The Lamplighter Group as a Group Generated by a 2-state Automaton, and its Spectrum

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Outline of the talk

We consider

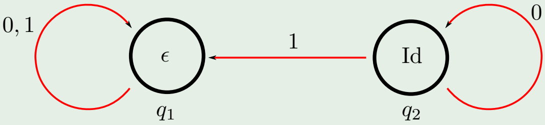
1. automata and automata groups
2. Lamplighter group
3. rooted regular trees and group action on them
4. Schreier graphs
5. Markov operators on Schreier graphs, and its spectra
6. spectrum of the Lamplighter group

Automata

- We consider *finite invertible automata* with the same input and output alphabet $\{0,1\}$.

Example

Denote the following (non-initial) automaton by A:



$$\mathrm{Id}: \begin{matrix} 0 \mapsto 0 \\ 1 \mapsto 1 \end{matrix}, \qquad \epsilon: \begin{matrix} 0 \mapsto 1 \\ 1 \mapsto 0 \end{matrix}$$

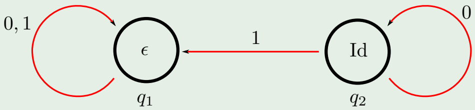
- We call A_{q_1} and A_{q_2} *initial automata*.

Examples of initial Automata

- Initial automata operate on (finite or infinite) sequences over the alphabet $\{0, 1\}$.

Example

Let A be the following (non-initial) automaton:



$$\text{Id} : \begin{array}{c} 0 \mapsto 0 \\ 1 \mapsto 1 \end{array}, \quad \epsilon : \begin{array}{c} 0 \mapsto 1 \\ 1 \mapsto 0 \end{array}$$

Then, for example,

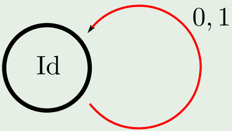
$$A_{q_1} : 00000 \mapsto 11111, \quad 10100 \mapsto 01011$$
$$A_{q_2} : 010 \mapsto 011, \quad 01000000\cdots \mapsto 01111111\cdots$$

Examples of Automata

- Automata producing the identity map on the set of strings are called *trivial* automata.

Example (trivial automaton)

Denote the following automaton by A:



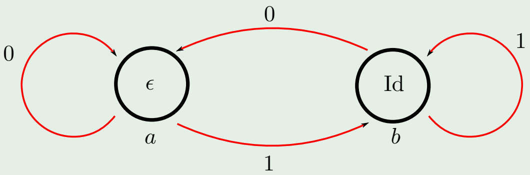
Then, for example,

$$A : 111111 \mapsto 111111, \quad 0100000\cdots \mapsto 0100000\cdots .$$

Examples of Automata

Example

- The following automaton A generates the *Lamplighter group*.



Note that

$$A_a(0x_1x_2\cdots) = 1A_a(x_1x_2\cdots), \quad A_a(1x_1x_2\cdots) = 0A_b(x_1x_2\cdots)$$
$$A_b(0x_1x_2\cdots) = 0A_a(x_1x_2\cdots), \quad A_b(1x_1x_2\cdots) = 1A_b(x_1x_2\cdots)$$

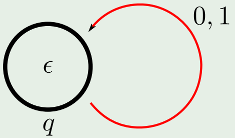
Composition of automata

- For any two initial automata A_q and B_s, joining the output of A_q with the input of B_s one gets a map which corresponds to an initial automaton $C_{q,s}$.
- We call $C_{q,s}$ the *composition* of A_q and A_s, and denote it by $A_q \star B_s$.

Composition of automata

Example

Denote the following automaton by A.



Then, for example, $10000 \mapsto 01111 \mapsto 10000$. Therefore,

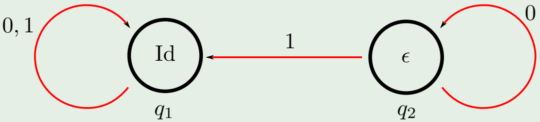
$$A_q \star A_q = \mathrm{Id}.$$

Equivalence of initial automata

- Two initial automata are called *equivalent* if they determine the same map on the set of strings.

Example

Denote the following automata by A:

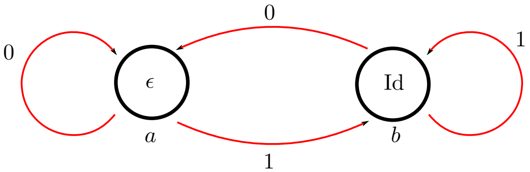


Then A_{q_1} is equivalent to the trivial automaton.

- The classes of equivalence of initial automata over the alphabet $\{0, 1\}$ constitute a group which is called the *finite automata group*.

Let A be a non-initial automaton, and let $Q = \{q_1, q_2, \cdots, q_\ell\}$ be the set of states of A. Then, the group $G(A) = \langle A_{q_1}, \cdots, A_{q_\ell} \rangle$ is called the *group generated by A*.

Reminder: the following automaton generates the Lamplighter group:

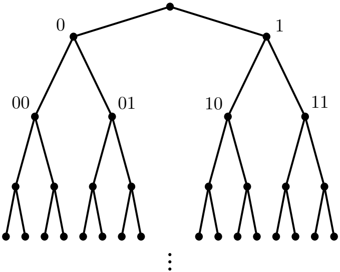


Remarks

- The study of automata groups led to the solution of a number of important problems in group theory (Burnside problem, Milnor problem, Atiyah problem, Day problem, Gromov problem, etc).
- The original definition of the Lamplighter group is $(\oplus_{\mathbb{Z}}\mathbb{Z}/2\mathbb{Z}) \rtimes \mathbb{Z}$. It can also be written as $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$.
- In general, it is not an easy task to recognize the group generated by a given automaton.
- A full classification of all automaton groups defined by automata with given number of states m and size of the alphabet k has been achieved only for $m = k = 2$. For the next smallest case $m = 3$ and $k = 2$ only a partial classification was obtained.

Rooted 2-regular tree

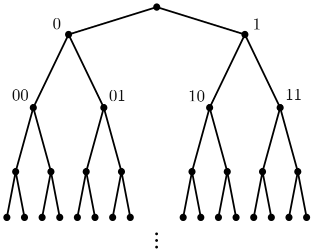
Consider the following *rooted 2-regular tree*:



- Denote by X the set of infinite rays joining the root vertex to infinity.
- Write the set of vertices of the n-th level as X_n.

Then, any finite automata group G acts on X and X_n in the natural way.

Action of G on X and X_n



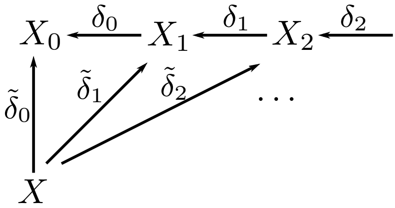
- Let $\delta_n : X_{n+1} \to X_n$ be the map given by deleting the last letter in each word. Define $\tilde{\delta}_n : X \to X_n$ in a similar way.

Then, δ_n and $\tilde{\delta}_n$ are surjective *G-equivariant map*, that is,

$$g\delta_n(x) = \delta_n(gx) \quad (g \in G, x \in X_n)$$
$$g\tilde{\delta}_n(x) = \delta_n(gx) \quad (g \in G, x \in X)$$

Action of G on X and X_n

The following diagram commutes, and δ_n and $\tilde{\delta}_n$ are surjective G-equivariant map.



Schreier graphs

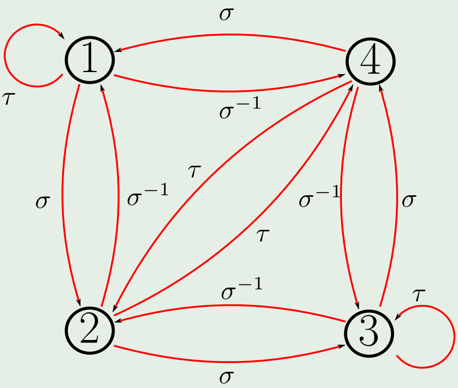
Let G be a group generated by a finite symmetric set S (S being symmetric means $S = S^{-1}$) which acts on a set Y. The *Shreier graph* Y can be defined by:

- the vertex set of the Schreier graph is Y
- the edge set is $S \times Y$
- for $s \in S$ and $y \in Y$, the edge (s, y) connects y to sy.

Example of a Schreier graph

Example (Schreier graph $Y = \{1, 2, 3, 4\}$)

Let G be the dihedral group D_8, and let Y be the set $\{1, 2, 3, 4\}$. Let $S = \langle \sigma, \sigma^{-1}, \tau \rangle$, where $\sigma = (1234)$ and $\tau = (24)$.



Markov operator and adjacency operator

The *Markov operator* on a Schreier graph Y is the operator

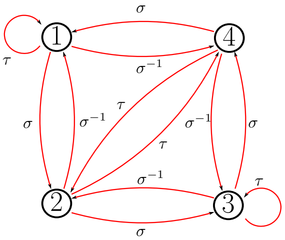
$$M : \ell^2(Y) \to \ell^2(Y)$$
$$(Mf)(y) = \frac{1}{|S|} \sum_{s \in S} f(sy).$$

Similarly, the *adjacency operator* A is defined by

$$(Af)(y) = \sum_{s \in S} f(sy).$$

Example of an adjacency operator

- Let $G = D_8$, $Y = \{1, 2, 3, 4\}$ and $S = \langle \sigma, \sigma^{-1}, \tau \rangle$.



Note that $\ell^2(Y)$ is isomorphic to $\mathbb{R}^4$. Therefore,

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

- Denote by $\mathrm{Sp}(T)$ the spectrum of a self-adjoint operator T.

Let G be a group generated by a finite symmetric set S, and assume that G acts on sets Y and $\widetilde{Y}$, and $\delta : \widetilde{Y} \to Y$ be a surjective G-equivalent map.

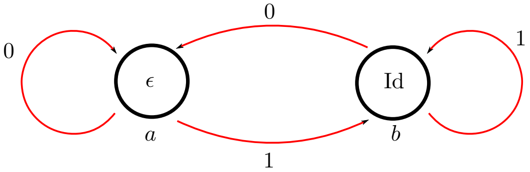
Then we have

$$\mathrm{Sp}(M_Y) \subseteq \mathrm{Sp}(M_{\widetilde{Y}}),$$

where M_Y and $M_{\widetilde{Y}}$ are the Markov operators on the Schreier graphs Y and $\widetilde{Y}$, respectively.

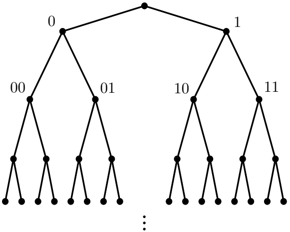
The Lamplighter group G and its symmetric set S

Reminder: the following automaton generates the Lamplighter group G:



Denote G_a, G_b simply by a, b, respectively. Let $S = \{a, b, a^{-1}, b^{-1}\}$.

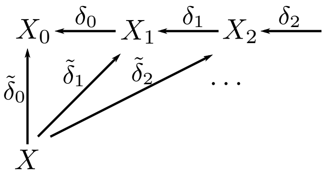
Action of G on X_n and G



- Recall that G acts on the rooted regular tree X, and also on the set of vertices of the n-th level X_n. It is known that action on X_n is transitive.
- Also, G acts on itself by left multiplication (*left regular representation*).
- Therefore, one can consider the spectrum of the Schreier graph G (which is precisely what we want to do! :)).

Nested sequence of spectra

Reminder: the following diagram commutes, and δ_n and $\tilde{\delta}_n$ are surjective G-equivariant map.



Therefore, we have

$$Sp(X_0) \subseteq Sp(X_1) \subseteq Sp(X_2) \subseteq \cdots$$

How to compute $Sp(G)$?

The following holds:

Theorem (L. Bartholdi & R. Grigorchuk, '00)

We have

$$\overline{\bigcup_{n\geq 0} Sp(X_n)} \subseteq Sp(G).$$

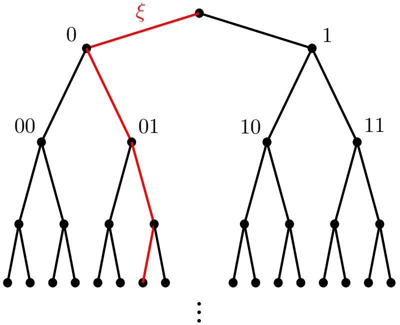
- We want to show that $Sp(G) = [-1, 1]$.
- By the above theorem, it is enough to show that

$$\overline{\bigcup_{n\geq 0} Sp(X_n)} = [-1, 1].$$

A little bit more detail...

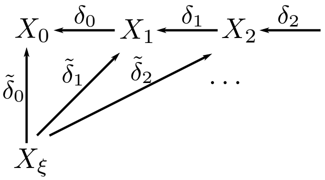
- Since X is uncountable, the action of G on X cannot be transitive.

Let $\xi \in X$ be an infinite ray.



Denote the orbit of ξ by X_ξ.

- The following diagram commutes:



Therefore, we have

$$Sp(X_0) \subseteq Sp(X_1) \subseteq Sp(X_2) \subseteq \cdots \subseteq Sp(X_\xi),$$

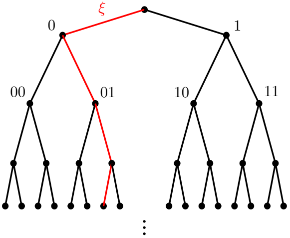
so

$$\overline{\bigcup_{n\geq 0} Sp(X_n)} \subseteq Sp(X_\xi).$$

In fact they coincide. By $Sp(X_\xi) = Sp(G)$, we have

$$\overline{\bigcup_{n\geq 0} Sp(X_n)} \subseteq Sp(G).$$

A tiny little bit more detail...

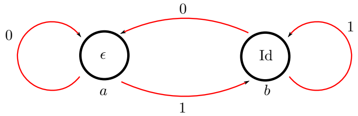


- Denote the stabilizer of of the ray ξ by P (*parabolic subgroup*).
- The parabolic subgroup P is cyclic, or trivial.
- Since G acts on X_ξ transitively, this action is equivalent to the action of G on G/P.
- Therefore, we have

$$Sp(X_0) \subseteq Sp(X_1) \subseteq Sp(X_2) \subseteq \cdots \subseteq Sp(X_\xi) = Sp(G/P).$$

The Lamplighter group

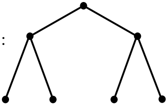
Reminder: the following automaton generates the Lamplighter group.



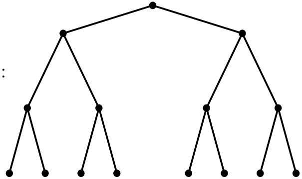
Recall that

$$a(0x_1x_2\cdots) = 1a(x_1x_2\cdots), \quad a(1x_1x_2\cdots) = 0b(x_1x_2\cdots)$$
$$b(0x_1x_2\cdots) = 0a(x_1x_2\cdots), \quad b(1x_1x_2\cdots) = 1b(x_1x_2\cdots)$$

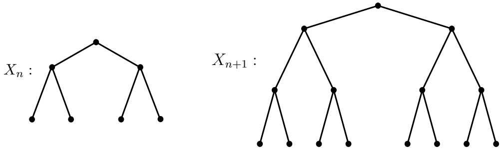
X_n :



X_{n+1} :



Operator recursion



$$a(0x_1x_2\cdots) = 1a(x_1x_2\cdots), \quad a(1x_1x_2\cdots) = 0b(x_1x_2\cdots)$$
$$b(0x_1x_2\cdots) = 0a(x_1x_2\cdots), \quad b(1x_1x_2\cdots) = 1b(x_1x_2\cdots)$$

Recall that G acts on X_n. Let a_n, b_n be the matrices corresponding to the action of a and b on X_n, respectively. Then we have

$$a_n = \begin{pmatrix} 0 & a_{n-1} \\ b_{n-1} & 0 \end{pmatrix}, \quad b_n = \begin{pmatrix} a_{n-1} & 0 \\ 0 & b_{n-1} \end{pmatrix}.$$

Spectrum of X_n

- Note that the spectrum of X_n is the set of eigenvalues of the matrix

$$a_n + b_n + a_n^{-1} + b_n^{-1}.$$

Theorem (R. Grigorchuk & A. Żuk, '01)

We have

$$Sp(a_n + b_n + a_n^{-1} + b_n^{-1}) = \left\{ 4 \cup 4\cos\left(\frac{p}{q}\pi\right) : 1 \leq p < q \leq n+1 \right\}.$$

Computation of the spectra (1)

Let us introduce the following matrix:

$$S_{n+1} = \begin{pmatrix} 0 & Id_{2^n} \\ Id_{2^n} & 0 \end{pmatrix}.$$

Define

$$\Phi_n(x_1, x_2) = \det(a_n + b_n + a_n^{-1} + b_n^{-1} - x_1 \mathrm{Id}_{2^n} - x_2 S_n),$$

where x_1 and x_2 are complex parameters.

- The same method also works for the first Grigorchuk group, the Hanoi Towers group, the tangled odometers group, etc etc...

Computation of the spectra (2)

We then obtain a recursive expression of the form

$$\Phi_n(x_1, \cdots, x_d) = P_n(x_1, \cdots, x_d)\Phi_{n-1}(F(x_1, \cdots, x_d)),$$

where P_n is a polynomial function and $F : \mathbb{R}^d \to \mathbb{R}^d$ is a rational function.

In our case,

$$P_n(x_1, x_2) = (x_1 - x_2)^{2^n} \text{ and } F(x_1, x_2) = \left(x_1 + x_2 + \frac{2}{x_2 - x_1}, -\frac{2}{x_2 - x_1} \right).$$

Therefore, we have

$$\Phi_{n+1}(x_1, x_2) = (x_2 - x_1)^{2^n} \Phi_n \left(x_1 + x_2 + \frac{2}{x_2 - x_1}, -\frac{2}{x_2 - x_1} \right).$$

Computation of the spectra (3)

So,

$$\Phi_n(x_1, x_2) = \det(a_n + b_n + a_n^{-1} + b_n^{-1} - x_1 \mathrm{Id}_{2^n} - x_2 S_n)$$

satisfies

$$\Phi_n(x_1, x_2) = P_n(x_1, x_2)\Phi_{n-1}(F(x_1, x_2)),$$

where P_n is a polynomial function and $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a rational function.

- If the point (x_1', x_2') is in the zero set of $\Phi_{n-1}(x_1, x_2)$ then any point in $F^{-1}(x_1', x_2')$ is in the zero set of $\Phi_n(x_1, x_2)$.
- Therefore, describing the joint spectrum leads us to consider iterations of the rational map F.

Computation of the spectra (4)

- We then find semi-conjugacy from the 2-dimensional rational function F to a polynomial function f in a single variable, that is,

$$\psi(F(x_1, x_2)) = f(\psi(x_1, x_2)).$$

- Then, since we have

$$\psi(F^m(x_1, x_2)) = f^m(\psi(x_1, x_2)),$$

  the iterations of F are related to the iterations of f and then the desired spectrum is described through the iterations of f.

In our case,

$$\psi(x_1, x_2) = x_1 + x_2, \quad \text{and} \quad f \text{ is the identity.}$$

Computation of the spectra (5)

Therefore, denoting

$$x_1' = x_1 + x_2 + \frac{2}{x_2 - x_1}, \quad x_2' = -\frac{2}{x_2 - x_1},$$

we have

$$\Phi_{n+1}(x_1, x_2) = (x_2 - x_1)^{2^n} \Phi_n \left(x_1', x_2'\right) \quad \text{and} \quad x_1' + x_2' = x_1 + x_2.$$

- Since

$$x_2' - x_1' = -(x_1 + x_2) - \frac{4}{x_2 - x_1},$$

one only needs to consider the iteration of the map

$$g : x \mapsto -(x_1 + x_2) - \frac{4}{x}.$$

Computation of the spectra (6)

- Denote $(\underbrace{g \circ g \circ \cdots \circ g}_{n})(x_2 - x_1)$ by P_k / Q_k.

Lemma (R. Grigorchuk & A. Żuk, '01)

We have

$$\Phi_n(x_1, x_2) = (4 - x_1 - x_2) \prod_{k=1}^{n} \left(\frac{P_k(x_1, x_2)}{Q_k(x_1, x_2)} \right)^{2^{n-k}}$$

Write $x_1 = 4\cos z$ for $z \in [0, \pi]$.

Then we get

$$\det(a_n + b_n + a_n^{-1} + b_n^{-1} - x_1 \mathrm{Id}_{2^n}) = \Phi_n(x_1, 0)$$
$$= (4 - 4\cos z)\left(\frac{1}{\sin z}\right)^{2^{n-1}} 2^n \prod_{k=2}^{n} (\sin(zk))^{2^{n-k}} \sin(z(n+1)).$$

This proves that

$$Sp(a_n + b_n + a_n^{-1} + b_n^{-1}) = \left\{4 \cup 4\cos\left(\frac{p}{q}\pi\right) : 1 \le p < q \le n+1\right\}.$$

Thank you! :)