A DYNAMICAL APPROACH TO STUDYING THE LEE-YANG ZEROS FOR THE POTTS MODEL ON THE CAYLEY TREE

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Abstract. Let $Z_n(z,t)$ denote the partition function of the q-state Potts Model on the rooted binary Cayley tree of depth n. Here, $z = \mathrm{e}^{-h/T}$ and $t = \mathrm{e}^{-J/T}$ with h denoting an externally applied magnetic field, T the temperature, and J a coupling constant. One can interpret z as a "magnetic field-like" variable and t as a "temperature-like" variable. Physical values $h \in \mathbb{R}$, $T > 0$, and $J \in \mathbb{R}$ correspond to $t \in (0,\infty)$ and $z \in (0,\infty)$. For any fixed $t_0 \in (0,\infty)$ and fixed $n \in \mathbb{N}$ we consider the complex zeros of $Z_n(z,t_0)$ and how they accumulate on the ray $(0,\infty)$ of physical values for z as $n \to \infty$. In the ferromagnetic case ($J > 0$ or equivalently $t \in (0,1)$) these Lee-Yang zeros accumulate to at most one point on $(0,\infty)$ which we describe using explicit formulae. In the antiferromagnetic case ($J < 0$ or equivalently $t \in (1,\infty)$) these Lee-Yang zeros accumulate to finitely many points of $(0,\infty)$, which we again describe with explicit formulae. The same results hold for the unrooted Cayley tree of branching number two.

These results are proved by adapting a renormalization procedure that was previously used in the case of the Ising model on the Cayley Tree by Müller-Hartmann and Zittartz (1974 and 1977), Barata and Marchetti (1997), and Barata and Goldbaum (2001). We then use methods from complex dynamics and, more specifically, the active/passive dichotomy for iteration of a marked point, along with detailed analysis of the renormalization mappings, to prove the main results.

1. Introduction

This paper concerns the Lee-Yang zeros for the $q \geq 2$ state Potts Model on the binary Cayley Tree. Because of the recursive nature in which the Cayley Tree is constructed, there is a suitable renormalization procedure which makes this problem amenable to methods from dynamical systems. This paper is intended for readers from statistical physics and also from mathematics (especially dynamical systems), so we will provide considerable background and motivation in Sections 1 and 2.

1.1. Lee-Yang zeros for the Ising Model.

Let $(\Gamma_n)_{n=1}^{\infty}$ be a sequence of graphs, and let $(V_n)_{n=1}^{\infty}$ and $(E_n)_{n=1}^{\infty}$ be the corresponding vertex set (sites) and edge set (bonds). Each such graph is interpreted as a finite approximation to a magnetic material and classically one might let Γ_n be an $n \times n$ piece of the $\mathbb{Z}^2$ square lattice or an $n \times n \times n$ piece of the $\mathbb{Z}^3$ cubical lattice.

In the Ising model, one magnetic particle (an electron, for example) is at each vertex, and two particles interact if and only if they are connected by an edge. Each particle is assigned a magnetic moment, called *spin*, which is represented in the model by the discrete variable $\sigma(i) \in \{-1,+1\}$ which describes the spin at vertex i. For each spin configuration, $\sigma : V_n \to \{\pm 1\}$ define the Hamiltonian (energy) of σ by

$$H_n(\sigma) := -J \cdot \sum_{<i,j> \in E_n} \sigma(i)\sigma(j) - h \cdot \sum_{i \in V_n} \sigma(i). \tag{1}$$

Here $J > 0$ is the ferromagnetic coupling constant and h is the strength of the external magnetic field.

For a fixed temperature $T > 0$, the Gibbs-Boltzman weight of the spin configuration σ is given by

$$W_n(\sigma) := e^{-H_n(\sigma)/T}. \tag{2}$$

(We set the Boltzman constant $k_B = 1$.) Summing over all the possible spin configurations gives the partition function

$$Z_n(h,T) := \sum_\sigma W_n(\sigma) = \sum_\sigma e^{-H_n(\sigma)/T}. \tag{3}$$

The probability $P_n(\sigma)$ of the system being in the state σ is proportional to $W_n(\sigma)$. Thus,

$$P_n(\sigma) = W_n(\sigma)/Z_n(h,T).$$

Since $Z_n(h,T)$ is a finite sum of exponentials, it can never be zero for real values of h and T. However, the zeros of $Z_n(h,T)$ at *complex* values of h or T do occur. The way in which they accumulate to real values of h and T in the limit as n tends to infinity gives information about the phase transitions in the model. This interpretation dates back to the works of Lee and Yang [43, 29]. We will use this interpretation as a motivation for studying the complex zeros of $Z_n(h,T)$ for the Cayley Tree and their limiting behavior as n tends to infinity, without pursuing more deeply how this relates to phase transitions.

By letting

$$z = e^{-h/T} \text{ (field-like) and } t = e^{-J/T} \text{ (temperature-like)}, \tag{4}$$

we get $Z_n(z,t)$ as a polynomial of z and t when multiplied by $z^{|V_n|}t^{|E_n|}$ to clear the denominator. For the physical values $T > 0$ and $h \in \mathbb{R}$, we must have $t \in (0,1)$ and $z \in (0,\infty)$. Because the partition function becomes a polynomial in the (z,t) variables we will study it exclusively in terms of z and t.

The initial studies of phase transitions for the Ising model considered what happens as T is varied with fixed $h = 0$. This corresponds to setting $z = 1$ and studying the zeros of $Z_n(1,t)$ in the complex t plane and how they accumulate to points on the interval $(0,1)$ of physically relevant values of t. Such zeros in the complex t plane are called *Fisher zeros* in honor of Michael Fisher [21, 8].

It is also interesting to fix $T = T_0 > 0$ and to vary h. This corresponds to fixing a value of $t = t_0 \in (0,1)$ and studying the the zeros of $Z_n(z,t_0)$ in the complex z plane and also how they accumulate to the real ray $(0,\infty)$ of physically relevant values of z. In 1952, Tsung-Dao Lee and Chen-Ning Yang published two important papers in statistical mechanics and proved a series of Lee-Yang theorems [43, 29]. The most interesting theorem is the following:

Theorem 1 (Lee-Yang Circle Theorem). *For $t \in [0,1]$, the complex zeros in z of the partition function $Z(z,t)$ of the Ising model on any graph lie on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.*

Interpreting parameters $z_0 \in (0,\infty)$ at which zeros of $Z_n(t_0,z)$ accumulate when $n \to \infty$ as corresponding to phase transitions, this implies that for any fixed $t_0 \in [0,1]$, the only physical parameter at which we could potentially observe a phase transition is when $z_0 = 1$ (equivalently when $h_0 = 0$). Because of this impressive theorem, the zeros of $Z_n(z,t_0)$ in the complex z plane are called *Lee-Yang* zeros.

1.2. **Potts Model with an external magnetic field and its Lee-Yang zeros.** Much of the discussion of the previous section carries over directly to the Potts model. Let $q > 2$ be a natural number. Now one can consider configurations $\sigma : V_n \longrightarrow \{0,\cdots,q-1\}$. If $J > 0$ is the coupling constant, then the energy of such a configuration when exposed to external magnetic field $h \in \mathbb{R}$ is defined to be

$$H_n(\sigma) = -J \sum_{(i,j)\in E_n} \delta(\sigma(i),\sigma(j)) - h \sum_{i\in V_n} \delta(\sigma(i),0). \tag{5}$$

See, for example, Equation 20 in [41]. (Here, $\delta(i,j) = 1$ if $i = j$ and $\delta(i,j) = 0$ otherwise.) The partition function $Z_n(z,t)$ and probability $P(\sigma)$ with which a spin configuration occurs are defined exactly in the same way as in the previous section, except that the new Formula (5) for the energy $H_n(\sigma)$ is used instead of the one in the Ising model.

Like for the Ising Model, it is convenient to express the partition function in terms of the "field-like" and "temperature-like" variables z and t that were defined in (4). In the context of the Potts

Model, one again calls the zeros of $Z_n(1,t)$ in the complex t plane the *Fisher zeros* and, for fixed $t_0 \in [0,1]$, the zeros of $Z_n(z,t_0)$ in the complex z plane, the *Lee-Yang zeros.*

Remark that for $t=0$ and $t=1$ it is easy to compute the partition function. For any connected graph Γ with k vertices one finds

$$Z_\Gamma(z,0) = z^{-k} + (q-1) \qquad \text{and} \qquad Z_\Gamma(z,1) = \left(q-1+z^{-1}\right)^k. \tag{6}$$

In particular the Lee-Yang zeros when $t=0$ are the k-th roots of $1/(1-q)$ and the Lee-Yang zeros when $t=1$ all k Lee-Yang zeros are equal to $1/(1-q)$. When $t \in (0,1)$ the Lee-Yang zeros are much more difficult to compute and they depend on the graph Γ and the temperature-like variable t in a non-trivial way.

Because of its complexity, the q-state Potts model with the presence of an external magnetic field has been researched by just a few authors, as described in Section 1.8.

1.3. **The antiferromagnetic case.** As in Sections 1.1 and 1.2 above, it is customary to choose the coupling constant J to be positive, making it energetically favorable for spins at neighboring vertices to be aligned. It corresponds to the *ferromagnetic* materials. However, there are some physical systems for which the opposite phenomenon holds. They are called *antiferromagnetic* and for such systems one assumes that $J<0$.

1.4. **Binary Cayley Tree.** The n^{th}*-level rooted binary Cayley tree* is a tree (a simple, undirected, connected, and finite graph in which any two vertices are connected by exactly one path) in which one vertex, called the root, is of degree two with all leaves (vertices of degree one) at a distance (minimum number of edges to connect) n from the root, and all the other vertices are of degree three. The n^{th}*-level unrooted binary Cayley tree* is a tree in which all vertices have degree 3 or 1 and for which there exists a unique vertex of degree 3 such that all of the leaves are of distance n from that specified vertex. The remainder of this paper we will denote the rooted binary Cayley Tree of level n by Γ_n and the unrooted binary Cayley Tree of level n by $\hat{\Gamma}_n$. We will consider the q-state Potts Model on these two families of graphs, as n approaches infinity, for the remainder of this paper.

1.5. **Plots of some Lee-Yang zeros for the Ising and Potts models on the Cayley tree.** There is a convenient recursive formula that allows one to compute the Lee-Yang zeros of the n^{th} rooted Cayley tree for the relatively small values of n. See Theorem D in Section 1.6 below. It applies to both the Ising and Potts models. Using that theorem, we have generated figures plotting Lee-Yang zeros for the Ising model and 3-state Potts model.

The left side of Figure 2 shows the Lee-Yang zeros for the Ising model on the binary Cayley tree when $n=5$ and when $t=0.0625$. Note that the zeros appear to lie perfectly on the unit circle $|z|=1$ as described by the Lee-Yang circle theorem. Moreover, there exists a critical temperature $t_{\text{crit}} = 1/3 > 0$ such that when $t < t_{\text{crit}}$ the Lee-Yang zeros accumulate to 1 on the positive real

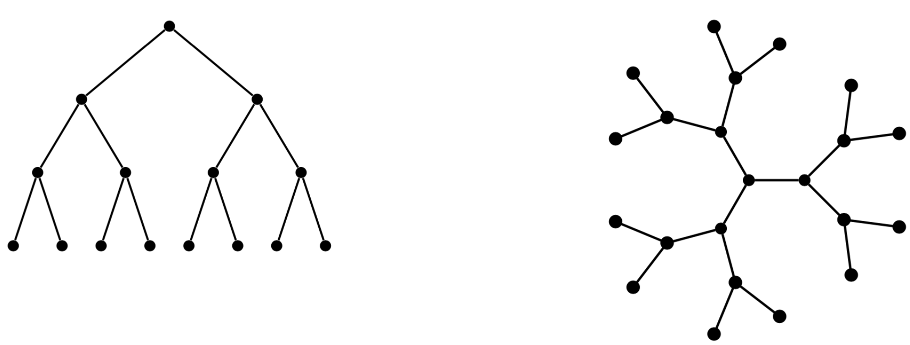

FIGURE 1. Left: rooted binary Cayley Tree Γ_4 of depth 4. Right: unrooted binary Cayley tree $\hat{\Gamma}_4$ of depth 4.

axis and such that when $t > t_{\text{crit}}$, the Lee-Yang zeros stay away from the positive real axis (right side of Figure 2). (For a rigorous justification of this phenomenon, see [13, Theorem A].)

The situation is dramatically different for the 3-state Potts model on the binary Cayley tree because the Lee-Yang zeros no longer lie on the unit circle. In fact, the Lee-Yang zeros of the 5^{th} rooted binary Cayley tree for the 3-state Potts model seem to lie inside the unit circle (left side of Figure 3). Furthermore, there is a critical temperature $t_{\text{crit}} = \frac{1+\sqrt{73}}{36} \approx 0.265$ such that for any $t \leq t_{\text{crit}}$, these zeros accumulate to a point on the positive real axis and such that when $t > t_c$, the Lee-Yang zeros stay away from the positive real axis (right side of Figure 3). (This will be rigorously justified in Theorem A.)

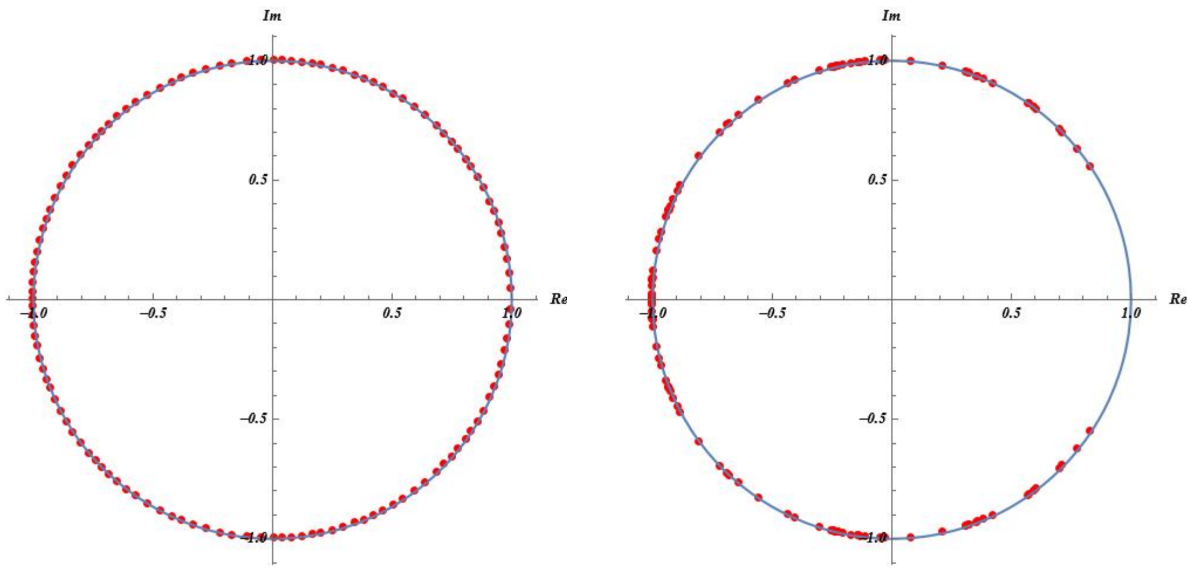

FIGURE 2. The Lee-Yang zeros for the 5^{th} rooted Cayley Tree with branching number 2 (Ising model). Left: $t = 0.0625 < t_c = 1/3$. Right: $t = 0.5 > t_c = 1/3$.

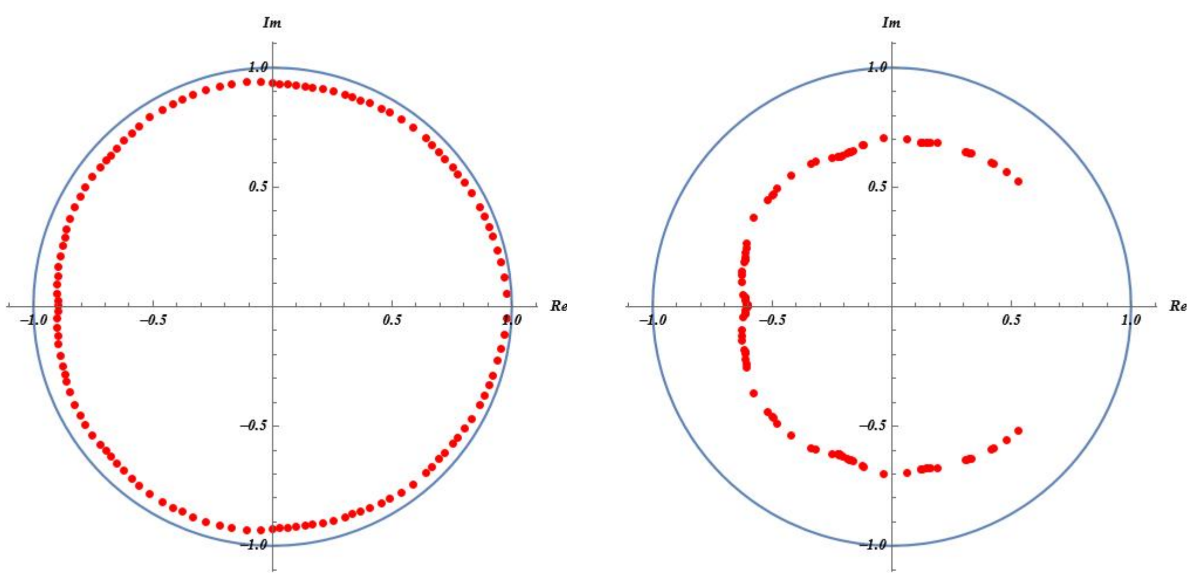

FIGURE 3. The Lee-Yang zeros for the 5^{th} rooted Cayley Tree with branching number 2 for 3-state Potts model. Left: $t = 0.0625 < t_c = \frac{1+\sqrt{73}}{36} \approx 0.265$. Right: $t = 0.5 > t_c = \frac{1+\sqrt{73}}{36} \approx 0.265$.

1.6. **Main Results.** For the remainder of the paper we will denote by $Z_n(z,t,q)$ the partition function for the q state Potts model on the rooted binary Cayley tree of depth n. The partition function for the unrooted binary Cayley tree of depth n will be denoted by $\hat{Z}_n(z,t,q)$. Often we will drop the dependence of q from the notation when it is clear: $Z_n(z,t) \equiv Z_n(z,t,q)$ and $\hat{Z}_n(z,t) \equiv \hat{Z}_n(z,t,q)$.

Our results consider the sets

$$\begin{aligned} \mathcal{B}(t,q) &:= \overline{\{z \in \mathbb{C} \,:\, Z_n(z,t,q) = 0 \,\text{for some}\, n \in \mathbb{N}\}}, \quad \text{and} \\ \hat{\mathcal{B}}(t,q) &:= \overline{\{z \in \mathbb{C} \,:\, \hat{Z}_n(z,t,q) = 0 \,\text{for some}\, n \in \mathbb{N}\}}. \end{aligned} \tag{7}$$

Following the Lee-Yang approach to studying phase transitions (see Section 1.1) we will be interested in where the sets $\mathcal{B}(t,q)$ and $\hat{\mathcal{B}}(t,q)$ intersect the ray $z \in (0,\infty)$ for various fixed choices of $t \geq 0$ and $q \in \mathbb{N}_{\geq 2}$.

We first consider the ferromagnetic case $J > 0$ in which physical values of z and t correspond to $z \in (0,\infty)$ and $0 < t < 1$. Let

$$t_1(q) = \frac{1}{1+q} \qquad \text{and} \qquad t_2(q) = \frac{q-2+\sqrt{q^2+32q-32}}{18(q-1)}. \tag{8}$$

Note that $t_1(2) = t_2(2) = 1/3$ but that $t_1(q) < t_2(q)$ for all $q \geq 3$.

Theorem A (Ferromagnetic Case). *For any $t \in [0,1]$, as $n \to \infty$ the Lee-Yang zeros for the $q \geq 2$ state Potts Model on the (rooted or unrooted) binary Cayley Tree accumulate to $z \in (0,\infty)$ if and only if $t \in [0, t_2(q)]$. They do so at a single point*

$$z_c(t,q) = \begin{cases} 1 & \text{if } \ 0 \leq t \leq t_1(q) \\ \mathcal{Z}_q(t) & \text{if } \ t_1(q) < t \leq t_2(q) \end{cases}$$

with

$$\mathcal{Z}_q(t) := \frac{\begin{pmatrix} (-27(q-1)^2t^4+18\left(q^2-3q+2\right)t^3+\left(q^2+14q-14\right)t^2+2(q-2)t+1)) \\ -\sqrt{(t-1)((q-1)t+1)(9(q-1)t^2-(q-2)t-1)^3} \end{pmatrix}}{8t((q-2)t+1)^3}. \tag{9}$$

Remark 1. *Theorem A can be reformulated as saying that for any $q \in \mathbb{N}_{\geq 2}$ we have that $\mathcal{B}(t,q) \cap (0,\infty)$ consists of a single point $z_c(t,q)$ for $0 \leq t \leq t_2(q)$ and that $\mathcal{B}(t,q) \cap (0,\infty) = \emptyset$ for $t_2(q) < t \leq 1$. The same holds when $\mathcal{B}(t,q)$ (associated with the rooted Cayley Tree) is replaced with $\hat{\mathcal{B}}(t,q)$ (associated with the unrooted Cayley Tree).*

Remark 2. *In the case of the Ising Model we have $t_1(2) = t_2(2) = 1/3$ so that for $t \in [0,1]$ the only $z \in (0,\infty)$ at which the Lee-Yang zeros can accumulate is $z = 1$. This is consistent with the Lee-Yang circle theorem. However, when $q \geq 3$ an interesting new phenomenon occurs for the Potts Model. For the non-empty interval $t \in (t_1(q), t_2(q)]$ the Lee-Yang zeros accumulate at $z_c(t,q) = \mathcal{Z}_q(t) < 1$. See Figure 4 for the case when $q = 3$.*

We now consider the antiferromagnetic case $J < 0$ in which physical values of z and t correspond to $z \in (0,\infty)$ and $t > 1$. Let

$$t_3(q) = \frac{3\left(3q-6+\sqrt{9q^2-32q+32}\right)}{2(q-1)}. \tag{10}$$

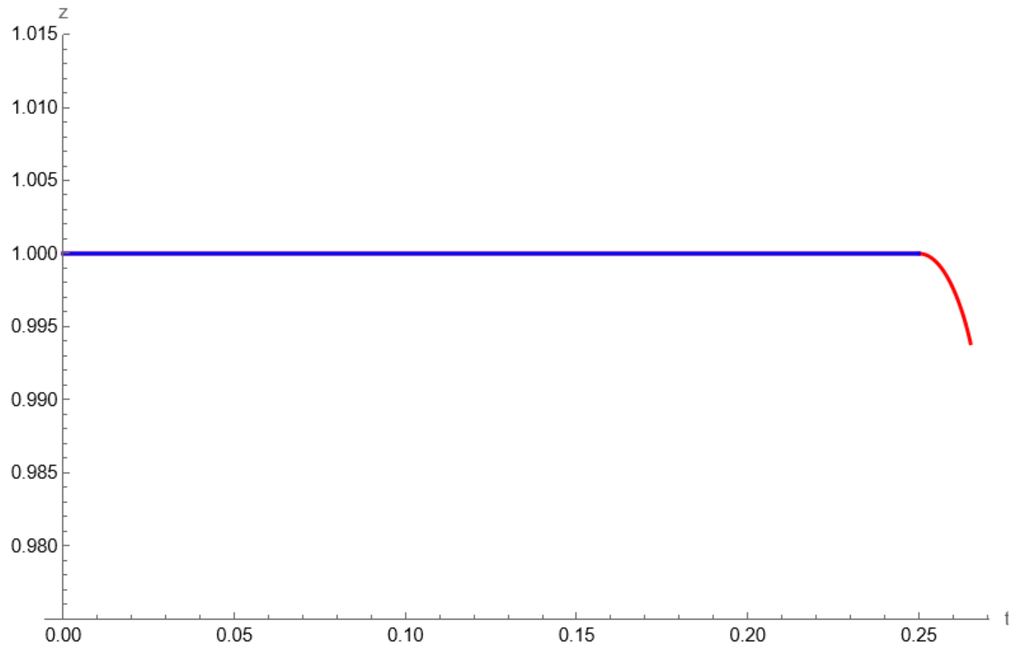

FIGURE 4. Graph of $z = z_c(t,3)$ (Theorem A). For $0 \le t \le 1/4$ we have $z_c(t,3) = 1$ (blue) and for $1/4 \le t \le t_2(q) \approx 0.2651$ we have $z_c(t,3) = \mathcal{Z}_q(t)$ (red).

Theorem B (Antiferromagnetic Case, $q \geq 3$). *For any $t > 1$, as $n \to \infty$ the Lee-Yang zeros for the $q \geq 3$ state Potts Model on the (rooted or unrooted) binary Cayley tree accumulate to $z \in (0,\infty)$ if and only if $t \geq t_3(q)$ and $z = z_c^{\pm}(t,q)$ where*

$$z_c^{\pm}(t,q) := \frac{\begin{pmatrix} (-3-6(-2+q)t-3(2+(-2+q)q)t^2-6(-2+q)(-1+q)t^3+(-1+q)^2t^4) \\ \pm\sqrt{(-1+t)^3(1-t+qt)^3(-9+18t-9qt-t^2+qt^2)} \end{pmatrix}}{8t(1-2t+qt)^3}. \tag{11}$$

A plot of $z_c^{\pm}(t,3)$ versus $t > 1$ is shown in Figure 5.

Remark 3. *Theorem B can be reformulated as saying that for any $q \in \mathbb{N}_{\geq 2}$ we have that $\mathcal{B}(t,q) \cap (0,\infty) = \emptyset$ for $1 \leq t < t_3(q)$ and that $\mathcal{B}(t,q) \cap (0,\infty)$ consists of the (one or) two points $z_c^{\pm}(t,q)$ for $t_3(q) \leq t < \infty$. The same holds when $\mathcal{B}(t,q)$ (associated with the rooted Cayley Tree) is replaced with $\hat{\mathcal{B}}(t,q)$ (associated with the unrooted Cayley Tree).*

The antiferromagnetic case for the Ising Model ($q = 2$) is slightly different than in Theorem B because one also has Lee-Yang zeros accumulating to $z = 1$ when $t \geq t_3(2) = 3$. We record this fact here:

Theorem C (Antiferromagnetic Case, $q = 2$). *For any $t > 1$, as $n \to \infty$ the Lee-Yang zeros for the Ising Model ($q = 2$) on the (rooted or unrooted) binary Cayley tree accumulate to $z \in (0,\infty)$ if and only if $t \geq t_3(q)$ and $z = 1$ or $z = z_c^{\pm}(t,2)$ where*

$$z_c^{\pm}(t,2) := \frac{t^4 - 6t^2 - 3 \pm \sqrt{(t^2-1)^3(t^2-9)}}{8t}. \tag{12}$$

A plot of $z_{\pm}(t,2)$ and also $z = 1$ versus $t > 1$ that illustrates Theorem C is given in Figure 6.

The following theorem is what enables us to use the methods from dynamical systems to prove Theorems A and B above. It is a generalization to the Potts model of a result used by Müller-Hartmann-Zittartz [35], Barata-Marchetti [3], Chio-He-Ji-Roeder [13], and others for the Ising model on the Cayley tree.

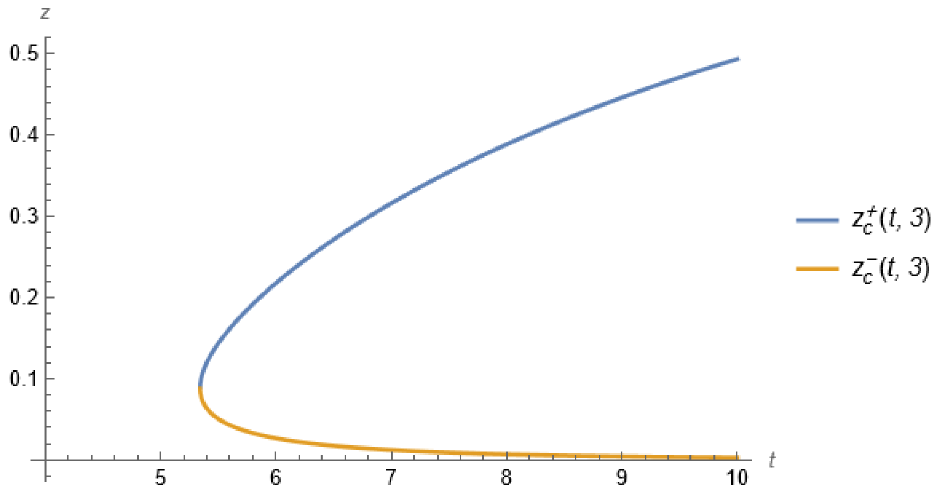

FIGURE 5. Graph of $z_c^-(t,3)$ and $z_c^+(t,3)$ (*Theorem B*).

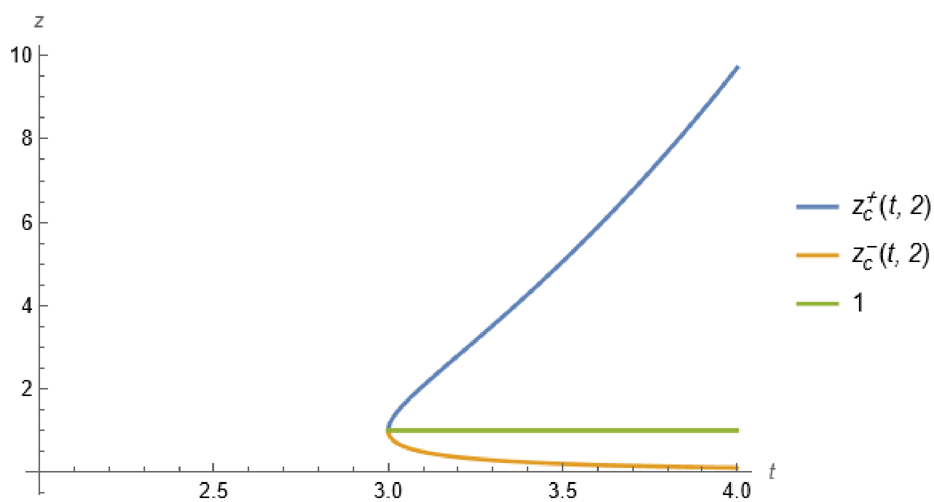

FIGURE 6. Graphs of $z_c^-(t,2)$ and $z_c^+(t,2)$ (Theorem C).

Theorem D (Renormalization Procedure). *For $q \in \mathbb{N}_{\geq 2}$, $t \in \mathbb{R}$, and $z, w \in \mathbb{C}$ define $R_{z,t,q}(w)$ and $\hat{R}_{z,t,q}(w)$ by*

$$R_{z,t,q}(w) := z\left[\frac{t+w+(q-2)tw}{1+(q-1)tw}\right]^2 \qquad \text{and} \qquad \hat{R}_{z,t,q}(w) := z\left[\frac{t+w+(q-2)tw}{1+(q-1)tw}\right]^3. \tag{13}$$

Then:

(i) *the Lee-Yang zeros of the q-state Potts model on the n^{th} level rooted Cayley tree with branching number two are the solutions z to*

$$R^n_{z,t,q}(z) = \frac{1}{1-q}, \qquad \text{and}$$

(ii) *the Lee-Yang zeros of the q-state Potts model on the n^{th} level unrooted Cayley tree with branching number two are the solutions z to*

$$\left(\hat{R}_{z,t,q} \circ R^{(n-1)}_{z,t,q}\right)(z) = \frac{1}{1-q}.$$

Remark 4. *The expression $R^n_{z,t,q}(z)$ means that one first iterates $R_{z,t,q}(w)$ n-times with respect to w and then substitutes $w = z$. The expression in Claim (ii) is interpreted analogously.*

1.7. **Comparison of temperatures $t_1(q)$ and $t_2(q)$ to the works of Wang and Wu and of Müller-Hartmann–Zittartz.** In their 1976 paper [40], Wang and Wu considered the ferromagentic q-state Potts Model on the Cayley tree and they computed the temperature $T_c(q) > 0$ such that the zero magnetic field susceptibility diverges for $T < T_c(q)$ but converges for $T > T_c(q)$. (See also [42, Section I.E].) As they explain, this proves that the limiting free energy $f(T, h)$ is non-analytic as h is varied about 0, at least in the range $0 < T < T_c(q)$. Expressed in our notation, and specialized to the binary Cayley tree, their result is

$$\frac{J}{T_c(q)} = \ln\left(\frac{q+\sqrt{2}-1}{\sqrt{2}-1}\right) \qquad \text{corresponding to} \qquad t_c = \mathrm{e}^{-J/T_c} = \frac{\sqrt{2}-1}{q+\sqrt{2}-1}.$$

The reader can check that $0 < t_c(q) < t_1(q)$ for all $q \in \mathbb{N}_{q\geq 2}$. In fact, $t_c(q)$ is the unique value of $t \in (0,1)$ at which $R'_{1,t,q}(1) = \sqrt{2}$. The reason for this connection between $t_c(q)$ and the derivative of the renormalization map $R_{z,t,q}$ (in the context of the Ising model) is nicely explained in the papers of Müller-Hartmann–Zittartz [35] and of Müller-Hartmann [34].

Wang and Wu also explain that the higher derivatives of $f(T, h)$ with respect to h diverge at $h = 0$ over the wider range of temperatures $0 < T < T_{\mathrm{BP}}(q)$, where $T_{\mathrm{BP}}(q)$ is the "Bethe-Peierls" temperature. Thus it is more interesting to compare the Bethe-Peierls temperature $t_{\mathrm{BP}}(q) = \mathrm{e}^{-J/T_{\mathrm{BP}}(q)}$ with $t_1(q)$ and $t_2(q)$. As explained by the papers of Müller-Hartmann–Zittartz and of Müller-Hartmann in the context of the Ising Model, this corresponds precisely to the unique temperature $t_{\mathrm{BP}}(q)$ at which $R'_{1,t,q}(1) = 1$. A simple calculation shows that $t_{\mathrm{BP}}(q) = t_1(q) = \frac{1}{1+q}$. Indeed, this condition shows up naturally in our proof of Theorem A.

1.8. **Related Works.** An early study of the Lee-Yang zeros for the $q \geq 3$ state Potts Model was done numerically by Kim-Creswich [28] in 1998. Their observations included that the Lee-Yang zeros no longer lie on the unit circle $|z| = 1$ when $q \geq 3$. Much closer to our paper is the 2002 paper of Myshlyavtsev-Ananikian-Sloot [24] where the Lee-Yang zeros for the q-state Potts model on the Cayley Tree are studied using a renormalization mapping similar to the one used here, combined with physical reasoning and also numerical experiments. A main novelty of our paper that goes beyond this nice work of Ghulghazaryan-Ananikian-Sloot is the use of the active/passive dichotomy from complex dynamics; see Section 2.1 below.

Other studies regarding the Potts model with non-zero external magnetic field were done by Chang and Shrock in [11] and [12], Shrock and Xu in [38] and [39], McDonald and Moffat [31].

Studies of the Lee-Yang zeros for the Ising Model are more common, so we mention the ones that are closest to the present work. Studies of the Lee-Yang zeros for the Ising Model on the Cayley Tree date back to Müller-Hartmann [34], Müller-Hartmann and Zittartz [35], whose works were followed by that of Barata and Marchetti [3], Barata and Goldbaum [4]. Some of the most recent work was done by Chio, He, Ji, and Roeder [13]. Extensive studies regarding the Lee-Yang zeros for the Ising Model on the Diamond Hierarchical Lattice were done by Bleher, Lyubich, and Roeder [6]. Studies of Locations of Lee-Yang zeros can be found in the works of Bencs, Buys, Guerini, and Peters [5], Regts and Peters [37], Camia, Jiang, and Newman [9], Hou, Jiang, and Newman [25], and the references therein. Studies of the limiting measure of Lee-Yang zeros for the Curie-Weiss Model were done by Kabluchko [27].

Note that studying statistical physics on the Cayley tree falls within the wider context of statistical physics on hierarchical lattices. For a sample of recent works see: Akin and Berker [1], De Simoi and Marmi [17], De Simoi [18], Jiang, Qiao, and Lan [26], Myshlyavtsev, Myshlyavtsev, and Akimenko [36], Alvarez [2], Chio and Roeder [14], Chang, Roeder, and Shrock [12], and Bleher, Lyubich, and Roeder [7].

1.9. **Structure of the paper.** In Section 2 we present some tools from complex dynamics that will be needed to prove Theorems A-C. Section 3 is devoted to the proof of Theorem A and Section 4 is devoted to the proof of Theorem B. It is followed by a short Section 5 which sketches the proof of Theorem C. A technical lemma that is used in the proofs of Theorem A is presented in Appendix A. Even though Theorem D is used in the proofs of Theorems A-C, we have relegated its proof to the Appendix B because it is relatively standard. In particular, it is similar to the derivation of the renormalization mapping for the Ising Model on the Cayley tree that was presented in [13].

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2. Background from complex dynamics and initial consequences for our problem.

The proofs of Theorem A-C are based on the recursive formula in Theorem D and some ideas in complex dynamics, which we present in Section 2.1, below. Section 2.2 concerns application of the methods from Section 2.1 to the Renormalization mapping $R_{z,t,q}(w)$. The ideas presented in Sections 2.1 and 2.2 allow us to plot high-quality computer-generated images of the accumulation set of Lee-Yang zeros of the Potts model on the n^{th}-level binary Cayley tree with branching number two as $n \to \infty$. This will be done in Section 2.3.

2.1. **Active-Passive Dichotomy in Complex Dynamics.**

We refer the reader to [33] for general background on holomorphic dynamics and to [10, Section V] for a similar discussion of the topics presented below that is also written for an audience from statistical physics.

Let $\Lambda \subset \mathbb{C}$ be open and let $\mathbb{C}_\infty$ be the Riemann sphere . Let $(f_\lambda)_{\lambda\in\Lambda}$ be a family of rational maps from the Riemann sphere to itself that depends holomorphically on the parameter λ. Let $a(\lambda)$ be a choice of initial conditions for the iterates of f_λ which also depends holomorphically on λ. It is called a *marked point* for f_λ.

Definition 1. *A marked point $a(\lambda)$ is called* passive *for f_λ at the parameter λ_0 if the sequence $(g_n)_{n=1}^\infty$ of functions defined by $g_n(\lambda) := f_\lambda^n(a(\lambda))$ forms a normal family in some neighborhood of λ_0. Else, we say $a(\lambda)$ is* active *for f_λ at the parameter λ_0.*

See [15, Chapter VII] for the definition of normal family.

Remark 5. *Historically, the above definitions were applied in the case where the marked point $a(\lambda)$ was a critical point of $f_\lambda(z)$ for each $\lambda \in \Lambda$, to study bifurcations in the dynamics of f_λ. See, for example,* [32]. *However, recently there has been considerable interest in the dynamical properties of noncritical marked points, with some of the motivations coming from problems in arithmetic dynamics. As a sample of such recent papers, we refer the reader to* [19, 16, 23, 20] *and the references therein.*

The set of all passive parameters for the marked point $a(\lambda)$ is an open set. It is called the *passive locus for* $a(\lambda)$. Similarly, the set of active parameters for the marked point $a(\lambda)$ is called the *active locus for* $a(\lambda)$.

Lemma 1. *Let ℓ be a natural number. Then, marked point $a(\lambda)$ is passive for $f_\lambda(z)$ at $\lambda = \lambda_0$ if and only if it is passive for the ℓ-th iterate $f_\lambda^\ell(z)$ at $\lambda = \lambda_0$.*

Proof. Suppose that the marked point $a(\lambda)$ is passive for $f_\lambda^\ell(z)$ at λ_0. To show that $a(\lambda)$ is passive for $f_\lambda(z)$ at λ_0, consider any subsequence $(n_k)_k$ of the sequence (n) of all natural numbers. There must then be some $m \in \{0, \dots, \ell-1\}$ such that for infinitely many indices k we have $n_k \equiv m (\mathrm{mod}\, \ell)$. We denote this subsequence by $(n_{k_j})_j$ and for each j we write $n_{k_j} = \ell\, q_j + m$ with $(q_j)_j$ an increasing sequence of natural numbers. Since $a(\lambda)$ is passive for $f_\lambda^\ell(z)$ we can find an open neighborhood U of λ_0 and a further subsequence $(q_{j_p})_p$ of $(q_j)_j$ such that $\left(f^\ell\right)^{q_{j_p}}(a(\lambda))$ converges uniformly on compact subsets of U to some holomorphic function $g(\lambda)$. Then, $f^{\ell\cdot q_{j_p}+m}(a(\lambda))$ converges uniformly on compact subsets of U to $f^m \circ g(\lambda)$. Therefore, given the original subsequence $(n_k)_k$ of (n) we have found a further subsequence $(\ell \cdot q_{j_p} + m)_p$ along which $f^{\ell\cdot q_{j_p}+m}(a(\lambda))$ converges uniformly on compact subsets of U. Therefore, the family of functions $\{f^n(a(\lambda))\}_{n=1}^{\infty}$ is a normal family on U implying that $\lambda_0 \in U$ is a passive parameter for $f_\lambda(z)$.

The converse implication immediately follows from the definition of normal family. $\square$

Definition 2. *Let $z_{rep}(\lambda_0)$ be a repelling fixed point of f_{λ_0}. Because f_λ varies holomorphically with λ, we can find a holomorphic map $z_{rep}(\lambda)$ defined in a neighborhood U of λ_0 such that $z_{rep}(\lambda)$ is a repelling fixed point of f_λ for all $\lambda \in U$. Let $n_0 \in \mathbb{N}$. We say $f_{\lambda_0}^{n_0}$ maps $a(\lambda_0)$* non-persistently *onto the repelling fixed point $z_{rep}(\lambda_0)$ if*

$$f_{\lambda_0}^{n_0}(a(\lambda_0)) = z_{rep}(\lambda_0) \text{ and } f_\lambda^{n_0}(a(\lambda)) \not\equiv z_{rep}(\lambda) \text{ on } U. \tag{14}$$

The definition that f_{λ_0} maps the marked point $a(\lambda_0)$ non-persistently onto a repelling periodic point of period $p > 1$ is defined analogously by replacing f_{λ_0} with $f_{\lambda_0}^p$.

Lemma 2. *Suppose $f_{\lambda_0}^{n_0}$ maps $a(\lambda_0)$ non-persistently onto a repelling periodic point for $f_{\lambda_0}(z)$. Then λ_0 is an active parameter for the marked point $a(\lambda)$ under f_λ.*

If $a(\lambda_0)$ is in the basin of attraction of an attracting periodic point for f_{λ_0}, then λ_0 is a passive parameter for the marked point $a(\lambda)$ under iteration of f_λ.

Even though this lemma is well-known in complex dynamics, for the convenience of the reader we include a sketch of the proof. Those who wish to see a more sophisticated proof can consider [22, Lemma 3.1(2)].

Proof. It suffices to prove the statement when the (repelling or attracting) periodic cycle is a fixed point. Indeed, Lemma 1 allows for a simple adaptation of both of the proofs below the case of a periodic point of higher period.

It follows from the definition of normal families that we can assume without loss of generality that $n_0 = 0$. By the Implicit Function Theorem and conjugation of f_λ by a suitable holomorphically varying Möbius transformation we can also assume without loss of generality that that $z = 0$ is a repelling fixed point of $f_\lambda(z)$ for all λ in some small open neighborhood $U \subset \Lambda$ of λ_0. If we let $r > 0$ be sufficiently small then for each $\lambda \in U$ the topological annulus

$$A_\lambda = f_\lambda(\mathbb{D}_r) \setminus \mathbb{D}_r$$

will serve as a fundamental domain for the dynamics of f_λ near 0. In other words, for any $\lambda \in U$ and any z with $0 < |z| < r$ there is a unique natural number $k(\lambda, z)$ such that $f_\lambda^{k(\lambda,z)}(z) \in A_\lambda$. Because f_λ is Lipshitz continuous for any $\lambda \in U$ we have that $k(\lambda, z) \to \infty$ as $z \to 0$.

Our assumption that $f_\lambda^{n_0}$ maps $a(\lambda_0)$ non-persistently onto the repelling fixed point and our assumption that $n_0 = 0$ implies that $a(\lambda_0) = 0$ and $a(\lambda) \neq 0$ for all $\lambda \neq \lambda_0$ chosen sufficiently close to λ_0. In particular for any λ sufficiently close to λ_0 there exists a natural number $\ell(\lambda) = k(\lambda, a(\lambda))$ such that $f_\lambda^{\ell(\lambda)}(a(\lambda)) \in A_\lambda$. For each $\lambda \in U$ and any $n \in \mathbb{N}$ we have that $f_\lambda^{-n}(A_\lambda)$ is also a topological annulus surrounding 0. Therefore, it follows from the intermediate value theorem (or other basic topological considerations) that for any $n \in \mathbb{N}$ there exists $\lambda_n \in U$ such that $\ell(\lambda_n) = n$.

We now suppose for contradiction that λ_0 is a passive parameter. Restricting U to a smaller neighborhood of λ_0, if necessary, we can suppose that $\lambda \mapsto f_\lambda^n(a(\lambda))$ is a normal family on U. In particular this implies that there is some subsequence $(n_j)_j$ of the sequence of all natural numbers such that $f_\lambda^{n_j}(a(\lambda))$ converges uniformly on compact subsets of U to a holomorphic function $g(\lambda)$. Note that for $\lambda = \lambda_0$ we have that $f_{\lambda_0}^{n_j}(a(\lambda_0)) = f_{\lambda_0}^{n_j}(0) = 0$ for all j implying that $g(0) = 0$. On the other hand, as explained in the previous paragraph, for each j there exists $\lambda_j \in U$ such that $\ell(\lambda_j) = n_j$. Moreover we must have $\lambda_j \to 0$ as $j \to \infty$ because for all λ outside of a given neighborhood of λ_0 we have that $|a(\lambda_0)|$ is uniformly bounded below, implying that $\ell(\lambda)$ is uniformly bounded above for such parameters. Because the convergence of $f_\lambda^{n_j}(a(\lambda))$ to $g(\lambda)$ is uniform on compact subsets of U and because $f_{\lambda_k}^{n_k}(\lambda_k) \in A_\lambda$ for each index k we therefore have

$$|g(0)| = \lim_{k\to\infty} |f_{\lambda_k}^{n_k}(\lambda_k)| \geq r > 0.$$

This contradicts that $g(0) = 0$.

Now suppose that $a(\lambda_0)$ is in the basin of attraction of an attracting fixed point for $f_{\lambda_0}(z)$. As in the previous case, we can assume without loss of generality that $z = 0$ is an attracting fixed point for $f_\lambda(z)$ for all λ in some sufficiently small neighborhood of λ_0. Restricting U further if necessary, there is a uniform choice of $r > 0$ such that for all $\lambda \in U$ the disc $\mathbb{D}_r(0)$ is in the basin of attraction of 0 under iteration of $f_\lambda(z)$. By hypothesis there is a natural number n_0 such that $|f^{n_0}_{\lambda_0}(a(\lambda))| < r/2$. By continuity we will then have that $|f^{n_0}_{\lambda_0}(a(\lambda))| < 3r/4$ for all λ is some open neighborhood $V \subset U$ of λ_0. It is then immediate that the sequence of functions $\lambda \mapsto f^n_\lambda(a(\lambda))$ converges uniformly to $g(\lambda) \equiv 0$ on all of V. Therefore λ_0 is a passive parameter. $\square$

Lemma 3. *Let k and ℓ be distinct natural numbers. Suppose that in any open neighborhood of parameter λ_0 there exist parameters λ_1 and λ_2 such that*

(i) *$a(\lambda_1)$ is in the basin of attraction for an attracting periodic point of prime period k for $f_{\lambda_1}(z)$ and*
(ii) *$a(\lambda_2)$ is in the basin of attraction for an attracting periodic point of prime period ℓ for $f_{\lambda_2}(z)$.*

Then λ_0 is an active parameter for the marked point $a(\lambda)$ under iteration of f_λ.

Proof. Without loss of generality and by Lemma 1, we can assume $k = 1$ and $\ell > 1$. For a contradiction, let's assume that λ_0 is a passive parameter for the marked point $a(\lambda)$ under iteration of $f_\lambda(z)$. Then, there is a connected open neighborhood U of λ_0 on which $(f^n_\lambda(a(\lambda)))^\infty_{n=1}$ forms a normal family. Then, $(f^{\ell\cdot n}_\lambda(a(\lambda)))^\infty_{n=1}$ has a sub-sequence $(f^{\ell\cdot n_k}_\lambda(a(\lambda)))^\infty_{k=1}$ that converges locally uniformly to a holomorphic function $g(\lambda)$ on U. Denote by $z_\bullet$ the attracting fixed point for f_{λ_1} that has $a(\lambda_1)$ in its basin of attraction. By the Implicit Function Theorem, there is an open neighborhood $V \subset U$ of λ_1 and a holomorphic function $z_{\mathrm{attr}} : V \to \hat{\mathbb{C}}$ such that $z_{\mathrm{attr}}(\lambda_1) = z_\bullet$ and $z_{\mathrm{attr}}(\lambda)$ is an attracting fixed point of f_λ for all $\lambda \in V$. It follows for some potentially smaller open neighborhood $W \subset V$ of λ_1 that we have $g(\lambda) = z_{\mathrm{attr}}(\lambda)$ for all $\lambda \in W$. Meanwhile, $g(\lambda_2)$ is one of the points in the attracting cycle of period $\ell > 1$ for $f_{\lambda_2}(z)$.

Now consider the family $(f^{\ell\cdot n+1}_\lambda(a(\lambda)))^\infty_{n=1}$. The sub-sequence $(f^{\ell\cdot n_k+1}_\lambda(a(\lambda)))^\infty_{k=1}$ converges locally uniformly to the holomorphic function $f_\lambda(g(\lambda))$ on U. Notice that

$$f(g(\lambda)) = g(\lambda) = z_{\mathrm{attr}}(\lambda)$$

on W and hence $f(g(\lambda)) = g(\lambda)$ on all of U by the Identity Theorem. On the other hand $f(g(\lambda_2)) \neq g(\lambda_2)$ because $g(\lambda_2)$ is a periodic point of period greater than 1. This is a contradiction. $\square$

Definition 3. *We say that a point $b \in \mathbb{C}_\infty$ is* exceptional *for a rational function $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ if the cardinality of the set $f^{-m}(\{b\})$ is less than or equal to two for all $m \in \mathbb{N}$. A marked point $b(\lambda)$ is called* persistently exceptional *for a holomorphic family of rational maps $\{f_\lambda\}_{\lambda\in\Lambda}$ if $b(\lambda)$ is exceptional for f_λ for all $\lambda \in \Lambda$.*

The following lemma plays a key role in our proofs of Theorems A and B.

.

Lemma 4 (Lyubich, [30])**.** *Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a family of rational maps which depends holomorphically on the parameter λ. Let $a(\lambda)$ and $b(\lambda)$ be marked points for f_λ with $b(\lambda)$ not being persistently exceptional for f_λ. If λ_0 is an active parameter for $a(\lambda)$ under f_λ, then*

$$\lambda_0 \in \overline{\{\lambda \in \Lambda : f^m_\lambda(a(\lambda)) = b(\lambda) \text{ for some } m \in \mathbb{N}\}}.$$

In addition to the original source [30], we also refer the reader to [10, Lemma V.2] for a proof of Lemma 4

2.2. **Applications of the Active-Passive dichotomy to the renormalization map $R_{z,t,q}(w)$.** We will now interpret our renormalization mapping $R_{z,t,q}(w)$ and Theorem D in the context of Section 2.1. Note that the degree of $R_{z,t,q}(w)$ in w drops if either $z = 0$ or $t \in \{1, 1/(1-q)\}$. Throughout this paper we will only work with $R_{z,t,q}(w)$ for fixed choices of $t \in (0,1) \cup (1,\infty)$ and $q \in \mathbb{N}_{\geq 2}$. For such fixed t, q we will let $z \in \mathbb{C} \setminus \{0\}$ serve the role of the varying parameter. These restrictions on the parameters allow us to avoid any possible drop in degree and therefore we obtain a holomorphic family of rational maps in the sense of Section 2.1 with z serving the role of the parameter λ varying over the parameter space $\Lambda = \mathbb{C} \setminus \{0\}$.

Consider two marked points $a(z) := z$ and $b(z) := 1/(1-q)$. Theorem D gives that the Lee-Yang zeros for the level n rooted and unrooted Cayley tree correspond to

$$\{z \in \mathbb{C} \setminus \{0\} \,:\, R^n_{z,t,q}(a(z)) = b(z)\} \quad \text{and} \quad \left\{z \in \mathbb{C} \setminus \{0\} \,:\, \left(\hat{R}_{z,t,q} \circ R^{(n-1)}_{z,t,q}\right)(a(z)) = b(z)\right\},$$

respectively. Note that we have used that $z = 0$ is never a Lee-Yang zero for when $t > 0$ which is why we are able to assume $z \in \mathbb{C} \setminus \{0\}$ in the above formula. Therefore, one has the following expressions for the sets $\mathcal{B}(t,q)$ and $\hat{\mathcal{B}}(t,q)$ that were defined in Equation (7) above:

$$\mathcal{B}(t,q) = \overline{\{z \in \mathbb{C} \setminus \{0\} \,:\, R^n_{z,t,q}(a(z)) = b(z) \text{ for some } n \in \mathbb{N}\}} \quad \text{and}$$
$$\hat{\mathcal{B}}(t,q) = \overline{\left\{z \in \mathbb{C} \setminus \{0\} \,:\, \left(\hat{R}_{z,t,q} \circ R^{(n-1)}_{z,t,q}\right)(a(z)) = b(z) \text{ for some } n \in \mathbb{N}\right\}},$$

respectively.

For fixed $t \in (0,1) \cup (1,\infty)$ and $q \in \mathbb{N}_{\geq 2}$ we denote the active locus of the marked point $a(z) = z$ under the holomorphic family $R_{z,t,q}(w)$ by $\mathcal{A}(t,q)$.

Lemma 5. *For any $t \in (0,1) \cup (1,\infty)$ and $q \in \mathbb{N}_{\geq 2}$ we have $\mathcal{A}(t,q) \subset \mathcal{B}(t,q)$ and $\mathcal{A}(t,q) \subset \hat{\mathcal{B}}(t,q)$.*

Proof. We will first prove the containment $\mathcal{A}(t,q) \subset \mathcal{B}(t,q)$. By Lemma 4 it suffices to show that $b(z) = 1/(1-q)$ is not persistently exceptional for $R_{z,t,q}(w)$. We will show that for $z = 1$ the point $b(1)$ is not exceptional for $R_{1,t,q}(w)$. The critical values of $R_{1,t,q}(w)$ are 0 and ∞ and therefore the critical values of the second iterate $R^2_{1,t,q}(w)$ consist of

$$0, \quad \infty, \quad R_{1,t,q}(0) = t^2, \quad \text{and} \quad \frac{((q-2)t+1)^2}{(q-1)^2 t^2},$$

each of which is non-negative (or infinite). Therefore $b(1) = 1/(1-q) < 0$ is not a critical value of $R^2_{1,t,q}(w)$ implying that it has four preimages under $R^2_{1,t,q}(w)$. We conclude that $b(1)$ is not exceptional for $R_{1,t,q}(w)$ and therefore that $b(z)$ is not persistently exceptional for $R_{z,t,q}(w)$.

The containment $\mathcal{A}(t,q) \subset \hat{\mathcal{B}}(t,q)$ will also follow from Lemma 4. Note that for any $z \neq 0$ the critical values of $\hat{R}_{z,t,q}(w)$ are 0 and infinity. Since $\hat{R}_{z,t,q}(w)$ is a rational map of degree three and $b(q) = 1/(1-q)$ is not a critical value, one of the three preimages of $b(q)$ will not be exceptional for $R_{z,t,q}(w)$. Denoting that preimage by $c(q)$ we apply Lemma 4 to see that

$$\mathcal{A}(t,q) \subset \overline{\{z \in \mathbb{C} \setminus \{0\} \,:\, R^{(n-1)}_{z,t,q}(a(z)) = c(z) \text{ for some } n \in \mathbb{N}\}} \subset \hat{\mathcal{B}}(t,q).$$

□

Lemma 6. *For any $t \in (0,1) \cup (1,\infty)$ and $q \in \mathbb{N}_{\geq 2}$ we have*

$$\mathcal{B}(t,q) \cap (0,\infty) = \hat{\mathcal{B}}(t,q) \cap (0,\infty) = \mathcal{A}(t,q) \cap (0,\infty).$$

Proof. Lemma 5 gives $\mathcal{A}(t,q) \cap (0,\infty) \subset \mathcal{B}(t,q) \cap (0,\infty)$ and $\mathcal{A}(t,q) \cap (0,\infty) \subset \hat{\mathcal{B}}(t,q) \cap (0,\infty)$. Therefore, it suffices to show that if $z \in (0,\infty) \setminus \mathcal{A}(t,q)$ then $z \notin \mathcal{B}(t,q)$ and $z \notin \hat{\mathcal{B}}(t,q)$.

We first show that if $z \in (0,\infty) \setminus \mathcal{A}(t,q)$ then $z \notin \mathcal{B}(t,q)$. Let $z_\bullet \in (0,\infty)$ be a passive parameter for the marked point $a(z) = z$ under $R_{z,t,q}(w)$. Then, by definition, there is an open neighborhood U of $z_\bullet$ such that the sequence $g_n : U \to \mathbb{C}_\infty$ of functions defined by

$$g_n(z) := R^n_{z,t,q}(a(z)) = R^n_{z,t,q}(z)$$

forms a normal family. For the sake of contradiction, let's assume that $z_\bullet \in \mathcal{B}(t,q)$. Then we can find a sub-sequence $(n_k)_{k\in\mathbb{N}}$ and points $z_{n_k} \in U$ such that:

(i) (g_{n_k}) converges uniformly on compact subsets of U,
(ii) $g_{n_k}(z_{n_k}) = R^{n_k}_{z_{n_k},t,q}(z_{n_k}) = 1/(1-q)$ for all $k \in \mathbb{N}$, and
(iii) $\lim_{k\to\infty} z_{n_k} = z_\bullet$.

Let $g(z)$ be the uniform limit of $g_{n_k}(z)$. Notice $t > 0, q \geq 2$ and $z_\bullet > 0$ imply that $g_{n_k}(z_\bullet) > 0$ for each k and hence that the limit satisfies $g(z_\bullet) \geq 0$. On the other hand, because g_{n_k} converges uniformly to g, and $z_{n_k} \to z_\bullet$, we have $g_{n_k}(z_{n_k}) \to g(z_\bullet)$. This is a contradiction because $g(z_\bullet) \geq 0$ and $g_{n_k}(z_{n_k}) = 1/(1-q) < 0$ for each k.

The proof that if $z \in (0,\infty) \setminus \mathcal{A}(t,q)$ then $z \notin \hat{\mathcal{B}}(t,q)$ is quite similar, except that one replaces the sequences of functions $g_n(z) = R^n_{z,t,q}(a(z))$ with a new sequence $\hat{g}_n(z) = \left(\hat{R}_{z,t,q} \circ R^{(n-1)}_{z,t,q}\right)(z)$. We leave the details to the reader. □

2.3. **Plotting Numerical Approximations to Lee-Yang Zeros Using The Active-Passive Dichotomy.** Lemma 5 gives that $\mathcal{A}(t,q) \subset \mathcal{B}(t,q)$ and $\mathcal{A}(t,q) \subset \hat{\mathcal{B}}(t,q)$. Although we are primarily interested in the sets $\mathcal{B}(t,q)$ and $\hat{\mathcal{B}}(t,q)$, the active locus $\mathcal{A}(t,q)$ for the marked point $a(z) = z$ under the holomorphic family $R_{z,t,q}(w)$ can be studied dynamically. Moreover, we can also use the computer to make plans of $\mathcal{A}(t,q)$ that are far more detailed than the plots of Lee-Yang zeros given in Figure 3. While the plots of $\mathcal{A}(t,q)$ may be missing some points of $\mathcal{B}(t,q)$ or $\hat{\mathcal{B}}(t,q)$ they still may be informative about those two sets of Lee-Yang zeros.

Arnaud Chéritat helped us by using his computer software to plot the active and passive parameters of the marked point $a(z) = z$ for $R_{z,t,q}$, for fixed q and t, in the complex z-plane. Figures 7 and 8 show the active loci related to $t = 0.26$ and $t = 0.5$ for the 3-state Potts model. The points in black are supposed to be active parameters and the points in white are supposed to be passive parameters. The reader should compare Figures 7 and 8 with Figure 3.

It is difficult to numerically compute the Lee-Yang zeros of the partition function when $t > 1$. Therefore, we did not include any images analogous to Figure 3 when $t > 1$ (the antiferromagnetic case). Instead, we include the plots of the active locus for the marked point $a(z) = z$ for $R_{z,t}(w)$ at $t = 6$ in Figure 9 and Figure 10. The Lee-Yang zeros for Γ_n accumulate to this active locus as $n \to \infty$.

We briefly describe here the method used by Chéritat to numerically compute the active locus for a marked point $a(\lambda)$ under a holomorphic family of mappings $f_\lambda(z)$. A rectangular region in the complex λ plane is divided into pixels. Each pixel is interpreted as a small square in the complex plane whose side length is $r > 0$. One also picks a parameter $0 < \theta < 1$, a small parameter $\epsilon > 0$, and a threshold $m_0 \in \mathbb{N}$. These are "tuned" by the user to get a reasonably looking plot.

To determine if the pixel associated to $\lambda = \lambda_0$ should be colored black (active), white (passive), or red (undecided) the program checks each of the following three conditions for $m = 1, 2, 3, \ldots$. Once one of the conditions below is met for the pixel associated to λ_0 the program moves on to check the next pixel.

(1) The program computes $F_m(\lambda) := f^m_\lambda(a(\lambda))$ together with a numerical approximation to the derivative $\left|\frac{d(F_m(\lambda)-\lambda)}{d\lambda}\right|_{\lambda=\lambda_0}$. If
$$|F_m(\lambda_0) - a(\lambda_0)| < r \cdot \theta \left|\frac{d(F_m(\lambda)-\lambda)}{d\lambda}|_{\lambda=\lambda_0}\right|$$
then it is assumed that $a(\lambda_0)$ is close to being periodic and, when that happens it is likely that $a(\lambda_0)$ is periodic repelling. In this case the program declares λ_0 to be active and the pixel is colored black.

(2) If Step 1 failed at iterate m then the program checks if
$$\left|\frac{df^m_\lambda(z)}{dz}|_{z=a(\lambda_0)}\right| < \epsilon.$$
If this holds then the program decides that $a(\lambda_0)$ is probably in the basin of attraction of an attracting periodic cycle. In this case, the program declares λ_0 to be passive and the pixel is colored white.

(3) If Steps 1 and 2 both fail and if m has reached the threshold, m_0 then the program stops and colors the pixel red, indicating that it could not decide if $a(\lambda)$ should be interpreted as being active or passive at the parameter λ_0.

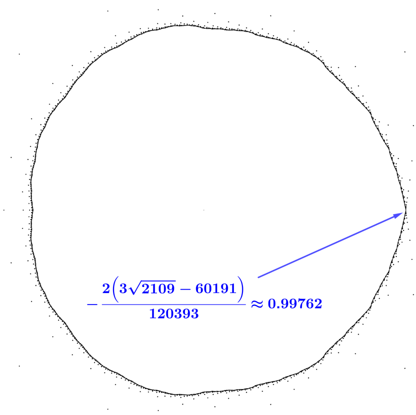

FIGURE 7. The active locus for the marked point $a(z) = z$ for $R_{z,t,3}(w)$ at $t = 0.26$. Points in the active locus are black and the points in the passive locus are white. As explained in Section 2.3 the Lee-Yang zeros at $t = 0.26$ accumulate to the active locus (black) as $n \to \infty$.

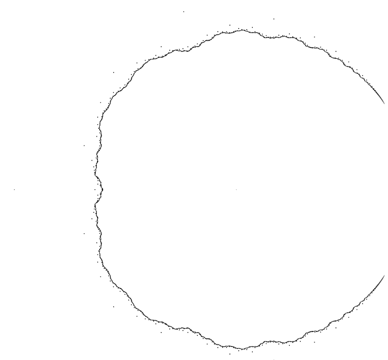

FIGURE 8. The active locus for the marked point $a(z) = z$ for $R_{z,t,3}(w)$ at $t = 0.5$. Points in the active locus are black and the points in the passive locus are white. As explained in Section 2.3 the Lee-Yang zeros at $t = 0.25$ accumulate to the active locus (black) as $n \to \infty$.

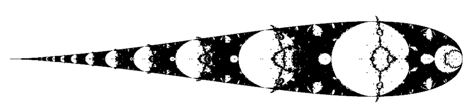

FIGURE 9. The active locus for the marked point $a(z) = z$ for $R_{z,t,3}(w)$ at $t = 6$. Points in the active locus are black and the points in the passive locus are white. The following Figure 10 is a zoomed-in version of this image around $z = 1$.

3. PROOF OF THEOREM A

Note that the cases when $t = 0$ and $t = 1$ are easily handled using the explicit formula (6). We therefore restrict our attention to $0 < t < 1$ throughout the remainder of this section. We refer the reader to Equation (8) for the definitions of $t_1(q)$ and $t_2(q)$. By Lemma 6 the proof of Theorem A reduces to the proof of the following theorem.

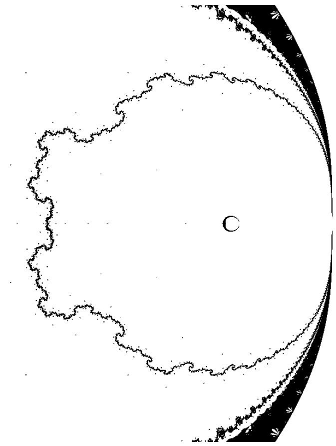

FIGURE 10. The active locus for the marked point $a(z) = z$ for $R_{z,t,3}(w)$ at $t = 6$. Points in the active locus are black and the points in the passive locus are white. This is a zoomed-in version of above Figure 9 around $z = 1$.

Theorem 2. *Suppose $0 < t < 1$ and let $\mathcal{A}(t, q)$ be the active locus for the marked point $a(z) = z$ under $R_{z,t,q}(w)$. Then*

$$\mathcal{A}(t,q) \cap (0,\infty) = \begin{cases} \{1\} & \text{if } \ 0 < t \leq t_1(q), \\ \{\mathcal{Z}_q(t)\} & \text{if } \ t_1(q) < t \leq t_2(q), \\ \emptyset & \text{if } \ t_2(q) < t < 1. \end{cases}$$

Here $\mathcal{Z}_q(t)$ is given in Equation (9) in the statement of Theorem A.

We begin with several lemmas.

Lemma 7. *Let $0 < t < 1$, $z > 0$, and $q \geq 2$. Then the map $R_{z,t,q}(w)$ has the following properties:*
(i) *$R_{z,t,q}(w)$ is increasing on $[0, \infty)$,*
(ii) *$R_{z,t,q}(w)$ has at least one fixed point on $[0, \infty)$, and*
(iii) *$R_{z,t,q}([0, \infty)]) \subseteq [c, d]$ with $c, d \in \mathbb{R}$ and $0 < c < d$.*

Proof. Simple calculations show that

$$\frac{\partial R_{z,t,q}}{\partial w}(w) = \frac{2z(1-t)((q-1)t+1)((q-2)tw+t+w)}{((q-1)tw+1)^3}.$$

Because $t \in (0, 1)$, $z > 0$, and $q \geq 2$, we get $\dfrac{\partial R_{z,t,q}}{\partial w}(w) > 0$ for all $w > 0$. Thus, $R_{z,t,q}(w)$ is increasing on $[0, \infty)$. This proves Claim (i).

Notice that $R_{z,t,q}(0) = zt^2 > 0$. Furthermore, the horizontal asymptote of $R_{z,t,q}(w)$,

$$\lim_{w\to\infty} R_{z,t,q}(w) = z\left[\frac{1+(q-2)t}{(q-1)t}\right]^2$$

is positive and finite. Thus, $R_{z,t,q}(w)$ must intersect the diagonal at least once. Hence, $R_{z,t,q}(w)$ has at least one positive fixed point, thus proving Claim (ii).

Thus, if we let

$$c := R_{z,t,q}(0) = zt^2 \quad \text{and} \quad d := \lim_{w\to\infty} R_{z,t,q}(w) = z\left[\frac{1+(q-2)t}{(q-1)t}\right]^2,$$

then Claim (iiii) follows from Claims (i) and (ii). $\square$

Let $c, d \in \mathbb{R}$ with $c < d$. Let $f : [c, d] \to [c, d]$ be a differentiable function such that $f'(x) > 0$ for all $x \in [a, b]$, and with $f(a) > a$, and $f(b) < b$. A fixed point x_{fix} for f is *repelling* if $f'(x_{\text{fix}}) > 1$, *neutral* if $f'(x_{\text{fix}}) = 1$, and *attracting* if $f'(x_{\text{fix}}) < 1$. The following Lemma is well-known, so we omit the proof.

Lemma 8. *Let $f : [c,d] \to [c,d]$ be as described above. Let A_f, N_f and R_f be the set of all attracting, neutral, and repelling fixed points of f respectively. Then for any $x_0 \in (c,d)\backslash(R_f \cup N_f \cup A_f)$, the orbit of x_0 under f converges to a point in $A_f \cup N_f$. Furthermore, if there is a point $x_N \in N_f$ and an open interval $I \ni x_0, x_N$ with no fixed point of f in $I\backslash\{x_N\}$ such that*

(i) *$x_0 < x_N$ and $f|_I(x) > x$ or*
(ii) *$x_0 > x_N$ and $f|_I(x) < x$,*

then the orbit $\{f^n(z_0)\}_{n=0}^{\infty}$ of x_0 converges to x_N.

Lemma 9. *Let $t \in (0,1)$ be fixed. Suppose $z > 0$ is an active parameter for the marked point $a(z) = z$ under $R_{z,t,q}(w)$. Then one of the following two cases occurs.*

(i) *The marked point $a(z) = z$ is a repelling fixed point of $R_{z,t,q}(w)$.*
(ii) *$R_{z,t,q}(w)$ has a positive neutral fixed point.*

Proof. Let $t \in (0,1)$ be fixed. Then for any fixed $z > 0$, by Lemma 7, the function $R_{z,t,q}(w)$ is increasing on $[0,\infty)$ with at least one fixed point. Moreover, $R_{z,t,q}(w)$ maps the compact interval $[c,d]$ to itself. Therefore, from Lemma 8 we see that $a(z) = z$ is a repelling fixed point of $R_{z,t,q}(w)$, the orbit of z under $R_{z,t,q}$ converges to a neutral fixed point, or the orbit of z under $R_{z,t,q}$ converges to an attracting fixed point of $R_{z,t,q}(w)$. By Lemma 2, the third option corresponds to passive behavior of the marked point $a(z)$, so that (i) or (ii) must occur. □

Lemma 10. *Let $t \in [0,\infty) \setminus \{1\}$ and $z \geq 0$. Then the marked point $a(z) = z$ is a fixed point of $R_{z,t,q}(w)$ if and only if $z = 1$.*

Proof. A direct calculation gives $R_{1,t,q}(1) = 1$. Conversely, $t \geq 0$ and $z \geq 0$ imply that $t + z + (q-2)tz \geq 0$, and $1 + (q-1)tz > 0$ which allows us to simplify

$$R_{z,t,q}(z) = z\left[\frac{t+z+(q-2)tz}{1+(q-1)tz}\right]^2 = z$$

to

$$t + z + (q-2)tz = 1 + (q-1)tz$$

by taking a square root. This implies $z(1-t) = 1-t$. Because $t \neq 1$, we get $z = 1$. □

Lemma 11. *Let $z > 0$, $t \in (0,1)$ and $q \geq 2$. If $R_{z,t,q}(w)$ has a neutral fixed point at some $w = w_N(z) > 0$, then $t \in (0, t_2(q)]$ and $z = N_\pm(t,q)$, where*

$$N_\pm(t,q) := \frac{\begin{pmatrix} (-27(q-1)^2t^4+18\left(q^2-3q+2\right)t^3+\left(q^2+14q-14\right)t^2+2(q-2)t+1)) \\ \pm\sqrt{(t-1)((q-1)t+1)(9(q-1)t^2-(q-2)t-1)^3} \end{pmatrix}}{8t((q-2)t+1)^3}.$$

Moreover, for each value of $t \in (0, t_2(q)]$ both values $N_\pm(t,q)$ are positive.

Remark: $\mathcal{Z}_q(t)$ from Theorem A is equal to $N_-(t,q)$. Figure 11 shows a plot $N_\pm(t,q)$ in the case that $q = 3$.

Proof. Throughout the proof, fix $q \geq 2$. By definition, $w_N > 0$ is a neutral fixed point of $R_{z,t,q}(w)$ if and only if

$$R_{z,t,q}(w_N) = w_N \qquad \text{and} \qquad \frac{d}{dw}R_{z,t,q}(w)\mid_{w=w_N(z)} = \pm 1.$$

However, by Lemma 7 we find that $\frac{d}{dw}R_{z,t,q}(w) = 1$. Therefore the above conditions reduce to the condition that $R_{z,t,q}(w) - w$ has a multiple root at w_N. Clearing denominators, this is equivalent to the following polynomial in w having a multiple root at some w_N:

$$P_{z,t,q}(w) := z((q-2)tw + t + w)^2 - w((q-1)tw+1)^2.$$

Taking the discriminant of $P_{z,t,q}(w)$ with respect to the variable w we find that $R_{z,t,q}$ has a neutral fixed point if and only if

$$-(-1+t)^2(1-t+qt)^2 z\, Q(z,t,q) = 0,$$

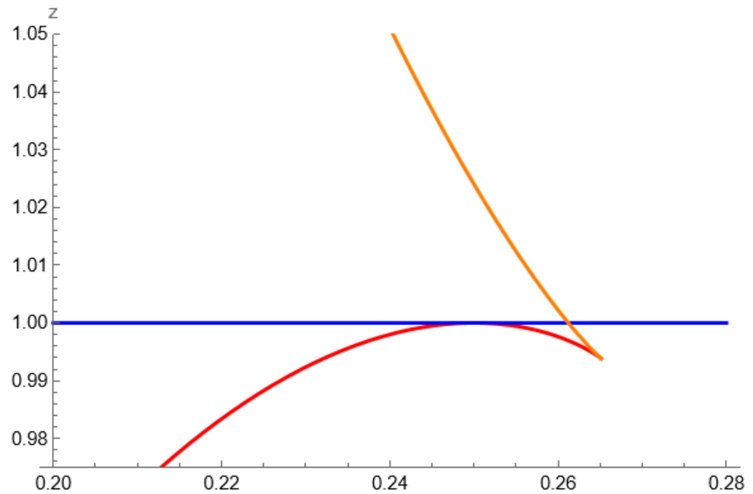

FIGURE 11. Plots of $N_+(t,3)$ (orange), $N_-(t,3)$ (red), and 1 (blue) versus t. Compare with Figure 4.

where the last factor $Q(z,t,q)$ is the following quadratic polynomial in z:

$$\begin{aligned}Q(z,t,q) := &\left(4q^3t^4 - 24q^2t^4 + 48qt^4 - 24(q-2)t^3 + 12(q-2)qt^3 + 12qt^2 - 32t^4 - 24t^2 + 4t\right) z^2 \\ &\left(-q^2t^4 + 3q^2t^2 + 2qt^4 - 6(q-2)t^3 + 6(q-2)qt^3 - 6qt^2 + 6qt - t^4 + 6t^2 - 12t + 3\right) z \\ &4qt - 4t = 0.\end{aligned}$$

Since $t \in [0,1)$, $z > 0$, and $q \geq 2$ the only way that the discriminant of $P_{z,t,q}(w)$ vanishes is when $Q(z,t,q)$ vanishes. The reader can check that $Q(z,t,q)$ vanishes precisely under the conditions stated in the conclusion of this lemma.

The square root in the formula for $N_\pm(t)$ is non-negative if and only if $t \in (0, t_2(q)]$. Since the denominator of $N_\pm(t,q)$ does not vanish for $t > 0$ we have that $N_\pm(t,q)$ both vary continuously as t varies over the interval $(0, t_2(q)]$. When $t = 1/(1+q)$ one finds that

$$N_-(t,q) = 1 \qquad \text{and} \qquad N_+(t,q) = \frac{(q-1)(q+1)^3}{(2q-1)^3},$$

both of which are positive. Meanwhile, substituting $z = 0$ into the polynomial $P_{z,t,q}(w)$ yields $-w(1 + (-1+q)tw)^2$ which has no solutions when $w, t > 0$. Therefore $N_\pm(t,q)$ remain positive for all $t \in (0, t_2(q)]$. □

We will need the following lemma to complete the proof of Theorem 2. Since the proof involves some technical calculations, we will give it in Appendix A.

Lemma 12. *We have:*

(i) *Suppose $z = N_+(t,q)$. Then, for any $t \in (0, t_2(q))$, the orbit of the marked point $a(z) = z$ under iteration of $R_{z,t,q}$ converges to an attracting fixed point $w_A(z)$ of $R_{z,t,q}(w)$.*

(ii) *Suppose $z = N_-(t,q)$. Then:*
   (a) *If $t \in (0, t_1(q))$, then the orbit of the marked point $a(z) = z$ under iteration of $R_{z,t,q}$ converges to an attracting fixed point $w_A(z)$ of $R_{z,t,q}(w)$.*
   (b) *If $t \in (t_1(q), t_2(q))$, then the orbit of the marked point $a(z) = z$ under iteration of $R_{z,t,q}$ converges to a neutral fixed point $w_N(N_-(t,q))$ of $R_{z,t,q}(w)$.*

(iii) *If $t \in (t_1(q), t_2(q))$ and $z < N_-(t,q)$ is sufficiently close to $N_-(t,q)$, then the orbit of z under iteration of $R_{z,t,q}$ converges to an attracting fixed point $w_A(z)$ of $R_{z,t,q}(w)$.*

Proof of Theorem 2. Suppose $0 < t < 1$. By Lemma 9, if $z > 0$ is an active parameter for the marked point $a(z) = z$ under $R_{z,t,q}(w)$, then either

(i). $a(z) = z$ is a repelling fixed point for $R_{z,t,q}(w)$, and in this case, that repelling fixed point must be $w = 1$ by Lemma 10, or (ii). $R_{z,t,q}(w)$ has a neutral fixed point $w_N(z) > 0$. Then by Lemma 11, $t \in (0, t_2(q)]$ and $z = N_{\pm}(t,q)$.

In Case (i), a simple calculation gives

$$R'_{1,t,q}(1) = \frac{2-2t}{(q-1)t+1}.$$

By solving the inequality $R'_{1,t,q}(1) > 1$ for t we get $0 \leq t < t_1(q)$. Moreover, when $0 \leq t < t_1(q)$, $R_{z,t,q}(w)$ maps $a(z) = z$ non-persistently onto this repelling fixed point $w = 1$ as $a(z) = z$ is not fixed for $z \neq 1$. Then by Lemma 2, $z = 1$ is an active parameter for the marked point $a(z) = z$ under the map $R_{z,t,q}(w)$. If $t = t_1(q)$, then $z = 1$ continues to be an active parameter for the marked point $a(z) = z$ as the active locus is closed. If $t > t_1(q)$, then $w = 1$ is an attracting fixed point. So, in that case, $a(z) = z$ is passive at $z = 1$.

We now consider Case (ii). Note that when $t = t_2(q)$ we have $N_+(t,q) = N_-(t,q)$. To finish the proof of Theorem 2 we must check:

(I). For any $0 < t < t_2(q)$ we have that $z = N_+(t,q)$ is passive for the marked point $a(z) = z$ under $R_{z,t,q}(w)$.

(II). For any $0 < t < t_1(q)$ we have that $z = N_-(t,q)$ is passive for the marked point $a(z) = z$ under $R_{z,t,q}(w)$, and for any $t_1(q) \leq t \leq t_2(q)$ we have that $z = N_-(t,q)$ is active for the marked point $a(z) = z$ under $R_{z,t,q}(w)$.

Suppose $0 < t < t_2(q)$. If $z = N_+(t,q)$, then Lemma 12(i) gives that the orbit of z converges to an attracting fixed point of $R_{z,t,q}(w)$. Thus, by Lemma 2, $z = N_+(t,q)$ is a passive parameter for the marked point $a(z) = z$ under $R_{z,t,q}(w)$. Similarly, when $0 < t < t_1(q)$ and $z = N_-(t,q)$, Lemma 12(ii(a)) gives that the orbit of z converges to an attracting fixed point of $R_{z,t,q}(w)$, making it a passive parameter for the marked point $a(z) = z$ under $R_{z,t,q}(w)$ by Lemma 2.

Suppose $t_1(q) < t < t_2(q)$ and $z = N_-(t,q)$. By Lemma 12(ii(b)), the orbit of $z_0 = N_-(t,q)$ converges to the neutral fixed point $w_N(N_-(t,q))$. Furthermore, there exists an attracting fixed point $w_A < w_N(N_-(t,q))$. This attracting fixed point varies continuously with respect to z around $N_-(t,q)$. We will denote this dependence on z by $w_A \equiv w_A(z)$. By Lemma 12(iii) if $z < z_0$ is sufficiently close to z_0, then the orbit of z converges to $w_A(z)$. Thus, for any subsequence $\left(R^{n_k}_{z,t,q}(z)\right)_{k=1}^{\infty}$ of $\left(R^{n}_{z,t,q}(z)\right)_{n=1}^{\infty}$ and any neighborhood U of $z_0 = N_-(t,q)$, we can find a further subsequence $\left(R^{n_{k_j}}_{z,t,q}(z)\right)_{j=1}^{\infty}$ of $\left(R^{n_k}_{z,t,q}(z)\right)_{k=1}^{\infty}$ and a point $z_1 \in U$ such that for all $j \in \mathbb{N}$

$$\left|R^{n_{k_j}}_{z_1,t,q}(z_1) - R^{n_{j_k}}_{z_0,t,q}(z_0)\right| > \frac{w_N(z_0) - w_A(z_1)}{2}.$$

Because $w_N(z_0) - w_A(z_0) > 0$ and because $w_A(z)$ depends continuously on z we can choose $z_1 \in U$ sufficiently close to z_0 so that the right-hand side of the above inequality is positive.

Therefore the family $\left\{R^{n}_{z,t,q}(z)\right\}_{n=1}$ of functions is not normal at $z_0 = N_-(t,q)$. Hence, the marked point $a(z) = z$ under $R_{z,t,q}(w)$ is active at $z_0 = N_-(t,q)$. Because the active locus is a closed set, the marked point $a(z) = z$ under $R_{z,t,q}(w)$ is also active at $z_0 = N_-(t,q)$ when $t = t_1(q)$ and $t = t_2(q)$.

This completes the proof of Theorem 2. □

4. Proof of Theorem B (Antiferromagnetic Case, $q \geq 3$)

Recall that Theorem B concerns the antiferromagnetic regime $J < 0$. Since the physical values of temperature are $T > 0$, this corresponds to the temperature-like parameter $t = e^{-\frac{J}{T}} > 1$ and the field-like parameter $z = e^{-\frac{h}{T}} > 0$. By Lemma 6 the proof of Theorem B reduces to the proof of the following theorem.

Theorem 3. *Suppose $t > 1, q \geq 3$ and let $\mathcal{A}(R_{z,t,q})$ be the active locus for the marked point $a(z) = z$ under $R_{z,t,q}(w)$. Then*

$$\mathcal{A}(R_{z,t,q}) \cap (0, \infty) = \begin{cases} \emptyset & \text{if } \; 1 \leq t < t_3(q) \\ \{z_c^{\pm}(t,q)\} & \text{if } \; t_3(q) \leq t. \end{cases} \tag{15}$$

Here $t_3(q) > 1$ and $z_c^{\pm}(t,q)$ are given in Equations (10) and (11) as in the statement of Theorem B.

Therefore, the rest of this section is devoted to the proof of Theorem 3. The reader can see the plots of $z_c^{\pm}(t,3)$ in Figure (5). The main challenge is that $R_{z,t,q}(w)$ is not increasing on $[0,\infty)$ when $t > 1$. However, the second iterate $R^2_{z,t,q}(w) = R_{z,t,q}\left(R_{z,t,q}(w)\right)$ of $R_{z,t,q}(w)$ is increasing on $[0,\infty)$ when $t > 1$. By Lemma 1 the active locus for the marked point $a(z) = z$ under $R_{z,t,q}(w)$ is the same as for the second iterate $R^2_{z,t,q}(w)$ so we can therefore work extensively with $R^2_{z,t,q}(w)$ throughout the proof of Theorem 3.

Lemma 13. *Let $z > 0$, $t > 1$, and $q \geq 2$. The map*

$$R^2_{z,t,q}(w) := R_{z,t,q}\left(R_{z,t,q}(w)\right) = \frac{z\left(z((q-2)t+1)((q-2)tw+t+w)^2 + t((q-1)tw+1)^2\right)^2}{\left((q-1)tz((q-2)tw+t+w)^2 + ((q-1)tw+1)^2\right)^2}$$

has the following properties.

(i) *$R^2_{z,t,q}(w)$ is increasing on $[0,\infty)$,*
(ii) *$R^2_{z,t,q}(w)$ has at least one fixed point on $[0,\infty)$, and*
(iii) *$R^2_{z,t,q}\left([0,\infty)]\right) \subseteq [c,d]$ with $c, d \in \mathbb{R}$ and $0 < c < d$.*

Proof. The interval $[0,\infty)$ is forward invariant under $R_{z,t,q}(w)$ so, by the chain rule it suffices to prove that $\frac{\partial}{\partial w} R_{z,t,q}(w) < 0$ when $z > 0,\ t > 1$, and $q \geq 2$ and when $w \geq 0$. Expressing everything in terms of $z,\ s = t - 1 > 0,\ p = q - 2 > 0$, and $w > 0$ we find:

$$\frac{\partial}{\partial w} R_{z,t,q}(w) = \frac{-2zs(2+p+s+ps)(1+s+w+pw+psw)}{(1+w+pw+sw+psw)^3} < 0.$$

This proves Claim (i). To prove Claim (ii), notice that

$$R^2_{z,t,q}(0) = \frac{z\left(t^2 z((q-2)t+1) + t\right)^2}{\left((q-1)t^3 z + 1\right)^2} > 0,$$

and

$$\lim_{w\to\infty} R^2_{z,t,q}(w) = \frac{z\left(t^3\left(q^3 z + q^2(1-6z) + 2q(6z-1) - 8z + 1\right) + 3(q-2)^2 t^2 z + 3(q-2)tz + z\right)^2}{(q-1)^2 t^2\left((q-2)^2 t^2 z + t(2qz + q - 4z - 1) + z\right)^2}$$

which is positive and finite. Hence, $R^2_{z,t,q}(w)$ has at least one positive fixed point by the Intermediate Value Theorem. Since $R^2_{z,t,q}(w)$ is increasing on $[0,\infty)$ we can prove Claim (iii) by letting $c := R^2_{z,t,q}(0)$ and $d := \lim_{w\to\infty} R^2_{z,t,q}(w)$. □

Lemma 14. *For fixed $z > 0$, $t > 1$ and $q \geq 2$, if the map $R^2_{z,t,q}(w)$ has a neutral fixed point, then*

$$t \geq t_3(q) > 1 \qquad \text{and} \qquad z = z_c^{\pm}(t,q),$$

where the formulae for $t_3(q)$ and $z = z_c^{\pm}(t,q)$ are given in Equations (10) and (11).

Furthermore, for all $t \geq t_3(q)$ we have

(a) *$0 < z_c^-(t,q) \leq z_c^+(t,q)$ with the equality $z_c^-(t,q) = z_c^+(t,q)$ only when $t = t_3(q)$.*
(b) *If $q \geq 3$ then $z_c^+(t,q) < 1$.*

Proof. The proof is quite similar to the proof of Lemma 11 given in the previous section. We express the condition that $R^2_{z,t,q}(w)$ has a neutral fixed point as the existence of a multiple root of a polynomial $P_{z,t,q}(w)$ which is obtained from the equation $R^2_{z,t,q}(w) = w$ by clearing denominators. The discriminant of $P_{z,t,q}(w)$ with respect to w equals

$$z^8(t-1)^{16}((q-1)t+1)^{16}\, Q_1(z,t,q)Q_2(z,t,q)^3, \tag{16}$$

where

$$Q_1(z,t,q) = z[t(27(q-1)^2t^3 - 18(q-2)(q-1)t^2 - q(q+14)t - 2q + 14t + 4) - 1] + 4tz^2((q-2)t+1)^3 + 4(q-1)t,$$

and

(17)

$$Q_2(z,t,q) = z(-(q-1)^2t^4 + 6(q-2)(q-1)t^3 + 3((q-2)q+2)t^2 + 6(q-2)t + 3) + 4tz^2((q-2)t+1)^3 + 4(q-1)t.$$

Therefore, we find that $R_{z,t,q}$ has a neutral fixed point if and only if one of the factors in Equation (16) vanishes. The first three factors clearly don't vanish for $z > 0, t > 1$, and $q \geq 2$. We claim that $Q_1(z,t,q)$ also does not vanish for this range of the parameters z, t, q. Expressing everything in terms of $z, s = t-1 > 0, p = q-2 \geq 0$, and $w > 0$ we find:

$$\begin{aligned} Q_1 = {} & 4p^3s^4z^2 + 16p^3s^3z^2 + 24p^3s^2z^2 + 16p^3sz^2 + 4p^3z^2 + 27p^2s^4z + 12p^2s^3z^2 + 90p^2s^3z \\ & + 36p^2s^2z^2 + 107p^2s^2z + 36p^2sz^2 + 52p^2sz + 12p^2z^2 + 8p^2z + 54ps^4z + 198ps^3z \\ & + 12ps^2z^2 + 252ps^2z + 24psz^2 + 124psz + 4ps + 12pz^2 + 16pz + 4p + 27s^4z \\ & + 108s^3z + 144s^2z + 4sz^2 + 72sz + 4s + 4z^2 + 8z + 4 > 0. \end{aligned} \tag{18}$$

Note that $Q_2(z,t,q)$ is quadratic in z with the coefficients depending on t and q. Solving $Q_2(z,t,q) = 0$ for z in terms of t and q we find precisely the values $z_c^{\pm}(t,q)$ given in Formula (11). Note that the term under the square root in formula for $z_c^{\pm}(t,q)$ vanishes at $t = t_3(q)$ and that it is negative for $1 < t < t_3(q)$ and strictly positive for $t > t_3(q)$. In particular, when $t > t_3(q)$ we have $z_c^-(t,q) < z_c^+(t,q)$.

We now prove the additional claim (a) by checking that for all $t \geq t_3(q)$ we have $0 < z_c^-(t,q)$. When $t = t_3(q)$ we have

$$z_c^+(t,q) = z_c^-(t,q) = \frac{-27q^3 + 153q^2 - 297q + 198 + \left(9q^2 - 35q + 35\right)\sqrt{9q^2 - 32q + 32}}{2(q-1)}, \tag{19}$$

which one can check is positive for all $q \geq 2$. It is clear from the formulae for $z_c^{\pm}(t,q)$ that for fixed $q \geq 2$ they are continuous functions of $t \geq t_3(q)$. Suppose for contradiction that there were some $t_4 > t_3(q)$ with $z_c^-(t_4,q) < 0$. In this case the Intermediate Value Theorem would give that there exists $t_5 \in [t_3(q), t_4]$ with $z_c^-(t_5,q) = 0$. We obtain a contradiction by noting that we would then have $Q_2\left(z_c^-(t_5,q), t_5, q\right) = Q_2(0,t_5,q) = 0$ while a direct calculation gives that $Q_2(0,t,q) = 4(q-1)t > 0$ for all $q \geq 2$ and $t > 0$.

To prove additional claim (b) it suffices to show that $Q_1(z,t,q) > 0$ for all $z \geq 1$, $t \geq 1$, and $q \geq 3$. If one computes $Q_1(1+r, 1+s, 3+p)$ one finds a polynomial whose constant term is 72 and all of whose monomials appear with a "plus" sign. (We have omitted the many line expression which is similar to Equation (18) above, but the reader can check it in a computer algebra software package.). □

In the following statement, we extend the range of allowable values of q from $\mathbb{N}_{\geq 3}$ to the interval $[3, \infty)$ so that methods involving continuity can be used in the proof.

Lemma 15. *Define sets $\mathcal{R}_1$ and $\mathcal{R}_2$ as follows:*

$$\mathcal{R}_1 := \{(z,t,q) \in \mathbb{R}^3 : t > 1,\ q \geq 3,\ \text{and if } t \geq t_3(q) \text{ then } 0 < z < z_c^-(t,q) \text{ or } z > z_c^+(t,q)\}$$

$$\mathcal{R}_2 := \{(z,t,q) \in \mathbb{R}^3 : t \geq t_3(q),\ q \geq 3,\ z_c^-(t,q) < z < z_c^+(t,q)\}.$$

Then,

(i) *For all $(z,t,q) \in \mathcal{R}_1$ the second iterate of the renormalization map, $R^2_{z,t,q}(w)$, has only one real fixed point. This fixed point is in $(0,\infty)$ and it is an attracting fixed point of $R_{z,t,q}(w)$.*

(ii) *For all $(z,t,q) \in \mathcal{R}_2$ the second iterate of the renormalization map, $R^2_{z,t,q}(w)$, has three real fixed points each of which is in $(0,\infty)$. One of these fixed points is a repelling fixed point of $R_{z,t,q}(w)$. The other two fixed points correspond to an attracting period 2-cycle of $R_{z,t,q}(w)$.*

Proof. We first prove that $\mathcal{R}_1$ and $\mathcal{R}_2$ are each path connected. We begin with $\mathcal{R}_1$. One can easily check from the formula (10) that for $q \geq 3$ one has $t_3(q) \geq 3$. Therefore, we have the containment

$$B := \{(z,t,q) \in \mathbb{R}^3 \,:\, z > 0, 1 < t < 3, \text{ and } q \geq 3\} \subset \mathcal{R}_1.$$

Note that B is convex and hence path connected. Moreover, for each fixed choice of real $q = q_0 \geq 3$ the slice

$$\mathcal{R}_1 \cap \{q = q_0\} := \{(z,t,q_0) \in \mathbb{R}^3 : t > 1 \text{ and if } t \geq t_3(q_0) \text{ then } 0 < z < z_c^-(t,q_0) \text{ or } z > z_c^+(t,q_0)\}$$

has non-trivial intersection with B. Therefore it suffices to show for each real $q_0 \geq 3$ that the slice $\mathcal{R}_1 \cap \{q = q_0\}$ is path connected. We will describe an explicit path within $\mathcal{R}_1 \cap \{q = q_0\}$ from any point $(z_0, t_0, q_0) \in \mathcal{R}_1 \cap \{q = q_0\}$ to the point $(1, 2, q_0) \in B$. There are three possibilities:

Option 1: $1 < t_0 < t_3(q_0)$. The straight line path between (z_0, t_0, q_0) and $(1, 2, q_0)$ remains in $\mathcal{R}_1 \cap \{q = q_0\}$.

Option 2: $t_0 \geq t_3(q_0)$ and $z_0 > z_c^+(t_0, q_0)$. Since $q_0 \geq 3$, Lemma 14 gives that $z_c^+(t_0, q_0) < 1$. We can therefore form a path from (z_0, t_0, q_0) to $(1, 2, q_0)$ by first connecting (z_0, t_0, q_0) to $(1, t_0, q_0)$ using a straight-line path (varying only z) and then connecting from $(1, t_0, q_0)$ to $(1, 2, q_0)$ by a straight line path (varying only t). Each of these line segments is in $\mathcal{R}_1 \cap \{q = q_0\}$.

Option 3: $t_0 \geq t_3(q_0)$ and $0 < z_0 < z_c^-(t_0, q_0)$. By Lemma 14 we have that $z_c^-(t, q_0) > 0$ for all $t \geq t_3(q_0)$. Moreover, it is clear from the expression for $z_c^-(t, q_0)$ that it is a continuous function of $t \geq t_3(q)$. Therefore, there exists a positive $M > 0$ such that for all $t \in [t_3(q_0), t_0]$ we have $z_c^-(t, q_0) \geq M$. We can therefore form a path within $\mathcal{R}_1 \cap \{q = q_0\}$ from (z_0, t_0, q_0) to $(1, 2, q_0)$ by a concatenation of three straight line paths. One first connects from (z_0, t_0, q_0) to $(M/2, t_0, q_0)$ by varying only z. One then connects from $(M/2, t_0, q_0)$ to $(M/2, 2, q_0)$ by varying only t. One finally connects from $(M/2, 2, q_0)$ to $(1, 2, q_0)$ by varying only z.

We have therefore proved that $\mathcal{R}_1$ is path connected. We will now prove that $\mathcal{R}_2$ is path-connected. For each $t \geq t_3(q)$ let

$$z_c^{\text{mid}}(t,q) = \frac{1}{2}(z_c^-(t,q) + z_c^+(t,q)).$$

For each $t > t_3(q)$ one has $z_c^{\text{mid}}(t,q) \in \mathcal{R}_2$. In particular, given any $3 \leq q_0 < q_1$ we can form a path in $\mathcal{R}_2$ connecting the slice $\mathcal{R}_2 \cap \{q = q_0\}$ to the slice $\mathcal{R}_2 \cap \{q = q_1\}$ using the mapping $q \mapsto (z_c^{\text{mid}}(t_3(q)+1, q), t_3(q)+1, q)$ where q varies over the interval $[q_0, q_1]$. Therefore, it suffices to prove for each $q = q_0 \geq 3$ that $\mathcal{R}_2 \cap \{q = q_0\}$ is path-connected. Given any $(z_0, t_0, q_0) \in \mathcal{R}_2 \cap \{q = q_0\}$ we can connect it using a path within $\mathcal{R}_2 \cap \{q = q_0\}$ to $(z_c^{\text{mid}}(t_3(q_0)+1), t_3(q_0)+1, q_0)$ as follows. One first varies just the z-coordinate to connect from (z_0, t_0, q_0) to $(z_c^{\text{mid}}(t_0, q_0), t_0, q_0)$. One then connects from $(z_c^{\text{mid}}(t_0, q_0), t_0, q_0)$ to $(z_c^{\text{mid}}(t_3(q_0)+1), t_3(q_0)+1, q_0)$ by varying t between t_0 and $t_3(q_0)+1$ and letting $z = z_c^m(t, q_0)$ as one does so.

We now prove Statements (i) and (ii). The regions $\mathcal{R}_1$ and $\mathcal{R}_2$ were chosen so that the second iterate of the renormalization mapping $R^2_{z,t,q}(w)$ has only attracting or repelling (but not neutral) real fixed points for all parameters $(z,t,q) \in \mathcal{R}_1 \cap \mathcal{R}_2$. Therefore, as one varies the parameters (z,t,q) within a given region $\mathcal{R}_1$ or $\mathcal{R}_2$ these real fixed points each vary continuously and their nature (attracting or repelling) remains unchanged. (This also implies that the same holds for fixed points of the first iterate $R_{z,t,q}(w)$.) Moreover, Lemma 13(iii) gives for all $(z,t,q) \in \mathcal{R}_1 \cup \mathcal{R}_2$ that $R^2_{z,t,q}((0,\infty))$ is contained in a compact subset of $(0,\infty)$. Therefore, as (z,t,q) varies over $\mathcal{R}_1$ or over $\mathcal{R}_2$ any fixed point of $R^2_{z,t,q}(w)$ that lies in $(0,\infty)$ of one choice of parameters (z,t,q) moves to a fixed point of $R^2_{z,t,q}(w)$ that again lies in $(0,\infty)$ for the other choice of parameters. Therefore it suffices to check the assertions of each of the claims (ii) and (iii) for a single choice of parameters in each region.

Region $\mathcal{R}_1$: Note that $(z,t,q)=(1,3,3)\in\mathcal{R}_1$. One finds that

$$R_{1,3,2}(w)=\frac{(4w+3)^2}{(6w+1)^2} \qquad \text{and} \qquad R^2_{1,3,2}(w)=\frac{\left(172w^2+132w+39\right)^2}{\left(132w^2+156w+55\right)^2}.$$

By solving $R^2_{1,3,3}(w)=w$ for w one finds that the only real solution is $w=1$ and that it is also a fixed point of $R_{1,3,3}(w)$. Moreover, $|R'_{1,3,2}(w)|=\frac{4}{7}<1$, so the only real fixed point of $R^2_{1,3,3}(w)$ is actually an attracting fixed point of the first iterate $R_{1,3,3}(w)$.

Region $\mathcal{R}_2$: Note that $(z,t,q)=(1/5,8,3)\in R_2$ because

$$z_c^-(8,3)=\frac{9229-119\sqrt{5593}}{46656}\approx 0.0071 \qquad \text{and} \qquad z_c^+(8,3)=\frac{9229-119\sqrt{5593}}{46656}\approx 0.3886.$$

We have

$$R_{1/5,8,3}(w)=\frac{(9w+8)^2}{5(16w+1)^2} \quad \text{and} \qquad R^2_{1/5,8,3}(w)=\frac{\left(1567w^2+368w+88\right)^2}{5\left(368w^2+352w+147\right)^2}.$$

By solving $R^2_{1/5,8,3}(w)=w$ for w one finds the following three real roots:

$$w_1:=\frac{939-17\sqrt{2165}}{1058}\approx 0.1399, \qquad w_2:=\frac{939+17\sqrt{2165}}{1058}\approx 1.6352 \quad \text{and}$$
$$w_3\approx 0.4412, \quad \text{which is the unique real root of } p(x)=-64-139x+79x^2+1280x^3.$$

The only real root of $R_{1/5,8,3}(w)=w$ is $w=w_3$ and one finds that $\left|(R_{1/5,8,3})'(w_3)\right|\approx 1.088>1$. Thus, w_3 is a repelling fixed point of $R_{1/5,8,3}(w)$. Furthermore, one can check that $R_{1/5,8,3}(w_1)=w_2$, $R_{1/5,8,3}(w_2)=w_1$, and

$$\left|(R^2_{1/5,8,3})'(w_1)\right|=\left|(R^2_{1/5,8,3})'(w_2)\right|\approx 0.7003<1.$$

Thus, w_1 and w_2 correspond to an attracting period 2-cycle of $R_{1/5,8,3}(w)$. □

Proof of Theorem 3. We will first prove that if $(z,t,q)\in\mathcal{R}_1\cup\mathcal{R}_2$ then the marked point $a(z)=z$ is passive under iteration of $R_{z,t,q}(w)$. By Lemma 1 it suffices to prove that $a(z)$ is passive under the second iterate $R^2_{z,t,q}(w)$. Moreover, by Lemma 2, it suffices to prove that for such choices of parameters we have that $a(z)$ is in the basin of attraction of an attracting fixed point for $R^2_{z,t,q}(w)$.

Suppose $(z,t,q)\in\mathcal{R}_1\cup\mathcal{R}_2$. By Lemma 13, we see that $R^2_{z,t,q}(w)$ satisfies the hypotheses of Lemma 8. Lemma 15 implies that $R^2_{z,t,q}(w)$ has no neutral fixed points and also that if $R^2_{z,t,q}(w)$ has a repelling fixed point then it is actually a repelling fixed point of the first iterate $R_{z,t,q}(w)$. Meanwhile, Lemma 10 gives that if $a(z)$ is a fixed point for $R_{z,t,q}(w)$ then $z=1$. As in the proof of Theorem 2 we compute that

$$R'_{1,t,q}(1)=\frac{2-2t}{(q-1)t+1}.$$

One can then check that for $t>1$ and $q\geq 3$ we have $-1<R'_{1,t,q}(1)<0$. Therefore, $a(z)=z$ cannot be a repelling fixed point of $R_{z,t,q}(w)$. By Lemma 8 we therefore have that if $(z,t,q)\in\mathcal{R}_1\cup\mathcal{R}_2$ then the marked point $a(z)=z$ is in the basin of attraction of an attracting fixed point of $R^2_{z,t,q}(w)$, as desired.

It remains to show that if $t\geq t_3(q)$ and $z=z_c^\pm(t,q)$ then the marked point $a(z)=z$ is active under $R_{z,t,q}(w)$. Since the active locus is closed, we can suppose that $t>t_3(q)$. Then, for any $z>z_c^+(t,q)$ we have that $(z,t,q)\in\mathcal{R}_1$ and Lemma 15 and the discussion in the previous paragraph gives that $a(z)$ is in the basin of attraction of an attracting fixed point for $R_{z,t,q}(w)$. On the other hand for any $z_c^-(t,q)<z<z_c^+(t,q)$ we have that $a(z)$ is in the basin of attracting of a period two attracting cycle of $R_{z,t,q}(w)$. Lemma 3 then implies that the marked point $a(z)$ is active at the parameter $z=z_c^+(t,q)$.

The proof that $a(z)$ is active at $z=z_c^-(t,q)$ is completely analogous. □

5. Proof of Theorem C (Antiferromagnetic Case, $q = 2$)

Like the proof of Theorem B, the proof of Theorem C reduces to the following statement.

Theorem 4. *Suppose $t > 1, q = 2$ and let $\mathcal{A}(R_{z,t,2})$ be the active locus for the marked point $a(z) = z$ under $R_{z,t,2}(w)$. Then*

$$\mathcal{A}(R_{z,t,2}) \cap (0,\infty) = \begin{cases} \emptyset & \text{if } 1 \le t < 3 \\ \{1\} \cup \{z_c^{\pm}(t,2)\} & \text{if } 3 \le t. \end{cases} \tag{20}$$

Here $t_3(2) = 3$ and $z_c^{\pm}(t,2)$ are given in Equations (10) and (12) as in the statement of Theorem C.

Sketch of proof. The proof is quite similar to the proof of Theorem 3 with a couple of small variations. First note that Lemmas 13 and 14 both hold for $q = 2$. We did not include the case $q = 2$ in the statement of Lemma 15 because $z_c^+(t,2)$ crosses $z = 1$ when $t = 3$. This would have complicated our proof that R_1 and R_2 are connected (as q is varied), which played an important role in the proof of Lemma 15. However, if one considers the following sets

$$\mathcal{S}_1 := \{(z,t) \in \mathbb{R}^2 : t > 1, \text{ and if } t \ge 3 \text{ then } 0 < z < z_c^-(t,2) \text{ or } z > z_c^+(t,2)\}$$
$$\mathcal{S}_2 := \{(z,t) \in \mathbb{R}^2 : t > 3,\ z_c^-(t,q) < z < z_c^+(t,2)\}.$$

then it is straightforward to prove the analogous claims for them as in Lemma 15. (As before, one proves that each set is connected and then checks the claims (i) and (ii) for a single choice of parameters in each set. The proof that S_1 and S_2 are connected is rather straightforward, since one can directly check that $z_c^+(t,2)$ is an increasing function and $z_c^-(t,2)$ is a decreasing function of $t \ge 3$.)

The proof of Theorem 4 then concludes in almost the same way as the proof of Theorem 3, except that since $q = 2$ one can have that the marked point $a(z) = z$ does non-persistently land on a repelling fixed point $w = 1$ when $z = 1$. It does this for all $t \ge 3$, leading to the additional point on $(0,\infty)$ at which the Lee-Yang zeros can accumulate that was described in Theorem C. □

Appendix A. Proof of Lemma 12.

We start with two simple lemmas that will be used in the proof. In each of these statements we extend the range of allowable values of q from $\mathbb{N}_{\ge 2}$ to the interval $[2,\infty)$ so that methods involving continuity can be used in the proofs.

Lemma 16. *Suppose $z > 0$, $t \in [0,1)$, and $q \ge 2$. If the renormalization mapping $R_{z,t,q}(w)$ has triple fixed point (i.e. w is a triple root of $R_{z,t,q}(w) = w$) then $t = t_2(q) = \frac{q-2+\sqrt{q^2+32q-32}}{18(q-1)}$.*

Proof. A polynomial $w^3 + aw^2 + bw + c = 0$ has a triple root if and only if $b = \frac{a^2}{3}$ and $c = \frac{a^3}{27}$. Using that $z > 0$, $t \in [0,1)$, and $q \ge 2$, we can rewrite the condition that $R_{z,t,q}(w) = w$ as a monic cubic polynomial of the above form. It has a triple root if and only if

$$\frac{[2tz((q-2)t+1)-1]}{-(q-1)^2t^2} = \frac{1}{3}\left(\frac{2tz((q-2)t+1)-1}{-(q-1)^2t^2}\right)^2 \text{ and } \frac{t^2z}{-(q-1)^2t^2} = \frac{1}{27}\left(\frac{2tz((q-2)t+1)-1}{-(q-1)^2t^2}\right)^3.$$

Clearing denominators, the above two equations can be simplified to the conditions that $P_1(z,t,q) = 0$ and $P_2(z,t,q) = 0$ where P_1 and P_2 are polynomials in z, q and t. One can then use elimination theory (resultants) to eliminate z from the above two equations. One finds the following factorized polynomial in q and t:

$$(q-1)^4(t-1)^2t^4(qt-t+1)^2\left(9qt^2 - qt - 9t^2 + 2t - 1\right) = 0.$$

Since $z > 0$, $t \in [0,1)$, and $q \ge 2$, the only relevant root comes from the last factor and it is $t = t_2(q)$. □

Lemma 17. *For any $q \ge 2$ and $t \in (0, t_2(q)]$ we have:*

(a) *when $z = N_+(t,q)$, the point $w = z$ is fixed by $R_{z,t,q}(w)$ if and only if $t = \frac{1-2\sqrt{q-1}}{5-4q}$, and*

(b) *when* $z = N_-(t,q)$*, the point* $w = z$ *is fixed by* $R_{z,t,q}(w)$ *if and only if* $t = t_1(q) = \frac{1}{q+1}$*.*

Proof. Note that the marked points $a(z) = N_\pm(t,q)$ are positive by Lemma 11. Lemma 10 therefore gives that they are fixed points of $R_{z,t,q}(w)$ if and only if they equal 1. The result then follows by solving $N_\pm(t,q) = 1$. $\square$

Lemma 18. *For any* $q \geq 2$ *and* $t \in (0, t_2(q))$ *if* $z = N_\pm(t,q)$ *then* $R_{z,t,q}(w)$ *has exactly two fixed points and the non-neutral fixed point is attracting.*

Proof. For $t \in (0, t_2(q))$ and $z = N_\pm(t,q)$ the mapping $R_{z,t,q}(w)$ has exactly two fixed points, with one corresponding to a double root of $R_{z,t,q}(w) = w$ and the other corresponding to a simple root of the same equation. (See the proof of Lemma 11.) With this range of values for the parameters t, q and $z = N_\pm(t,q)$ we have that $R'_{z,t,q}(w) > 0$ for $w > 0$. Therefore, the double root corresponds to a neutral fixed point $w_N(t,q)$ and for all w in a sufficiently small interval I containing $w_N(t,q)$ we either have $R_{z,t,q}(w) \geq w$ or for all $w \in I$ we have $R_{z,t,q}(w) \leq w$. In other words, the graph of $R_{z,t,q}(w)$ does not cross the diagonal at the neutral fixed point $w_N(t,q)$.

Note that $R_{z,t,q}(0) = zt^2 > 0$, and that

$$\lim_{w \to \infty} R_{z,t,q}(w) = \frac{z((q-2)t+1)^2}{(q-1)^2 t^2} \in (0, \infty).$$

Therefore, if we denote the fixed point corresponding to the simple root of $R_{z,t,q}(w) = w$ by $w_2(t,q)$ we must have a small interval J containing $w_2(t,q)$ such that $R_{z,t,q}(w) > w$ for all $w \in J$ satisfying $w < w_2(t,q)$ and such that $R_{z,t,q}(w) < w$ for all $w \in J$ satisfying $w > w_2(t,q)$. This implies that $0 < R'_{z,t,q}(w_2(t,q)) < 1$ and hence that w_2 is an attracting fixed point for $R_{z,t,q}(w)$. $\square$

For the remainder of the proof we will denote the neutral and attracting fixed points whose existence is given by Lemma 18 by $w_N(t,q)$ and $w_A(t,q)$.

Proof of Lemma 12. **Proof of Claim (i):** We suppose that $t \in (0, t_2(q))$ and that $z = N_+(t,q)$. Rather than considering $q \geq 2$ as a natural number we consider it as a real number. By Lemma 17 the marked point $z = N_+(t,q)$ is a fixed point of $R_{z,t,q}(w)$ if and only if $t = \frac{1-2\sqrt{q-1}}{5-4q} =: t_4(q)$. A direct calculation gives that $t_4(q) = t_2(q)$ if and only if $q = 2$ and that for $q > 2$ we have $0 < t_4(q) < t_2(q)$. Thus, the two curves $t = t_2(q)$ and $t = t_4(q)$ divide the qt-plane, when $t > 0$ and $q \geq 2$, into three regions:

Region I: $0 < t < t_4(q)$, Region II: $t_4(q) < t < t_2(q)$, and Region III: $t > t_2(q)$.

Refer to Figure 12 for an illustration. Note that Region III is not considered under the hypotheses of Lemma 12, so we will ignore it.

Note that every complex fixed point of $R_{z,t,q}(w)$ is real when $z = N_+(t,q)$ and when (t,q) varies over either Region I or Region II and that one of them is consistently a solution of multiplicity two to the equation $R_{z,t,q}(w) = w$. Since complex solutions to $R_{z,t,q}(w) = w$ vary continuously with respect to the parameters, this implies that $w_N(t,q)$ and $w_A(t,q)$ vary continuously as the parameters (t,q) are varied over Region I or over Region II and they are never equal since that would correspond to a fixed point of multiplicity 3 which only happens when $t = t_2(q)$, by Lemma 16. Moreover, by Lemma 7 we have that $R_{z,t,q}([0,\infty))$ is compactly contained in $(0,\infty)$ for each choice of parameters, so that any fixed points in $(0,\infty)$ for one choice of parameters cannot leave $(0,\infty)$ for a different choice of parameters in the same region.

Within Regions I and Region II we also have that the marked point $z = N_+(t,q)$ is not fixed by $R_{z,t,q}(w)$ because $t \neq t_4(q)$ in those regions. Therefore, the order with which $w_N(t,q), w_A(t,q)$, and $N_+(t,q)$ occur on $[0,\infty)$ is constant in each of the two regions.

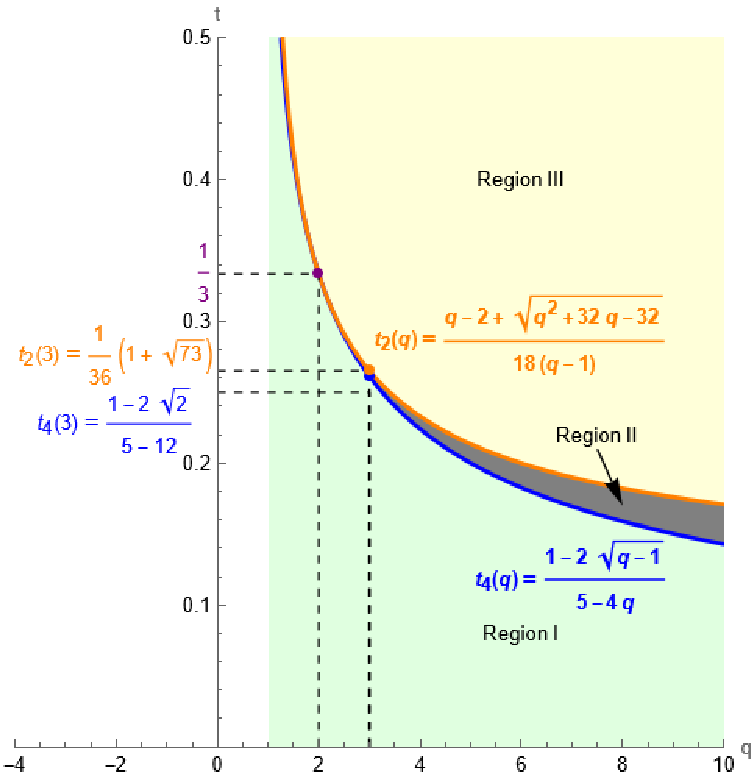

FIGURE 12. Plot of Regions I, II, and III in proof of Claim (i).

Region I: Considering $(t, q) = (0.1, 5)$, which lies in Region 1. For these parameters one can do explicit calculations in a computer algebra system (e.g. Mathematica) to find:

$$z = N_+(0.1, 5) = \frac{141\sqrt{329} + 6457}{4394} \approx 2.051,$$

$$w_N(0.1, 5) = \frac{1}{52}\left(59 - 3\sqrt{329}\right) \approx 0.0881717, \quad \text{and}$$

$$w_A(0.1, 5) = \frac{1}{416}\left(189\sqrt{329} + 3433\right) \approx 16.4931.$$

In particular, we find that for all (t, q) in Region I we have the following order

$$w_N(t, q) < z = N_+(t, q) < w_A(t, q).$$

In particular, since $R_{z,t,q}(0) > 0$ and since the graph of $R_{z,t,q}(w)$ does not cross the diagonal at $w_N(t, q)$ we find that $R_{z,t,q}(w) > w$ for all $w \in (w_N(t, q), w_A(t, q))$. This implies that the orbit of $z = N_+(t, q)$ under iteration of $R_{z,t,q}(w)$ converges to $w_A(t, q)$, as claimed.

Region II: Considering $(t, q) = (0.15, 10)$ which is in Region II. For these parameters one can again do explicit calculations to find:

$$z = N_+(0.15, 10) = \frac{151\sqrt{120649} + 1880293}{2044416} \approx 0.945376,$$

$$w_N(0.15, 10) = \frac{637 - \sqrt{120649}}{2376} \approx 0.121908, \quad \text{and}$$

$$w_A(0.15, 10) = \frac{799\sqrt{120649} + 327037}{769824} \approx 0.78533.$$

In particular, we find that for all (t, q) in Region II we have the following order

$$w_N(t, q) < w_A(t, q) < z = N_+(t, q).$$

Since $\lim_{w\to\infty} R_{z,t,q}(w)$ is finite we have that for all $w > w_A(t,q)$ that $R_{z,t,q}(w) < w$. This implies that the orbit of $z = N_+(t,q)$ under iteration of $R_{z,t,q}(w)$ converges to $w_A(t,q)$, as claimed.

Finally, suppose that $t = t_4(q)$, corresponding to the boundary between Regions I and II. It is clear from the above calculations and continuity that in this case $z = N_+(t,q) = w_A(t,q)$. Thus Claim (i) is proved.

Proof of Claim (ii): We suppose that $t \in (0, t_2(q))$ and that $z = N_-(t,q)$. By Lemma 17 the marked point $z = N_+(t,q)$ is a fixed point of $R_{z,t,q}(w)$ if and only if $t = t_1(q) = 1/(q+1)$. A direct calculation gives that $t_1(q) = t_2(q)$ if and only if $q = 2$ and that for $q > 2$ we have $0 < t_1(q) < t_2(q)$. Thus, the two curves $t = t_2(q)$ and $t = t_1(q)$ divide the qt-plane, when $t > 0$ and $q \geq 2$, into three regions:

$$\text{Region I:}\quad 0 < t < t_1(q), \qquad \text{Region II:}\quad t_1(q) < t < t_2(q), \qquad \text{and} \quad \text{Region III:}\quad t > t_2(q).$$

Refer to Figure 13 for an illustration. Note that Region III is not considered under the hypotheses

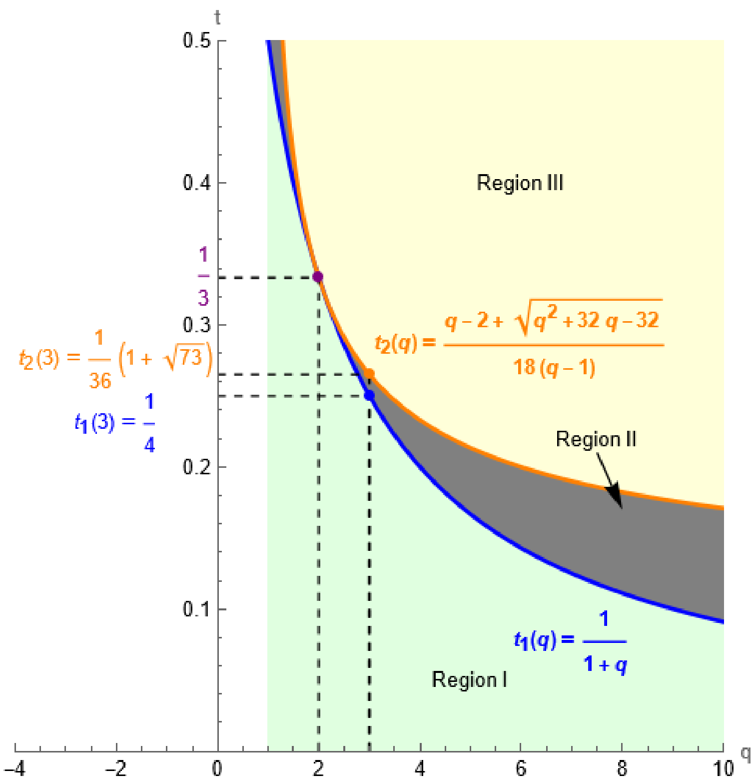

Figure 13. Plot of Regions I, II, and III in proof of Claim (ii).

of Lemma 12, so we will ignore it.

Just as in the proof of Claim (i) the fixed points $w_N(t,q), w_A(t,q)$, and the marked point $z = N_-(t,q)$ vary continuously with (t,q) over Regions I and II, that they never leave $(0,\infty)$, and on a given region they are never equal. Therefore, the order with which $w_N(t,q), w_A(t,q)$, and $N_-(t,q)$ occur on $[0,\infty)$ is constant in each of the two regions.

Region I: Consider $(t,q) = (0.2, 3)$, which lies in Region I. For these parameters one can do explicit calculations to find:

$$z = N_-(0.2,3) = \frac{1}{36}\left(39 - \sqrt{21}\right) \approx 0.95604,$$
$$w_N(0.2,3) = \frac{1}{6}\left(\sqrt{21}+6\right) \approx 1.76376, \quad \text{and}$$
$$w_A(0.2,3) = \frac{1}{12}\left(33 - 7\sqrt{21}\right) \approx 0.0768308.$$

In particular, we find that for all (t, q) in Region I we have the following order

$$w_A(t,q) < z = N_-(t,q) < w_N(t,q).$$

In particular, since $\lim_{w\to\infty} R_{z,t,q}(w)$ is finite and since the graph of $R_{z,t,q}(w)$ does not cross the diagonal at $w_N(t,q)$ we find that $R_{z,t,q}(w) < w$ for all $w \in (w_A(t,q), w_N(t,q))$. This implies that the orbit of $z = N_-(t,q)$ under iteration of $R_{z,t,q}(w)$ converges to $w_A(t,q)$, as claimed. This proves Claim (ii)(a).

Region II: Consider $(t,q) = (1/6, 6)$, which lies in Region II. For these parameters one finds:

$$\begin{aligned} z = N_-(1/6,6) &= \frac{1}{320}(-3)\left(\sqrt{33}-111\right) \approx 0.98677,\\ w_N(1/6,6) &= \frac{1}{20}\left(\sqrt{33}+9\right) \approx 0.737228, \quad \text{and}\\ w_A(1/6,6) &= \frac{1}{80}\left(69-11\sqrt{33}\right) \approx 0.0726226. \end{aligned}$$

In particular, we find that for all (t, q) in Region I we have the following order

$$w_A(t,q) < w_N(t,q) < z = N_-(t,q).$$

since $\lim_{w\to\infty} R_{z,t,q}(w)$ is finite we find that $R_{z,t,q}(w) < w$ for all $w \in (w_N(t,q), \infty)$. This implies that the orbit of $z = N_-(t,q)$ under iteration of $R_{z,t,q}(w)$ converges to $w_N(t,q)$, as claimed.

Note that if $t = t_1(q)$ it is clear from the above calculations and continuity that $z = N_+(t,q) = w_N(t,q)$. Combined with the analysis of the parameters in Region II, this proves Claim (ii)(b).

Proof of Claim (iii): First suppose that $z = N_-(t,q)$ for (t,q) in Region II, as discussed in Case (ii) above. Since the graph of $R_{z,t,q}(w)$ does not cross the diagonal at the fixed point $w_N(t,q)$ we have that for all $w \in (w_A(t,q), \infty)$ that $R_{z,t,q}(w) \leq w$ with equality occurring at $w = w_N(t,q)$. Decreasing the parameter $z > 0$ decreases the values of $R_{z,t,q}(w)$ and this will eliminate the neutral fixed point $w_N(t,q)$. Meanwhile, if the decrease of the parameter $z > 0$ is by a sufficiently small amount, then the then the attracting fixed point $w_A(t,q)$ will move continuously to an attracting fixed point w'_A for the perturbed map, and we will have that $R_{z,t,q}(w) < w$ for all $w \in (w'_A, \infty)$. This implies that the orbit of the marked point $w = z$ will now converge to the new attracting fixed point w'_A, as claimed in Part (iii) of the Lemma. □

Appendix B. Derivation of the Renormalization Mapping (Proof of Theorem D)

Proof of Theorem D. For each $0 \leq j \leq q-1$ we define the conditional partition functions of Γ_n conditioned on the spin $\sigma(r)$ at the root vertex equaling j as follows

$$Z_n^j \equiv Z_n^j(z,t) := \sum_{\sigma \ s.t \ \sigma(r)=j} W_n(\sigma).$$

Here $W_n(\sigma) := e^{-\frac{H_n(\sigma)}{T}}$ is the Boltzmann-Gibbs weight of the configuration σ.

(21) Notice that $Z_n^j = Z_n^k$ for any $1 \leq j,k \leq q-1$.

To see why this is true, let ρ be a permutation on $\{0, 1, \cdots, q-1\}$ which fixes 0. Then $H_n(\sigma) = H_n(\rho \circ \sigma)$, implying the claim. The term Z_n^0 is different from Z_n^k for $k = 1, \cdots, q-1$ because of the term $-h\sum_{i\in V_n} \delta(\sigma(i), 0)$ that corresponds to the interaction between the externally applied magnetic field and the spins $\sigma(i)$ in (5). The full partition function then equals to

$$Z_n \equiv Z_n(z,t) = Z_n^0 + (q-1)Z_n^1.$$

Computing Z_0^0 and Z_0^1: Notice that Γ_n is just the root vertex r when $n = 0$, so that $H(\sigma) = -h$ when $\sigma(r) = 0$ and $H(\sigma) = 0$ when $\sigma(r) = 1$. Hence, we have

$$Z_0^0 = e^{h/T} = z^{-1} \qquad \text{and} \qquad Z_0^1 = e^0 = 1. \tag{22}$$

Computing Z_{n+1} in terms of Z_n^0 and Z_n^1: We first compute Z_{n+1}^0 and Z_{n+1}^1 in terms of Z_n^0 and Z_n^1. Let $\sigma : V_{n+1} \longrightarrow \{0, \cdots, q-1\}$ be a spin configuration. Let Λ_n be the graph that can be obtained by joining a single new vertex (say α) to the root vertex r of Γ_n using a new edge (see Figure 14). By Λ_n^2 let us denote the graph that one can obtain by gluing two Λ_n graphs together at the vertex α (see Figure 15). In fact $\Gamma_{n+1} = \Lambda_n^2$. Recall from Equation (5) that $H_n(\sigma)$ denotes the Hamiltonian of the n^{th} level Cayley rooted tree with branching number two for the spin configuration σ. By $H_{\Lambda_n}(\sigma)$, let us denote the Hamiltonian of Λ_n with spin configuration σ. Let $\mathcal{W}_{n+1}(\sigma)$ be the Boltzmann-Gibbs weight of configuration σ for the graph Λ_n, and let $\mathcal{Z}_{n+1}^{\sigma(\alpha)}$ be the conditional partition function of Λ_n, supposing the spin at α is $\sigma(\alpha)$. Here we extend the domain of σ from V_{n+1} to $V_{n+1} \cup \{\alpha\}$. Thus, $\sigma(\alpha) \in \{0, \cdots, q-1\}$.

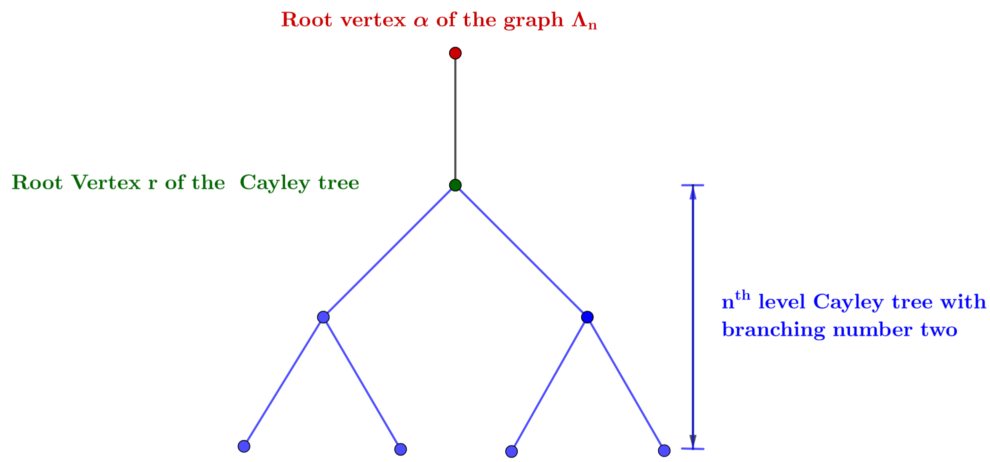

FIGURE 14. Graph Λ_n.

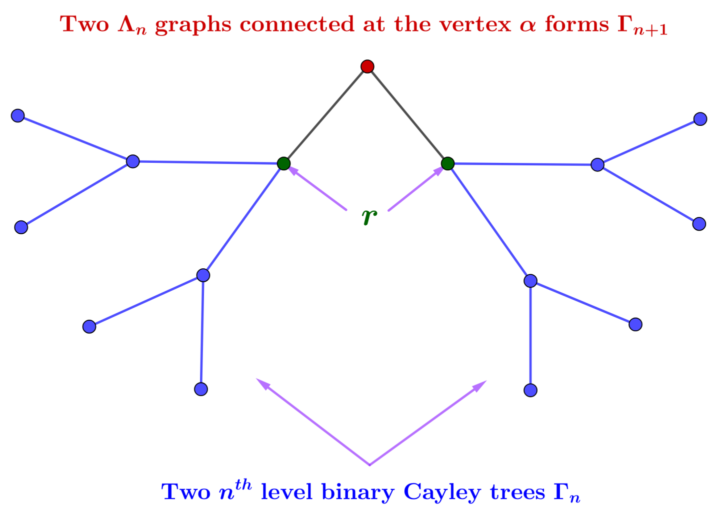

FIGURE 15. Graph $\Gamma_{n+1} = \Lambda_n^2$: two Λ_n connected at their respective vertices α.

Computing Z_{n+1}^0 in terms of Z_n^0 and Z_n^1: First, we compute the conditional partition function $\mathcal{Z}_{n+1}^0$ of Λ_n in terms of z, t, Z_n^0 and Z_n^1. Notice that the superscript 0 on Z_{n+1}^0 indicates that we will have $\sigma(\alpha) = 0$ throughout this entire subsection. We have two cases.

Case 1: $\sigma(r)=0$. Then $H_{\Lambda_n}(\sigma)=H_n(\sigma)-h$, implying that $\sum_{\sigma(r)=0}\mathcal{W}_{n+1}=z^{-1}Z_n^0$.

Case 2: $\sigma(r)\neq 0$. Then $H_{\Lambda_n}(\sigma)=H_n(\sigma)+J-h$, implying that $\sum_{\sigma(r)\neq 0}\mathcal{W}_{n+1}=(q-1)tz^{-1}Z_n^0$. Here we have used Equation (21).

Thus

$$\mathcal{Z}_{n+1}^0=z^{-1}\big(Z_n^0+(q-1)tZ_n^1\big). \tag{23}$$

Notice that, when $\sigma(\alpha)=0$,

$$H_{n+1}=\text{Hamiltonian of }\Gamma_n^2=H_{\Gamma_n}(\sigma)+H_{\Gamma_n}(\sigma)+h.$$

Here the term "h" comes from the Hamiltonian of the graph of a single vertex α (subtracting the energy of the single point α). Thus, $Z_{n+1}^0=z(\mathcal{Z}_{n+1}^0)^2$. This together with (23) this gives us

$$Z_{n+1}^0=z^{-1}\big(Z_n^0+(q-1)tZ_n^1\big)^2. \tag{24}$$

Computing Z_{n+1}^1 in terms of Z_n^0 and Z_n^1: First, we compute the conditional partition function $\mathcal{Z}_{n+1}^1$ of Λ_n in terms of z,t,Z_n^0 and Z_n^1. Notice that the superscript 1 on Z_{n+1}^1 indicates that we will have $\sigma(\alpha)=1$ throughout this entire subsection. We have three cases.

Case 1: $\sigma(r)=0$. Then $H_{\Lambda_n}(\sigma)=H_n(\sigma)+J$, implying that $\sum_{\sigma(r)=0}\mathcal{W}_{n+1}=tZ_n^0$.

Case 2: $\sigma(r)=1$. Then $H_{\Lambda_n}(\sigma)=H_n(\sigma)$, implying that $\sum_{\sigma(r)=1}\mathcal{W}_{n+1}=Z_n^1$.

Case 3: $\sigma(r)\neq 0,1$. Then $H_{\Lambda_n}(\sigma)=H_n(\sigma)+J$, implying that $\sum_{\sigma(r)\notin\{0,1\}}\mathcal{W}_{n+1}=(q-2)tZ_n^1$. Here we have used Equation (21).

Thus

$$\mathcal{Z}_{n+1}^1=tZ_n^0+Z_n^1+(q-2)tZ_n^1. \tag{25}$$

Notice that, when $\sigma(\alpha)=1$,

$$H_{n+1}=\text{Hamiltonian of }\Gamma_n^2=H_{\Gamma_n}(\sigma)+H_{\Gamma_n}(\sigma).$$

Thus $Z_{n+1}^1=\big(\mathcal{Z}_{n+1}^1\big)^2$. This together with Equation (25) gives us

$$Z_{n+1}^1=\big(tZ_n^0+Z_n^1+(q-2)tZ_n^1\big)^2. \tag{26}$$

Therefore, by (24) and (26) we have a formula for the full partition function of the rooted Cayley tree with branching number two at level $n+1$:

$$\begin{aligned} Z_{n+1}&:=Z_{n+1}^0+(q-1)Z_{n+1}^1\\ &=z^{-1}\big(Z_n^0+(q-1)tZ_n^1\big)^2+(q-1)\Big(tZ_n^0+Z_n^1+(q-2)tZ_n^1\Big)^2. \end{aligned}\tag{27}$$

We are interested in the zeros of Z_{n+1}. Notice that $Z_{n+1}=0$ when

$$\frac{Z_{n+1}^1}{Z_{n+1}^0}=z\frac{\big(tZ_n^0+Z_n^1+(q-2)tZ_n^1\big)^2}{\big(Z_n^0+(q-1)tZ_n^1\big)^2}=z\left[\frac{tZ_n^0+Z_n^1+(q-2)tZ_n^1}{Z_n^0+(q-1)tZ_n^1}\right]^2=\frac{1}{1-q}. \tag{28}$$

Let $w_n:=\frac{Z_n^1}{Z_n^0}$. Then by (28),

$$w_{n+1}=z\left[\frac{tZ_n^0+Z_n^1+(q-2)tZ_n^1}{Z_n^0+(q-1)tZ_n^1}\right]^2=z\left[\frac{t+\frac{Z_n^1}{Z_n^0}+(q-2)t\frac{Z_n^1}{Z_n^0}}{1+(q-1)t\frac{Z_n^1}{Z_n^0}}\right]^2=z\left[\frac{t+w_n+(q-2)tw_n}{1+(q-1)tw_n}\right]^2.$$

Thus if we define

$$R_{z,t,q}(w):=z\left[\frac{t+w+(q-2)tw}{1+(q-1)tw}\right]^2, \tag{29}$$

then

$$w_n = R^n_{z,t,q}(w_0).$$

Here, the superscript "n" indicates that we compose $R_{z,t,q}(w)$ with itself n times (with respect to the variable w while fixing parameters z, t and q). In other words, it denotes the n^{th} iterate of the function $R_{z,t,q}$.

By definition and (22) we have $w_0 = \frac{Z_0^1}{Z_0^0} = \frac{1}{z^{-1}} = z$. So, $w_n = R^n_{z,t,q}(z)$. Therefore, by (27) and (29) the Lee-Yang zeros of the q-state Potts model on the n^{th} level Cayley tree with branching number two are the solutions to the equation

$$R^n_{z,t,q}(z) = \frac{1}{1-q}. \tag{30}$$

This finishes the proof of Part (i) of Theorem D.

Case of the unrooted Cayley tree: We now explain how to prove Part (ii) of Theorem D about the unrooted Cayley $\hat{\Gamma}_n$. Denote by c the central vertex of $\hat{\Gamma}_n$, i.e. the vertex at distance n from the leaves of $\hat{\Gamma}_n$. For any $0 \le j \le q-1$ denote by $\hat{Z}_n^j$ the conditional partition function for $\hat{\Gamma}_n$ conditioned on $\sigma(c) = j$. Like in the proof for the rooted tree, we again have $\hat{Z}_n^j = \hat{Z}_n^k$ for any $1 \le j, k \le q-1$.

One obtains $\hat{\Gamma}_n$ by taking three copies of the rooted tree Γ_{n-1} and attaching each of their root vertices by an edge to the central vertex c. In much the same way as we handled the rooted tree above, one can prove that:

$$\hat{Z}_n^0 = z^{-1}\big(Z_{n-1}^0 + (q-1)tZ_{n-1}^1\big)^3 \qquad \text{and} \qquad \hat{Z}_n^1 = \big(tZ_{n-1}^0 + (1+(q-2)t)Z_{n-1}^1\big)^3.$$

Therefore, $\hat{Z}_n = \hat{Z}_n^0 + (q-1)\hat{Z}_n^1 = 0$ if and only if

$$\frac{\hat{Z}_n^1}{\hat{Z}_n^0} = z\frac{\big(tZ_{n-1}^0 + (1+(q-2)t)Z_{n-1}^1\big)^3}{\big(Z_{n-1}^0 + (q-1)tZ_{n-1}^1\big)^3} = \hat{R}_{z,t,q}(w_{n-1}) = \left(\hat{R}_{z,t,q} \circ R^{(n-1)}_{z,t,q}\right)(z) = \frac{1}{1-q}.$$

This finishes the proof of Part (ii) of Theorem D. □

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