Fractal Structure of Zeros in Hierarchical Models (after Derrida, De Seze, and Itzykson)

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I. Preliminaries.

Diamond hierarchical lattice.

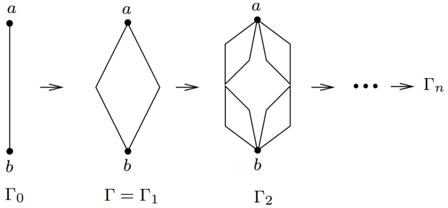

Figure : The first few graphs in the DHL

Let Γ be the diamond graph. The **diamond hierarchical lattice** is the sequence of graphs $\{ \Gamma_n \}_{n\in\mathbb{N}}$ such that

- $\Gamma_1 := \Gamma$.
- Γ_{n+1} has two marked vertices a, b and is obtained from Γ_n by replacing each edge of Γ_n by Γ_1.

Let $\Gamma_n = (V_n, \ E_n)$

Potts model on the DHL

A *configuration* of spins is a mapping

$$\sigma : V_n \longrightarrow \{1, 2, \ldots, q\}$$

The Ising model is the case $q = 2$. The *Energy* of σ is

$$\mathcal{H}_n(\sigma) := -J \sum_{(i,j)\in E_n} \delta_{\sigma_i,\ \sigma_j}$$

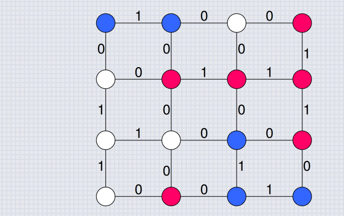

Figure : Here, $q = 3$ and $\mathcal{H} = -10J$.

$P_n(\sigma)$ and the partition function.

A configuration σ occurs with probability proportional to the *Gibbs weight*

$$W_n(\sigma) := e^{-H_n(\sigma)/T}$$

Note that

- When T is close to zero, then minimal energy configurations have much higher probabilities.

- When T is close to ∞, all configurations have more or less the same probability.

Hence,

$$P_n(\sigma) = \frac{W_n(\sigma)}{Z_n}$$

where $Z_n := \sum_\sigma W(\sigma)$ is the *partition function*.

we introduce the change of variables

$$y := e^{J/T}$$

so that Z_n becomes a polynomial in y of degree $|E_n|$:

$$Z_n(y) = \sum_{\sigma} y^{I(\sigma)}$$

where $I(\sigma) := \sum_{(i,j)\in E_n} \delta_{\sigma(i),\sigma(j)}$ is the *interaction of* σ. There are exactly q configurations such that the spins are alligned, so:

$$Z_n(y) = q \prod_{i=1}^{|E_n|} (y - y_i)$$

The zeros of Z_n, $\{y_i\}_{1\leqslant i\leqslant |E_n|}$ are called the *Fisher zeros*.

II. Computing the Fisher zeros: Migdal - Kadanoff renormalization equations.

two conditional partition functions

Let

$$U_n := \sum_{\substack{\sigma \text{ s.t.} \\ \sigma(a)=\sigma(b)=1}} W_n(\sigma)$$

$$V_n := \sum_{\substack{\sigma \text{ s.t.} \\ \sigma(a)=1,\ \sigma(b)=2}} W_n(\sigma)$$

(U_n and V_n are functions of y). Clearly,

$$Z_n = qU_n + q(q-1)V_n$$

Finding an expression for U_n and V_n in terms of U_{n-1} and V_{n-1} is not hard (see blackboard) and since

$$U_0 = y, \quad V_0 = 1$$

we can compute Z_n via an iterative procedure.

We have obtained:

$$Z_n(y) = L \circ R^n(y, 1)$$

where $R : \mathbb{C}^2 \to \mathbb{C}^2$ is given by

$$R(U, V) := \left(\left(U^2 + (q-1)V^2\right)^2, \; V^2(2U + (q-2)V)^2 \right)$$

and $L : \mathbb{C}^2 \to \mathbb{C}$ is

$$L(U, V) := qU + q(q-1)V$$

Another iteration for the Fisher zeros

In the paper of **Derrida, De Seze and Itzykson (1983)** a different iterative procedure is used: Define $T : \mathbb{C} \to \mathbb{C}$ as

$$T(y) := \left(\frac{y^2 + q - 1}{2y + q - 2} \right)^2$$

Then $Z_n(y)$ are the 4^{n-1} preimages of $1 - q$ by the $(n-1)$-th iterate of T.

Derrida, De Seze and Itzykson (1983) studies numerically what happens in the *thermodynamic limit* $n \to \infty$.

Fractal structure of Fisher zeros

Recall that

- The *Julia set* of T, $\mathcal{J}(T)$, is the closure of the set of repelling periodic points of T.
- **Mikhail Lyubich** and **Alexandre Freires, Artur Lopes, and Ricardo Mañé** have shown (1983) that if a point y_0 is not *exceptional* for T (see below) then the probability measures $\mu_n(y_0)$ supported on $\{\ T^{-n}(y_0)\ \}$ converge, as $n \to \infty$ to the measure of maximal entropy, which is supported on $\mathcal{J}(T)$.
- $y_0 = 1 - q$ is not exceptional, since it is not a critical value of T.

Hence, in the thermodynamic limit $n \to \infty$, the Fisher zeros converge, in the sense explained above, to $\mathcal{J}(T)$.

results of the numerical simulations (pictures)

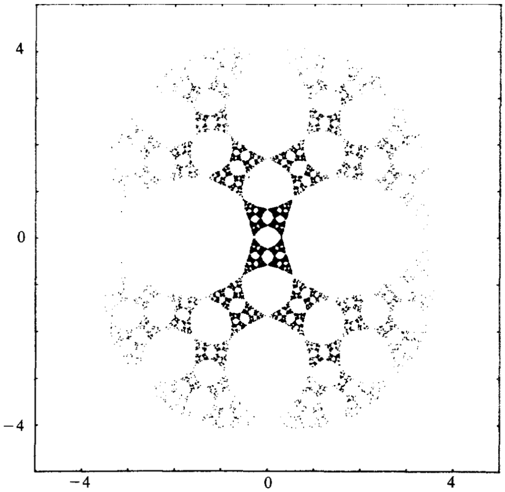

Figure : Here, $q = 2$. No bias in the Monte Carlo procedure.

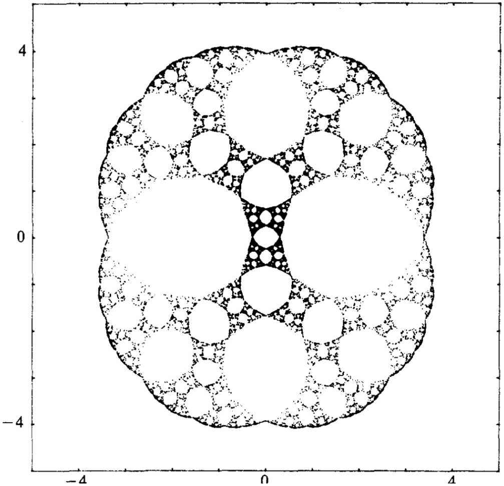

Figure : Here, $q = 2$, as before. Biased Monte Carlo procedure.

results of the numerical simulations (pictures)

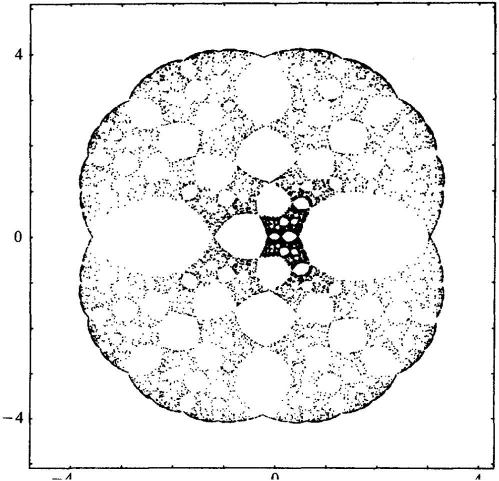

Figure : Here, $q = 1.5$. Biased Monte Carlo procedure.

results of the numerical simulations (pictures)

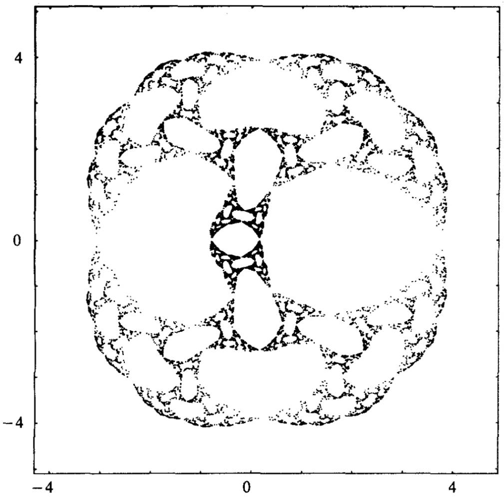

Figure : Here, $q = 2.5$. Biased Monte Carlo procedure.

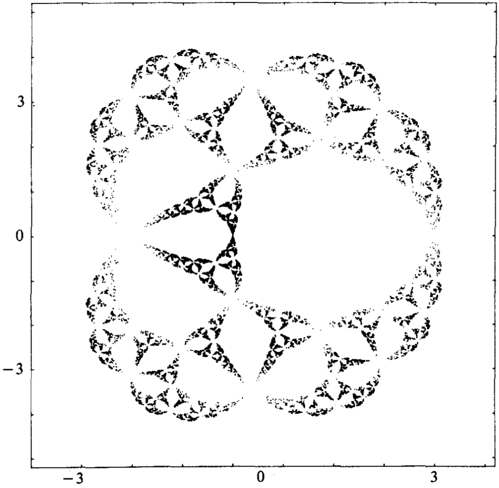

Figure : Here, $q = 3$. Biased Monte Carlo procedure.

results of the numerical simulations (pictures)

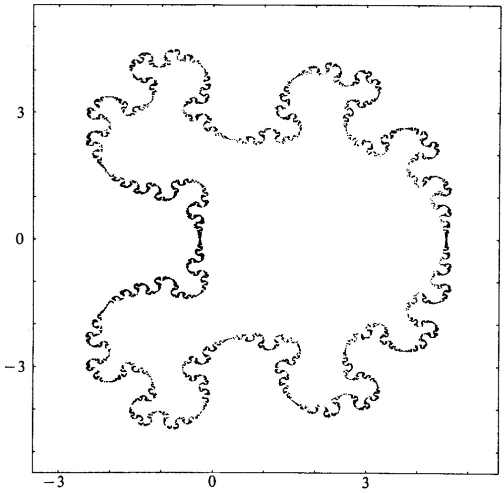

Figure : Here, $q = 4$. Biased Monte Carlo procedure.