(x) - x\| \geq \varepsilon$ so that γ_∞ does not coincide with the identity on U'. However, γ_∞ must coincide with the identity on $U \subset U'$ since $\{\gamma_n\}_{n=1}^{\infty}$ converges uniformly to the identity on U. The resulting contradiction proves our claim. □

Since $\{\gamma_n\}_{n=1}^{\infty}$ converges to the identity on compact subsets of V, it follows that the elements of $\mathrm{Aut}(V)$ obtained by restricting them to V actually converges to the identity as elements of $\mathrm{Aut}(V)$ equipped with its Lie group structure, cf. Corollary 10.3. Thus $\mathcal{G}_V$ is a non-discrete subgroup of $\mathrm{Aut}(V)$ and hence its closure must be a Lie group with strictly positive dimension.

It remains to check the second assertion. For this, recall that $\mathcal{G}$ preserves the real volume form associated with the holomorphic volume form Ω given in (10). This implies that the group of derivatives of elements of G_p is a subgroup of $\mathrm{SL}(2,\mathbb{C})$. On the other hand, Corollary 10.3, Part (3), informs us that G_p must be compact. Since $\mathrm{SU}\,(2)$ is a maximal compact subgroup [6] of $\mathrm{SL}(2,\mathbb{C})$, it follows from the classical Bochner Linearization Theorem that the local action of G_p around $p \in V$ is conjugate to the (local) linear action of a closed subgroup of $\mathrm{SU}\,(2)$ on a neighborhood of the origin. □

The possibility of having a point $p \in V$ whose stabilizer G_p is conjugate to all of $SU(2)$ is a challenge for us as it raises quite a few technical issues. To avoid get involved in a much longer argument and keep us focused on the situations of primary interest, we will work only with the following set of parameters:

$$\mathbb{C}^4_{\text{good}} = \{(A,B,C,D) \in \mathbb{C}^4 \,:\, \text{every fixed point of every element of } \mathcal{G}_{A,B,C,D} \setminus \{\mathrm{id}\} \text{ is in } \mathcal{J}_{A,B,C,D}\}.$$

In Propositions 10.9 and 10.11, it will be seen that the set $\mathbb{C}^4_{\text{good}}$ is "quite large" and contains several parameters of interest. In particular, it will be shown that $\mathbb{C}^4 \setminus \mathbb{C}^4_{\text{good}}$ is at worst a countable union of (proper) real-algebraic subsets of $\mathbb{C}^4$ and, in particular, it has null Lebesgue measure.

First, however, we have a simple and well-known lemma.

Lemma 10.5. *Let f be an automorphism of a Kobayashi hyperbolic domain V. Assume there is a point $p \in U$ that is fixed by f and where the differential of f coincides with the identity. Then f is the identity on all of V.*

Proof. Since f is an isometry of the Kobayashi metric, the statement would be immediate if the Kobayashi metric were a Riemannian metric which, however, is not always the case. To overcome this difficulty, we locally replace the Kobayashi metric by the Bergman one as follows. For small $r > 0$, let $B_r(p)$ denote the ball of radius r with respect to the Kobayashi metric. Since a holomorphic map cannot increase the Kobayashi distance, it follows that $f(B_r(p)) \subset B_r(p)$. The analogous argument applied to f^{-1} allows us to conclude that f induces an automorphism of $B_r(p)$. Now, if $r > 0$ is small enough, then $B_r(p)$ can be identified with a bounded domain in some space $\mathbb{C}^n$ so that the Bergman metric is well defined. Thus f induces an isometry of the resulting Riemannian metric on $B_r(p)$ and hence coincides locally with the identity. The lemma then follows. □

Proposition 10.6. *Suppose that $(A,B,C,D) \in \mathbb{C}^4_{\text{good}}$ and that $\mathcal{G}_{A,B,C,D}$ is locally non-discrete on a connected open $U \subset \mathcal{F}_{A,B,C,D}$. Let V denote the Fatou component containing U. Then,*

1. *The closure $G = \overline{\mathcal{G}_V}$ of $\mathcal{G}_V$ in $\operatorname{Aut}(V)$ is a real Lie Group of dimension ≥ 1.*
2. *The stabilizer G_p of any $p \in V$ is trivial.*

Proof. Beyond the proof of Proposition 10.4, it remains to show Claim (2). Suppose for contradiction there exists of some point $p \in V$ and some $g \in G \setminus \{\text{id}\}$ satisfying $g(p) = p$. Note that g may lie in $\overline{\mathcal{G}_V} \setminus \mathcal{G}_V$ so that the statement does not follow immediately from the definition of the parameter set $\mathbb{C}^4_{\text{good}}$.

Since $g \neq \text{id}$, Lemma 10.5 implies that $Dg(p)$ is not the identity either. On the other hand, $Dg(p)$ is conjugate to a matrix in $SU(2)$ so that the preceding discussion shows that $Dg(p) - \text{id}$ is, in fact, invertible.

On the other hand, g lies in $G = \overline{\mathcal{G}_V}$ so that there are elements $\gamma_n \in \mathcal{G}$ with $\gamma_n \to g$ locally uniformly on V. Moreover, since the functions are holomorphic, this implies C^∞ convergence on compact subsets of V. Since $Dg(p) - \text{id}$ is invertible, it follows from the implicit function theorem (for Banach spaces) that for sufficiently large n the mappings γ_n have fixed points p_n converging to p. Therefore $D\gamma_n(p_n) \to Dg(p) \neq \text{id}$, implying that $\gamma_n \neq \text{id}$ for sufficiently large n. Thus, we have found non-trivial $\gamma_n \in \mathcal{G}$ having fixed points in the Fatou component V, contradicting the choice of parameters $(A,B,C,D) \in \mathbb{C}^4_{\text{good}}$. We conclude that $G_p = \{\text{id}\}$. □

Recall from Definition/Proposition 9.3 that, bar the identity, all elements of $\mathcal{G}$ are split in hyperbolic maps and parabolic maps. Furthermore an element of $\mathcal{G}$ is parabolic if and only if it is conjugate to a non-trivial power of the generators g_x, g_y or g_z.

Lemma 10.7. *For any (A,B,C,D), any fixed point of a parabolic element $\gamma \in \mathcal{G}_{A,B,C,D}$ lies in the Julia set of the corresponding $\mathcal{G} = \mathcal{G}_{A,B,C,D}$-action.*

Proof. Since parabolic maps are conjugate to a non-trivial power of one of the generators g_x, g_y, or g_z, it suffices to prove the statement for a non-trivial power of, say, g_x. It follows from Proposition 5.1 and Lemma 4.2 that for all but finitely many values of $x_0 \in \mathbb{C} \setminus [-2,2]$ the action of g_x on the fiber $S_{x=x_0}$ is loxodromic, with two distinct fixed points at infinity. Consider an iterate g_x^ℓ for some $\ell \neq 0$. Any point on any such $S_{x=x_0}$ will have orbit under g_x^ℓ that tends to infinity. These points form an open dense subset of $S_{A,B,C,D}$, implying that any point having bounded orbit under g_x^ℓ (and hence any fixed point of g_x^ℓ) must be in the Julia set. □

Now we will need a significantly more elaborate result.

Lemma 10.8. *There is a countable union $\mathcal{H} \subset \mathbb{C}^4$ of real algebraic hypersurfaces such that if $(A,B,C,D) \in \mathbb{C}^4 \setminus \mathcal{H}$ then $S_{A,B,C,D}$ is smooth and any hyperbolic $\gamma \in \mathcal{G}_{A,B,C,D}$ has every fixed point consisting of a hyperbolic saddle point.*

Proof. Let $\text{NS} \subset \mathbb{C}^4$ be the set of parameters (A,B,C,D) for which $S_{A,B,C,D}$ is not smooth. It consists of finitely many complex algebraic hypersurfaces and we immediately include NS as part of $\mathcal{H}$.

Fix a hyperbolic map $f_{A,B,C,D} \in \mathcal{G}_{A,B,C,D}$ and recall that the hyperbolic nature of $f_{A,B,C,D}$ depends only on its spelling in terms of the generators g_x, g_y, and g_z. In particular, the notion of hyperbolic map does not depend on the parameters (A,B,C,D). Since $\mathcal{G}$ is countable, we can then fix the spelling of $f_{A,B,C,D}$ and reduce the proof to checking that there are finitely many real-algebraic hypersurfaces $H \subset \mathbb{C}^4$ away from which every fixed point of $f_{A,B,C,D}$ is a hyperbolic saddle. The remainder of the proof consists of showing that this is, indeed, the case.

Consider the set of of 7-tuples $(A,B,C,D,x,y,z) \in \mathbb{C}^7$ and the subset $\widetilde{H} \subset \mathbb{C}^7$ consisting of points (A,B,C,D,x,y,z) such that

1. $S_{A,B,C,D}$ is smooth, i.e. $(A,B,C,D) \in \mathbb{C}^4 \setminus \text{NS}$,

(2) $(x,y,z) \in S_{A,B,C,D}$,
(3) $f_{A,B,C,D}(x,y,z) = (x,y,z)$, and
(4) $Df_{A,B,C,D}(x,y,z)$ is not a hyperbolic saddle.

In our setting, Condition (4) is equivalent to requiring that $\mathrm{tr}(Df_{A,B,C,D}(x,y,z)) \in [-2,2]$. Therefore, $\widetilde{H}$ is a semi-algebraic subset of $\mathbb{C}^7$, i.e. it is a set given by finitely many polynomial equations and polynomial inequalities with real coefficients.

Notice that $H = \mathrm{pr}(\mathcal{H})$, where $\mathrm{pr}(A,B,C,D,x,y,z) = (A,B,C,D)$. It follows from the Tarski-Seidenberg Theorem [5, Theorem 2.2.1] that H is also semi-algebraic. By definition, the dimension of a semi-algebraic set is the (real) dimension of the (real) Zariski closure of the set. Therefore, if we prove that $\dim(H) \leq 7$ it will follow that H is contained in a finite union of real-algebraic hypersurfaces of $\mathbb{C}^4$, which is sufficient for our purposes.

The Cylindrical Algebraic Decomposition Theorem [5, Theorem 2.3.6] asserts that a semi-algebraic set can be decomposed into finitely many sets, each of which is homeomorphic to $[0,1]^{d_i}$ for some d_i. Moreover, the dimension of the set (in the sense of the previous paragraph) equals the maximum of the d_i, see [5, Section 2.8]. In particular, if $\dim(H) = 8$, then H would have non-empty interior. We will prove that this is not the case.

First recall that the fixed points of $f_{A,B,C,D}$ are all isolated (see Corollary 9.7 or Lemma 16 in [39]). Let us first prove the following:

Claim. There is an open $U \subset \mathbb{C}^4 \setminus (H \cup \mathrm{NS})$.

Proof of the claim. Consider a sequence of parameters $(A_n, B_n, C_n, Z_n) \in \mathbb{C}^4 \setminus \mathrm{NS}$ converging to the Picard Parameters $(0,0,0,4)$. Suppose for contradiction that for every n the mapping f_{A_n,B_n,C_n,D_n} has a fixed point $p_n \in S_{A_n,B_n,C_n,D_n}$ that is not a hyperbolic saddle. Then, since f_{A_n,B_n,C_n,D_n} preserves the volume form ω, both eigenvalues of $Df_{A_n,B_n,C_n,D_n}(p_n)$ have modulus equal to 1.

For sufficiently large n the fixed points p_n remain away from some fixed neighborhood of Δ_∞ (Proposition 9.6). Therefore, we can extract a subsequence so that p_{n_k} converges to some point $p_\infty \in S_{0,0,0,4}$. Since $f_{A,B,C,D}$ is continuous and depends continuously on the parameters we have that p_∞ is a fixed point of $f_{0,0,0,4}$. For any $(A,B,C,D) \in \mathbb{C}^4$ let $F_{A,B,C}$ denote the extension of $f_{A,B,C,D}$ as an automorphism of $\mathbb{C}^3$. The points p_{n_k} and p_∞ are also fixed points for $F_{A_{n_k},B_{n_k},C_{n_k}}$ and $F_{0,0,0}$, respectively. For each k all three of the eigenvalues of $DF_{A_{n_k},B_{n_k},C_{n_k}}(p_{n_K})$ have modulus equal to 1, with the third eigenvalue corresponding to a direction transverse to S_{A_n,B_n,C_n,D_n}. Since the eigenvalues of a matrix depend continuously on its entries and since the derivative of $DF_{A,B,C}(q)$ depends continuously on the parameters (A,B,C) and on the point q, we conclude that each eigenvalue of $DF_{0,0,0}(p_\infty)$ has modulus equal to 1. Since $p_\infty \in S_{0,0,0,4}$ this contradicts Corollary 6.8.

We conclude that there is some n such that every fixed point of f_{A_n,B_n,C_n,D_n} is a hyperbolic saddle. Since we chose the parameters $(A_n,B_n,C_n,D_n) \in \mathbb{C}^4 \setminus \{\mathrm{NS}\}$ the surface S_{A_n,B_n,C_n,D_n} is also smooth. Both of these are open conditions and therefore they hold on some small neighborhood U of $(A_n,B_n,C_n,D_n) \in \mathbb{C}^4$. The claim follows at once. □

Consider now the set $\widetilde{M} \subset \mathbb{C}^7$ consisting of 7-tuples $(A,B,C,D,x,y,z) \in \mathbb{C}^7$ such that

(1) $(x,y,z) \in S_{A,B,C,D}$,
(2) $f_{A,B,C,D}(x,y,z) = (x,y,z)$, and
(3) $Df_{A,B,C,D}(p) - \mathrm{id}$ is singular.

It is an complex algebraic subset of $\mathbb{C}^7$. The projection $M = \mathrm{pr}(\widetilde{\mathrm{M}}) \subset \mathbb{C}^4$ onto the first four coordinates is therefore *constructible*, see [49]. Alternately, up to replacing the initial algebraic set by the corresponding projective scheme, the so-called main theorem of elimination theory tells us that the resulting projection on the coordinates (A,B,C,D) yields an algebraic set M. In any case, the fundamental result to be used here is the fact that the Zariski-closure of the constructible set M must coincide with its closure for the standard topology, see [49]. Since it was shown that the complement of M has non-empty interior in the standard topology, it follows that M is contained in a *proper* Zariski-closed subset of $\mathbb{C}^4$.

Consider the Zariski-open set $Z = \mathbb{C}^4 \setminus (\mathrm{NS} \cup \overline{M})$, where $\overline{M}$ stands for the closure of the constructible set M. In particular, Z is not empty. Suppose that $(A_0, B_0, C_0, D_0) \in Z$ and that p_{A_0,B_0,C_0,D_0} is a fixed point of f_{A_0,B_0,C_0,D_0}. Since p_{A_0,B_0,C_0,D_0} is simple ($Df_{A_0,B_0,C_0,D_0}(p_{A_0,B_0,C_0,D_0})-$ id is invertible), p_{A_0,B_0,C_0,D_0} varies holomorphically with the parameters (A, B, C, D). In other words, we have a locally defined holomorphic mapping $(A, B, C, D) \mapsto p_{A,B,C,D}$ for (A, B, C, D) close enough to (A_0, B_0, C_0, D_0). Similarly, the differential $Df_{A,B,C,D}(p_{A,B,C,D})$ also varies holomorphically with the parameters.

Moreover, the initial fixed point p_{A_0,B_0,C_0,D_0} can actually be (globally) continued along paths $c : [0, 1] \to Z$. Indeed, as the parameters vary in Z, two fixed points of $f_{A,B,C,D}$ cannot collide since they are all simple. Furthermore, they cannot hit Δ_∞ either, owing to Proposition 9.6.

Suppose for contradiction that the set H had non-empty interior. We can therefore choose some (A_0, B_0, C_0, D_0) and some $\epsilon > 0$ such that $\mathbb{B}_\epsilon((A_0, B_0, C_0, D_0)) \subset H \cap Z$. Since f has finitely many fixed points we can reduce $\epsilon > 0$, if necessary, so that there is some fixed point $p(A, B, C, D)$ varying holomorphically over $\mathbb{B}_\epsilon((A_0, B_0, C_0, D_0))$ such that $\mathrm{tr}(Df_{A,B,C,D}(p_{A,B,C,D})) \in [-2, 2]$ for all $(A, B, C, D) \in \mathbb{B}_\epsilon((A_0, B_0, C_0, D_0))$.

Let $(A_1, B_1, C_1, D_1) \in U \cap Z$, where U is the open set provided by the Claim above. Consider a simple path $c : [0, 1] \to Z$ with $c(0) = (A_0, B_0, C_0, D_0)$ and $c(1) = (A_1, B_1, C_1, D_1)$. Within Z there is a simply connected neighborhood V of $c([0, 1])$ on which $p(A, B, C, D)$ and $Df_{A,B,C,D}(p_{A,B,C,D})$ vary holomorphically. Since $\mathrm{tr}(Df_{A,B,C,D}(p_{A,B,C,D})) \in [-2, 2]$ on an open neighborhood of $c(0)$ the same holds on all of V. In particular, $\mathrm{tr}(Df_{A_1,B_1,C_1,D_1}(p_{A_1,B_1,C_1,D_1})) \in [-2, 2]$, contradicting that $(A_1, B_1, C_1, D_1) \in \mathbb{C}^4 \setminus H$.

We conclude that the (real) Zariski closure of H has real dimension equal to 7 and thus that H is contained in finitely many real-algebraic hypersurfaces. The lemma is proved. □

We summarize these two lemmas with the following proposition.

Proposition 10.9. *There is a countable union of real-algebraic hypersurfaces $\mathcal{H} \subset \mathbb{C}^4$ such that if $(A, B, C, D) \in \mathbb{C}^4 \setminus \mathcal{H}$ then every fixed point of every element of $\mathcal{G}_{A,B,C,D} \setminus \{\mathrm{id}\}$ is in $\mathcal{J}_{A,B,C,D}$.* □

Similarly, the argument used in Lemma 10.8 can be repeated word-for-word to yield:

Corollary 10.10. *For all but countably many $D \in \mathbb{C}$, every fixed point of every element in $\mathcal{G}_{0,0,0,D}$, bar the identity, lies in $\mathcal{J}_{0,0,0,D}$.* □

When (A, B, C, D) are all real, there is a simple sufficient condition for every fixed point of every element of $\mathcal{G}_{A,B,C,D} \setminus \{\mathrm{id}\}$ to be in $\mathcal{J}_{A,B,C,D}$. More specifically, for real parameters (A, B, C, D), the real slice $S_{A,B,C,D}(\mathbb{R}) = S_{A,B,C,D} \cap \mathbb{R}^3$ is invariant by the action of $\mathcal{G}_{A,B,C,D}$ and the resulting dynamics in this real 2-dimensional surface can be investigated in further detail. In particular, Cantat proved in [9, Theorem 5.1] that if the real slice $S_{A,B,C,D}(\mathbb{R}) = S_{A,B,C,D} \cap \mathbb{R}^3$ is connected, then for any hyperbolic mapping f the set of bounded orbits $\mathcal{K}_f$ of f is contained in $S_{A,B,C,D}(\mathbb{R})$ and that f is uniformly hyperbolic on $\mathcal{K}_f$. Moreover, according to Benedetto-Goldman [4], the real slice $S_{A,B,C,D}(\mathbb{R})$ is connected if and only if the product $ABCD < 0$ and none of these (real) parameters lies in the interval $(-2, 2)$. Taking into account Lemma 10.7, the combination of Cantat's and Benedetto-Goldman's theorems then yield:

Proposition 10.11. *If (A, B, C, D) are real and $S_{A,B,C,D}(\mathbb{R})$ is connected, then then every fixed point of every element of $\mathcal{G}_{A,B,C,D} \setminus \{\mathrm{id}\}$ is in $\mathcal{J}_{A,B,C,D}$.* □

11. Ruling out Fatou components: Proofs of Theorems H and K

Let $V \subset S_{A,B,C,D}$ be a connected component of the Fatou set $\mathcal{F}_{A,B,C,D}$ and denote by $\mathcal{G}_V \leq \mathcal{G}$ its stabilizer. The purpose of this section is to study the dynamics of $\mathcal{G}_V$ on V and, in particular, to derive sufficient conditions to ensure that certain open sets of $S_{A,B,C,D}$ must be contained in the Julia set. We also remind the reader that V contains only smooth points of $S_{A,B,C,D}$, owing to Remark 4.1. In our discussion, we will have to distinguish between bounded and unbounded Fatou component.

Let us first consider the case of unbounded Fatou components which relies heavily on Lemma 9.8. More generally, we resume the notation employed in Section 9. Every vertex v_i, $i \in \{1,2,3\}$, of Δ_∞ is contained in a neighborhood $\mathcal{U}_\infty(i) \subset \overline{S} = \overline{S}_{A,B,C,D}$ where "standard coordinates" are defined. The neighborhoods $\mathcal{U}_\infty(i)$ are assumed to be pairwise disjoint. The distance of a point in $\mathcal{U}_\infty(i)$ to v_i is measured with the Euclidean metric arising from the "standard coordinates". We then set $\mathcal{V}_\infty = \{v_1, v_2, v_3\}$ and $\mathcal{U}_\infty = \mathcal{U}_\infty(1) \cup \mathcal{U}_\infty(2) \cup \mathcal{U}_\infty(3)$. Finally, the distance from $p \in \mathcal{U}_\infty$ to $\mathcal{V}_\infty$ is equal to the distance in $\mathcal{U}_\infty(i)$ of p to v_i where $i \in \{1,2,3\}$ is chosen so that $p \in \mathcal{U}_\infty(i)$.

We are now ready to prove Theorems H and K. We repeat the statements here for the convenience of the reader.

Theorem H. *Suppose that for some parameters A, B, C there is a point $p \in \mathbb{C}^3$ and $\epsilon > 0$ such that for any two vertices $v_i \neq v_j \in \mathcal{V}_\infty$, $i \neq j$, there is a hyperbolic element $\gamma_{i,j} \in \mathcal{G}_{A,B,C}$ satisfying:*

(A) $\mathrm{Ind}(\gamma_{i,j}) = v_i$ *and* $\mathrm{Attr}(\gamma_{i,j}) = v_j$, *and*

(B) $\sup_{z \in B_\epsilon(p)} \|\gamma_{i,j}(z) - z\| < K(\epsilon)$.

Then, for any D, we have that $B_{\epsilon/2}(p) \cap S_{A,B,C,D}$ is disjoint from any unbounded Fatou components of $\mathcal{G}_{A,B,C,D}$. Here, $K(\epsilon) > 0$ denote the constant given in Proposition 8.1.

Proof. Let $S(0)$ be the set of all six elements $\gamma_{i,j} \in \mathcal{G}$ satisfying the hypotheses of Theorem H. As in Proposition 8.1 and Lemma 9.8, for every natural number n we will consider the inductively defined sets of iterated commutators $S(n)$ where, for every n, the set $S(n+1)$ contains every possible commutator of any two distinct elements of $S(n)$.

We assume aiming at a contradiction that for some parameter D there is an unbounded Fatou component $V \subset S_{A,B,C,D}$ for $\mathcal{G}_{A,B,C,D}$ such that $V \cap B_{\epsilon/2}(p) \neq \emptyset$.

Let $p' \in B_{\epsilon/2}(p) \cap V$ and let $\delta > 0$ be sufficiently small so that $B_\delta(p') \cap S_{A,B,C,D} \subset B_{\epsilon/2}(p) \cap V$. Since the elements $\gamma_{i,j}$ satisfy Hypothesis (B), it follows from Proposition 8.1 that there is some integer $N \geq 0$ such that for any $\gamma \in S(N')$, with $N' \geq N$, we have $\gamma(p') \in B_\delta(p')$ and hence that $\gamma(V) = V$.

For a fixed neighborhood $\mathcal{U}_\infty$ of the vertices of Δ_∞ as above, Lemma 9.8 gives us a sequence of elements $\{\eta_n\}_{n=0}^\infty \in \mathcal{G}$ satisfying Assertions (i) and (ii) of the lemma in question.

We claim that V intersects $\mathcal{U}_\infty$ non-trivially. Since V is unbounded, there is a sequence $\{q_k\}_{k=1}^\infty \subset V$ which accumulates to Δ_∞. Passing to a subsequence, if necessary, we can suppose that it converges to some $q_\infty \in \Delta_\infty$. If $q_\infty \in \mathcal{V}_\infty$ then the claim holds. Otherwise, we have that $\gamma_{2,1}^{(N)}(\Delta_\infty \setminus \{v_2\}) = \{v_1\}$ so that $\gamma_{2,1}^{(N)}(q_\infty) = v_1$. Since $\gamma_{2,1}^{(N)} \in S(N)$ it stabilizes V and we obtain a sequence $\{\gamma_{2,1}^{(N)}(q_k)\}_{k=1}^\infty \subset V$ that converges to v_1, thus implying the claim.

Now consider some point $r \in V \cap \mathcal{U}_\infty$. Because of Assertion (i), Proposition 8.1 implies that $\{\eta_n\}_{n=0}^\infty$ converges uniformly to the identity on the open set $B_\delta(p') \cap S_{A,B,C,D} \subset B_{\epsilon/2}(p) \cap V$. In turn, since V is a Fatou component, this implies that $\{\eta_n\}_{n=0}^\infty$ actually converges uniformly to the identity on any compact subset of V (see the claim in the proof of Proposition 10.4). Applying this to the singleton set $\{r\}$, we find that $\eta_n(r) \to r$. In contrast, Assertion (ii) from Lemma 9.8 implies that $\mathrm{dist}(\eta_n(r), \mathcal{V}_\infty) \to 0$, providing a contradiction.

We conclude that any Fatou component for $\mathcal{G}_{A,B,C,D}$ that intersects $B_{\epsilon/2}(p) \cap S_{A,B,C,D}$ non-trivially must be bounded. □

Theorem K. *Suppose that* $(A, B, C, D) \in \mathbb{C}^4_{\text{good}}$ *and that* V *is a bounded Fatou component for* $\mathcal{G}_{A,B,C,D}$*, then the stabilizer* $\mathcal{G}_V$ *of* V *is cyclic.*

It should be noticed that, in each of the examples from Section 8 where we prove that $\mathcal{G}_{A,B,C,D}$ is locally non-discrete on some open $U \subset S_{A,B,C,D}$, the proof was carried out by producing non-trivial elements of the sets $S(n)$ of iterated commutators that converge to the identity on U as n tends to infinity. The theorem above asserts that, if in addition we have $(A, B, C, D) \in \mathbb{C}^4_{\text{good}}$, then this set U does not intersect any bounded Fatou component of $\mathcal{G}_{A,B,C,D}$. Indeed, if V were to be a Fatou component intersecting U then for all sufficiently large n the elements of $S(n)$ would stabilize V so that $\mathcal{G}_V$ would not be Abelian.

Lemma 11.1. *Assume that* V *is a bounded Fatou component of* $\mathcal{G}_{A,B,C,D}$*. Then the closure* $G = \overline{\mathcal{G}_V}$ *of* $\mathcal{G}_V$ *in* $\mathrm{Aut}(V)$ *is a compact real Lie group.*

Proof. Since G is a closed subgroup of the Lie group $\mathrm{Aut}(V)$ it is a real Lie group. Thus we only have to show that G is compact.

We first notice that every element of $G = \overline{\mathcal{G}_V}$ preserves the holomorphic volume form Ω defined in (10). Indeed, by construction, Ω is invariant by elements in $\mathcal{G}_V$ and the condition of preserving Ω is clearly closed so that it has to hold for the closure of $\mathcal{G}_V$. Since V is bounded, the total (real) volume of V defined by means of Ω is finite. Hence, we can find a relatively compact open set $K_0 \subset V$ such that $\mathrm{vol}_\Omega(K_0) > \frac{1}{2}\mathrm{vol}_\Omega(V)$. This implies that for every $g \in G$ we have $g(K_0) \cap K_0 \neq \emptyset$.

Let $K = \overline{K_0}$. Since $\mathrm{Aut}(V)$ acts properly on V,

$$\{\alpha \in \mathrm{Aut}(V) \, : \, \alpha(K) \cap K \neq \emptyset\}$$

is a compact subset of $\mathrm{Aut}(V)$. It follows that the closed subset G is compact as well. □

Proof of Theorem K. As an abstract group, $\mathcal{G}$ is isomorphic to the congruence group Γ_2 that is defined in (18). Since any Abelian subgroup of a non-elementary Fuchsian group is cyclic, it suffices to prove that $\mathcal{G}_V$ is Abelian.

We begin by pointing out that every element in $\mathcal{G}_V$ is hyperbolic. Indeed, parabolic elements are conjugate to one of the generators g_x, g_y, or g_z, and hence are such that every open set of $S_{A,B,C,D}$ contains points whose orbit under (any) parabolic element is unbounded. Thus $\mathcal{G}_V$ cannot possess parabolic elements. Finally, it cannot possess elliptic elements either since $\mathcal{G}$ contains no elliptic element; see Definition/Proposition 9.3 and Remark 9.4.

Now, suppose for contradiction that there are two non-commuting elements $\eta, \tau \in \mathcal{G}_V$. By using again the isomorphism between $\mathcal{G}$ and Γ_2, we see that η and τ correspond to hyperbolic elements in Γ_2 which do not commute. In particular, their iterates also do not commute since two hyperbolic elements of $SL(2, \mathbb{Z})$ commute if and only if they have the same axes of translation in $\mathbb{H}^2$.

Owing to Lemma 11.1, the closure $G = \overline{\mathcal{G}_V}$ is a compact Lie Group. Since η and τ are hyperbolic elements, they have infinite order. We can therefore find subsequences of the iterates η^{n_k} and τ^{n_ℓ} converging to the identity and such that the elements of this subsequence are pairwise different. Thus the subgroup of G generated by η and τ non-trivially accumulates on the identity which implies that the dimension of G itself as a real Lie group is strictly positive. Furthermore, since η^{n_k} and τ^{n_ℓ} do not commute for any k and ℓ, we also have that the identity component G_0 of G is non-Abelian. On the other hand, the only compact connected real Lie groups of dimension one or two being Abelian (tori), it follows that, in fact, we have $\dim_{\mathbb{R}}(G) \geq 3$, where $\dim_{\mathbb{R}}(G)$ stands for the dimension of G as real Lie group.

Conversely, the condition that the parameters (A, B, C, D) are in the set $\mathbb{C}^4_{\text{good}}$ implies that for any point $p \in V$ the orbit $G_0(p)$ of p under G_0 is diffeomorphic to G_0. In particular, $\dim_{\mathbb{R}}(G_0) \leq 4$. However, if we had $\dim_{\mathbb{R}}(G_0) = 4$, then $G_0(p)$ would be four dimensional, implying that $G_0(p) = V$. This is clearly impossible since G_0 is compact and V is open. Summarizing, we must have $\dim_{\mathbb{R}}(G) = 3$.

The action of G_0 on V is smooth, proper, and free (since (A,B,C,D) lies in $\mathbb{C}^4_{\rm good}$). It follows that the quotient space V/G_0 can be given a structure of a smooth manifold with

$$\dim_{\mathbb{R}}(V/G_0) = \dim_{\mathbb{R}}(V) - \dim_{\mathbb{R}}(G_0) = 1$$

in such a way that the quotient map $\pi : V \to V/G_0$ is a submersion. Thanks to the classical result of Ehresmann, this gives V the structure of a fiber bundle $V \to V/G_0$ where the fibers are diffeomorphic to G_0, see, for example, [42]. In particular, the base V/G_0 is of dimension 1.

As a one-dimensional smooth manifold, it follows that V/G_0 is either $\mathbb{S}^1$ or $\mathbb{R}$. The former case is impossible because V is non-compact, while the total space of a fiber bundle with compact base and compact fibers is compact.

We will now show that the possibility of having $V/G_0 = \mathbb{R}$ cannot occur either. For this, note first that our assumption on parameters implies that $S_{A,B,C,D}$ is smooth and hence the closure $\overline{S}_{A,B,C,D}$ is smooth in $\mathbb{CP}^3$. It is therefore biholomorphic to the blow-up of $\mathbb{CP}^2$ at six points, implying that $\overline{S}_{A,B,C,D}$ is simply connected. If we choose some point $p_0 \in V$ then the orbit of G_0 through p_0 gives an embedding of G_0 into $S_{A,B,C,D}$. Since $\overline{S}_{A,B,C,D}$ is simply connected, it follows that $\overline{S}_{A,B,C,D} \setminus G_0(p_0)$ has two connected components U_1 and U_2, see, for example [14, Proposition 7.1.1]. Moreover, one of these components, say U_1, contains the triangle at infinity Δ_∞ and the other component U_2 is bounded in $S_{A,B,C,D}$.

By Theorem C, the Julia set $\mathcal{J}_{A,B,C,D}$ is connected. The Julia set is also unbounded since it contains the fibers S_{x_0} for $x_0 \in (-2,2)$ by virtue of Lemma 5.3. Therefore, $\mathcal{J}_{A,B,C,D} \subset U_1$ and $U_2 \subset V$.

Recalling that $\pi : V \to \mathbb{R}$ stands for the bundle projection, we clearly can assume without loss of generality that $\pi(p_0) = 0$. The fiber bundle structure implies that $V \setminus G_0(p_0)$ has two connected components. One of them corresponds to $U_1 \cap V$ and the other to $U_2 \subset V$. Since π is non-zero on each component we can suppose that $\pi(U_1 \cap V) = (0,\infty)$ and $\pi(U_2) = (-\infty, 0)$. However, notice that $U_2 \cup G_0(p_0)$ is closed and bounded (in $\mathbb{C}^3$) and hence it is compact. This implies that π attains a minimum value on $U_2 \cup G_0(p_0)$ which, in turn, contradicts the fact that $\pi(U_2) = (-\infty, 0)$.

We conclude that any two elements of the stabilizer $\mathcal{G}_V$ must commute, and hence that $\mathcal{G}_V$ is cyclic. □

12. Coexistence: Proof of Theorems F and G

In this section we combine the previous results to prove Theorem F about the coexistence of local discreteness and non-discreteness for an open set of parameters and Theorem G about coexistence of Fatou set and Julia set with non-empty interior for the same set of parameters, after removing countably many real-algebraic hypersurfaces $\mathcal{H}$; see Proposition 10.9.

Proof of Theorem F. Lemma 8.5 and Proposition 8.7 give that there is an open neighborhood $\mathcal{P}_1 \subset \mathbb{C}^4$ that contains $(0,0,0,0)$ and that contains each of the Dubrovin-Mazzocco parameters $(A(a), B(a), C(a), D(a))$ for $a \in (-2,2)$ such that for each $(A,B,C,D) \in \mathcal{P}_1$ we have that $\mathcal{G}_{A,B,C,D}$ is locally non-discrete on a non-empty open set $U \subset S_{A,B,C,D}$. Moreover, the proofs these results are obtained by showing that for arbitrarily large n there are non-trivial elements of the sets $S(n)$ of iterated commutators of "level n" from Proposition 8.1. Therefore, for each of these parameters values we have non-commuting elements arbitrarily close to the identity on U.

On the other hand, Theorem E and the proof of Proposition 7.1 ensure the existence of an open set $\mathcal{P}_2 \subset \mathbb{C}^4$ containing $(0,0,0,0)$ along with each of the Dubrovin-Mazzocco parameters $(A(a), B(a), C(a), D(a))$, with $a \in (-2,2)$, such that for each $(A,B,C,D) \in \mathcal{P}_2$ we have:

(i) A point $p = (u,u,u) \in \mathbb{C}^3$ and an $\epsilon > 0$ with the following property. For every point q in the ball $\mathbb{B}_\epsilon(p)$ and any non-trivial $\gamma \in \mathcal{G}_{A,B,C,D}$ one of the coordinates of $\gamma(q)$ has modulus greater than $|u| + \epsilon$. In particular, any non-trivial $\gamma \in \mathcal{G}_{A,B,C,D}$ satisfies $\gamma(q) \notin \mathbb{B}_\epsilon(p)$.

(ii) Some point $q_0 \in \mathbb{B}_\epsilon(p) \cap S_{A,B,C,D}$ which must therefore be in $\mathcal{F}_{A,B,C,D}$.

Let V_{BQ} be the Fatou component containing the point q_0 appearing in item (ii) above. According to Proposition 10.1, one of the following holds:

(a) there is a sequence of mappings $f_n \in \mathcal{G}_{A,B,C,D} \setminus \{\mathrm{id}\}$ that converges uniformly on compact subsets of V_{BQ} to the identity, or
(b) the action of $\mathcal{G}_{A,B,C,D}$ on V_{BQ} is properly discontinuous.

However, Case (a) is impossible because it would follows that $f_n(q_0) \to q_0$ and, in particular, that $f_n(q_0) \in \mathbb{B}_\epsilon(p)$ for sufficiently large n. This contradicts the preceding item (i). We must therefore have that Case (b) holds on V_{BQ}.

Therefore, the local non-discreteness of $\mathcal{G}_{A,B,C,D}$ on U and its local discreteness (and even the properly discontinuous action) on V_{BQ} coexist in $S_{A,B,C,D}$ for every $(A,B,C,D) \in \mathcal{P} = \mathcal{P}_1 \cap \mathcal{P}_2$. In other words, these groups are locally non-discrete without being "globally non-discrete". □

Proof of Theorem G. Let $\mathcal{P} \subset \mathbb{C}^4$ be the open set of parameters constructed in the proof of Theorem F. For any $(A,B,C,D) \in \mathcal{P}$ the group $\mathcal{G}_{A,B,C,D}$ has a non-trivial Fatou component $V_{\mathrm{BQ}} \subset S_{A,B,C,D}$, as proved in Theorem E.

Therefore, it remains to show that for any $(A,B,C,D) \in \mathcal{P} \setminus \mathcal{H}$ the open set $U \subset S_{A,B,C,D}$ on which $\mathcal{G}_{A,B,C,D}$ is locally non-discrete (from Theorem F) satisfies $U \subset \mathcal{J}_{A,B,C,D}$. Here, $\mathcal{H}$ is the countable union of real-algebraic hypersurfaces provided by Proposition 10.9. Recall from Theorem F that there are non-commuting pairs of elements of $\mathcal{G}_{A,B,C,D}$ arbitrarily close to the identity on U. Therefore, Theorem K gives that U is disjoint from any bounded Fatou component of $\mathcal{G}_{A,B,C,D}$. In fact, to ensure that U is disjoint from any bounded Fatou component is the only place in the proof where the parameters in $\mathcal{H}$ need to be removed from the set of parameters $\mathcal{P}$.

We will now use Theorem H to show that U is also disjoint from any unbounded Fatou component, and this for every $(A,B,C,D) \in \mathcal{P}$. Since this requires more specific details, the discussion will be split into two cases in order to make the argument more clear. Also, in the sequel, we are allowed to reduce the size of the open set U, if necessary.

Case 1: When (A,B,C,D) is sufficiently close to $(0,0,0,0)$.

We saw in the proof of Lemma 8.5 that that if A, B, and C are all sufficiently close to 0 and if $h_x = g_x^2, h_y = g_y^2$, and $h_z = g_z^2$, then there is some $\epsilon > 0$ such that for any $h \in \{h_x, h_y, h_z\}$ we have

$$\sup_{q \in \mathbb{B}_\epsilon(\mathbf{0})} \|h(q) - q\| < K(\epsilon).$$

Here, $K(\epsilon)$ is the constant from Proposition 8.1. Therefore, if we let $S(0) = \{h_x, h_x^{-1}, h_y, h_y^{-1}, h_z, h_z^{-1}\}$ and define the sets $S(n)$ of iterated commutators of "level n" for each $n \geq 0$, it then follows from Proposition 8.1 that for any $\gamma \in S(n)$ we have

$$\sup_{q \in \mathbb{B}_{\epsilon/2}(\mathbf{0})} \|\gamma(q) - q\| \leq \frac{K(\epsilon)}{2^n}.$$

Since the relationship between ϵ and $K(\epsilon)$ given by (29) is linear, it follows that for each $\gamma \in S(1)$ we have

$$\sup_{q \in \mathbb{B}_{\epsilon/2}(\mathbf{0})} \|\gamma(q) - q\| < K(\epsilon/2). \tag{37}$$

Let

$$\gamma_{1,2} = [h_x, h_z], \quad \gamma_{1,3} = [h_y, h_z], \quad \text{and} \quad \gamma_{2,3} = [h_y, h_x]$$

and let $\gamma_{2,1} = \gamma_{1,2}^{-1}$, $\gamma_{3,1} = \gamma_{1,3}^{-1}$, and $\gamma_{3,2} = \gamma_{2,3}^{-1}$. It is straightforward to check that these six mappings satisfy Hypothesis (A) of Theorem H. For example, one has

$$\begin{aligned}\gamma_{1,2} = [h_x, h_z] = h_x^{-1} h_z^{-1} h_x h_z &= (s_y s_z s_y s_z)(s_x s_y s_x s_y)(s_z s_y s_z s_y)(s_y s_x s_y s_x) \\ &= s_y s_z s_y s_z s_x s_y s_x s_y s_z s_y s_z s_x s_y s_x.\end{aligned}$$

Since the right-hand side represents a cyclically reduced composition containing all three mappings s_x, s_y, and s_z, Definition/Proposition 9.3 implies that $\gamma_{1,2}$ is hyperbolic. Moreover, since the first (right-most) mapping is s_x we have $\mathrm{Ind}(\gamma_{1,2}) = v_1$ and since the last (left-most) mapping is s_y we have $\mathrm{Attr}(\gamma_{1,2}) = v_2$.

Meanwhile, estimate (37) implies that these six mappings $\gamma_{i,j} \in S(1)$ satisfy Hypothesis (B) of Theorem H on the ball of radius $\epsilon/2$. Therefore, for all (A, B, C, D) close enough to the origin in $\mathbb{C}^4$ the ball $\mathbb{B}_{\epsilon/2}(0) \cap S_{A,B,C,D} \subset U$ is disjoint from any unbounded Fatou component of $\mathcal{G}_{A,B,C,D}$.

Case 2: When (A, B, C, D) is close to Dubrovin-Mazzocco parameters.

We saw in Section 8 that if we let

$$A(a) = B(a) = C(a) = 2a + 4, \quad \text{and} \quad D(a) = -(a^2 + 8a + 8)$$

then for any $a \in (-2, 2)$ the surface $S_a = S_{A(a),B(a),C(a),D(a)}$ has three singular points $p_1(a), p_2(a)$, and $p_3(a)$ given in (32). Each of the singular points is a common fixed point of s_x, s_y, and s_z.

Let us focus on the singular point $p_1(a)$ while pointing out that the entire discussion below applies to $p_2(a)$ and $p_3(a)$ as well. In the proof of Proposition 8.7 it was shown that for any $a \in (-2, 2)$ there is an $\epsilon > 0$, an open neighborhood W_0 of $(A(a), B(a), C(a))$ in $\mathbb{C}^3$, and a sufficiently high iterate k so that if we let

$$f_x = g_x^k, \quad f_y = g_y^{-1} g_x^k\, g_y, \quad \text{and} \quad f_z = g_z^{-1} g_x^k\, g_z$$

then for any $(A, B, C) \in W_0$ and any $f \in \{f_x, f_y, f_z\}$ we have

$$\sup_{q \in \mathbb{B}_\epsilon(p_1(a))} \|f(q) - q\| < K(\epsilon).$$

Let

$$\gamma_{1,2} = [f_x, f_z], \quad \gamma_{1,3} = [f_y, f_z], \quad \text{and} \quad \gamma_{2,3} = [f_y, f_x]$$

and let $\gamma_{2,1} = \gamma_{1,2}^{-1}$, $\gamma_{3,1} = \gamma_{1,3}^{-1}$, and $\gamma_{3,2} = \gamma_{2,3}^{-1}$. One can then check that these six mappings satisfy Hypothesis (A) of Theorem H. For example, one has that

$$\begin{aligned}\gamma_{1,2} = [f_x, f_z] = f_x^{-1} f_z^{-1} f_x f_z &= (s_y s_z)^k\ (s_x s_y)(s_y s_z)^k (s_y s_x)\ (s_z s_y)^k\ (s_x s_y)(s_z s_y)^k (s_y s_x) \\ &= (s_y s_z)^k\ (s_x s_z)(s_y s_z)^{k-1} (s_y s_x)\ (s_z s_y)^k\ (s_x s_y)(s_z s_y)^{k-1} (s_z s_x).\end{aligned}$$

This is a cyclically reduced composition containing all three mappings s_x, s_y, and s_z and therefore it represents a hyperbolic mapping thanks to Definition/Proposition 9.3. Moreover, since the first (right-most) mapping is s_x we have $\mathrm{Ind}(\gamma_{1,2}) = v_1$ and since the last (left-most) mapping is s_y we have $\mathrm{Attr}(\gamma_{1,2}) = v_2$.

As in the previous example, these six commutators may not be $K(\epsilon)$ close to the identity on $\mathbb{B}_\epsilon(p_1(a))$. However, we can again use the linearity of the dependence of $K(\epsilon)$ on ϵ to see that they satisfy Hypothesis (B) of Theorem H on $\mathbb{B}_{\epsilon/2}(p_1(a))$. Therefore, for all (A, B, C, D) close enough to $(A(a), B(a), C(a), D(a))$ the ball $\mathbb{B}_{\epsilon/2}(p_1(a)) \cap S_{A,B,C,D} \subset U$ is disjoint from any unbounded Fatou component of $\mathcal{G}_{A,B,C,D}$.

We have possibly reduced the size of the open set of parameters $\mathcal{P} \subset \mathbb{C}^4$, while still containing $(0, 0, 0, 0)$ and still containing each of the Dubrovin-Mazzocco parameters $(A(a), B(a), C(a), D(a))$ for $a \in (-2, 2)$. We have also possibly reduced the size of the open set U on which $\mathcal{G}_{A,B,C,D}$ is locally non-discrete in which a way that for any $(A, B, C, D) \in \mathcal{P} \setminus \mathcal{H}$ we have $U \subset \mathcal{J}_{A,B,C,D}$.

Therefore for all $(A, B, C, D) \in \mathcal{P} \setminus \mathcal{H}$ the group $\mathcal{G}_{A,B,C,D}$ has a non-empty Fatou component $V_{\mathrm{BQ}} \subset \mathcal{F}_{A,B,C,D}$ and a Julia set $\mathcal{J}_{A,B,C,D}$ with non-empty interior. $\square$

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