Lamplighter groups from affine automorphisms of rooted trees

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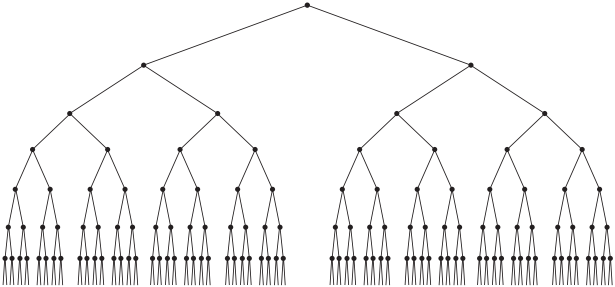
Automata – transducers

$V(T) = X^*, \qquad X = \{0, \ldots, d-1\}$ – alphabet

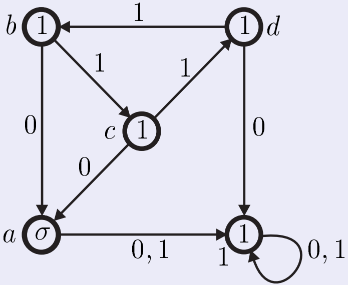


$$G < \mathrm{Aut}\, T$$

Action on T given by finite initial automaton

Definition (By Example)

$S_2 = \{\varepsilon, \sigma\}$ acts on $X = \{0, 1\}$.



$\mathcal{A}$ — noninitial automaton,
$\mathcal{A}_q$ — initial automaton, $q \in \{a, b, id\}$.

$\mathcal{A}_q$ acts on X^* (and on T)

Definition of automaton group

Given an automaton A every state q defines an automorphism A_q of X^*

Definition

The automaton group generated by automaton A is a group

$$G(A) = \langle A_q \mid q \text{ is a state of A} \rangle < \mathrm{Aut}\, X^*$$

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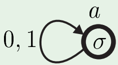
Example



$a(w) = \overline{w}$. Thus $a^2 = 1$ and $G(A) \simeq C_2$.

Motivation

- Burnside problem on infinite periodic groups
(Aleshin group, Grigorchuk group, Gupta-Sidki group,...)
- Milnor problem on groups of intermediate growth
(Grigorchuk group, Gupta-Sidki group,...)
- Day problem on amenability
(Grigorchuk group, Gupta-Sidki group,...)
- Atiyah conjecture on L^2 Betti numbers
(Lamplighter group)
- Connection to holomorphic dynamics via Iterated Monodromy Groups
- Connection to combinatorics via Hanoi Towers groups

History

Notation: (m, n)-automata — m-state automata over n-letter alphabet

- (2000 Grigorchuk, Nekrashevych, Sushchansky) Groups generated by (2,2)-automata were classified: $\{1\}$, $\mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}$, D_∞ and the Lamplighter group.
- (2001 Reznykov, Sushchansky) Semigroups generated by (2,2)-automata were classified: additionally 29 nonisomorphic semigroups, including a semigroup of intermediate growth
- (2007 Bondarenko,Grigorchuk, Kravchenko, Muntyan, Nekrashevych, S., Šunić) Groups generated by (3,2)-automata were studied: up to 115 non-isomorphic groups generated by 194 "non-symmetric" automata. No Burnside groups.
- (2014 Caponi, S.) There are 7471 "non-symmetric" (4,2)-automata. No Burnside groups (4 groups pending).

Lamplighter Group

Definition

The lamplighter group is

$$\begin{aligned} L_2 &= \mathbb{Z}_2 \wr \mathbb{Z} \\ &\cong (\cdots \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \cdots) \rtimes \mathbb{Z}, \\ &\cong \langle a, b \mid a^2 = [a, a^{b^i}] = 1, i \geq 1 \rangle \end{aligned}$$

where $\mathbb{Z}$ acts on $B = \oplus_{i \in \mathbb{Z}} \mathbb{Z}_2$ by "shifting the index".

We have $\langle a \rangle \cong \mathbb{Z}_2$ and

$$\langle a^{b^i}, i \in \mathbb{Z} \rangle \cong \cdots \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \cdots$$

and we have $\mathbb{Z}$-worth of commuting elements in L_2.

How do we get many commuting elements in $\mathrm{Aut}\, T$?

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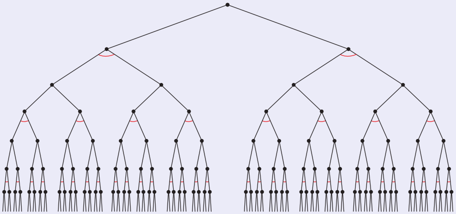
Definition

- An automorphism of X^* is called spherically homogeneous if it acts on the k-th letter of each word by a permutation depending only on k.



- $\mathrm{SHAut}(X^*) \cong \mathrm{Sym}(X)^\infty$ – spherically homogeneous automorphisms
- $\mathbb{Z}_d^\infty < \mathrm{SHAut}(X^*)$

Observation

Known instances of lamplighters have $\oplus_{i \in \mathbb{Z}} \mathbb{Z}_2$ inside $\mathrm{SHAut}(X^*)$.

So we need to understand what normalizes $\mathbb{Z}_d^\infty < \mathrm{SHAut}(X^*)$.

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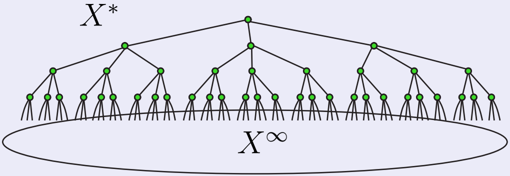
Automorphisms of X^* coming from boundary actions

Definition

The boundary X^∞ of the tree X^* consists of all infinite words over X (that correspond to all infinite paths from the root of X^*)



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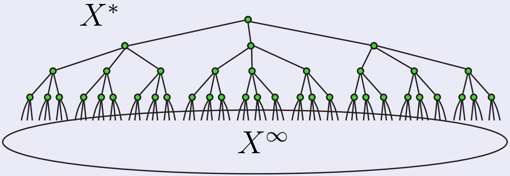

- Each automorphism of X^* induces a transformation of X^∞
- Some transformations of X^∞ induce automorphisms of X^*

Boundary as the ring of power series $\mathbb{Z}_d[[t]]$

To define transformations of X^∞ we can use different structures.

For $X = \mathbb{Z}_d$ the elements of X^∞ become power series in $\mathbb{Z}_d[[t]]$.

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For $X = \mathbb{Z}_d$ the elements of X^∞ become power series in $\mathbb{Z}_d[[t]]$.

$$X^\infty \ni a_0 a_1 a_2 \ldots \longleftrightarrow a_0 + a_1 t + a_2 t^2 + \cdots \in \mathbb{Z}_d[[t]]$$

Example

For a fixed power series $g(t) \in \mathbb{Z}_d[[t]]$ the transformation of $\mathbb{Z}_d[[t]]$

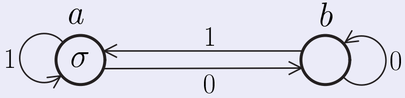
$$f(t) \mapsto f(t) + g(t)$$

induces a element of $\mathrm{SHAut}(X^*)$.

Starting point

Theorem (Grigorchuk, Nekrashevych, Sushchansky 2000)

The group generated by automaton



isomorphic to the lamplighter group is induced by transformations

$$\begin{aligned} f(t) &\mapsto (1+t)f(t)+1 \\ f(t) &\mapsto (1+t)f(t) \end{aligned}$$

of $\mathbb{Z}_2[[t]]$.

The language of power series turned out to be very fruitful:

- Silva and Steinberg (2005) realized $G \wr \mathbb{Z}$ as automaton group for each finite abelian G
- Similar ideas: Bartholdi and Šunić (2006) produced a different representation of $\mathbb{Z}_d^k \wr \mathbb{Z}$
- Gives rise to families of (bi)reversible automata generating $\mathbb{Z}_d^k \wr \mathbb{Z}$ (Bondarenko, S. – in preparation)

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But it still is not rich enough to describe all lamplighters.

For $X = \mathbb{Z}_d$, $X^\infty = \mathbb{Z}_d^\infty$ has a natural structure of right $\mathbb{CFM}(\mathbb{Z}_d)$-module (we can multiply "vectors" by column finite matrices on right).

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Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \dots \\ a_{21} & a_{22} & a_{23} & a_{24} & \dots \\ a_{31} & a_{32} & a_{33} & a_{34} & \dots \\ a_{41} & a_{42} & a_{43} & a_{44} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

be a **column finite matrix** over $\mathbb{Z}_d$ and $\mathbf{b} \in \mathbb{Z}_d^\infty$ be a **(row) vector**.

Definition

A transformation

$$\begin{array}{rccc} \pi_{A,\mathbf{b}}: & \mathbb{Z}_d^\infty & \longrightarrow & \mathbb{Z}_d^\infty \\ & \mathbf{x} & \mapsto & \mathbf{b} + \mathbf{x}A \end{array}$$

is called an affine transformation of $\mathbb{Z}_d^\infty$.

Proposition

Let

$$A = \begin{bmatrix} a_{11} & * & * & * & \dots \\ 0 & a_{22} & * & * & \dots \\ 0 & 0 & a_{33} & * & \dots \\ 0 & 0 & 0 & a_{44} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

be an **upper unitriangular** (a_{ii} **is a unit in** $\mathbb{Z}_d$) **matrix** *and* $\mathbf{b} \in \mathbb{Z}_d^\infty$ *be a* **vector**.

Then the affine transformation $\pi_{A,\mathbf{b}}$ *of* $\mathbb{Z}_d^\infty$ *induces an automorphism of* X^* *(also denoted by* $\pi_{A,\mathbf{b}}$*).*

Definition

- Automorphism $\pi_{A,\mathbf{b}}$ of X^* is called affine.
- $\mathrm{Aff}(X^*)$ – the group of all affine automorphisms of X^*.

Example

The group of shifts $\mathrm{Aff}_I(X^*) = \{\pi_{I,\mathbf{b}} \mid \mathbf{b} \in \mathbb{Z}_d^\infty\} \cong \mathbb{Z}_d^\infty$ is a normal abelian subgroup of $\mathrm{Aff}(X^*)$ consisting of spherically homogeneous automorphisms.

Example

For a power series $g(t) = a_0 + a_1 t + a_2 t^2 + \cdots$ the transformation of $\mathbb{Z}_d[[t]]$ defined by

$$f(t) \mapsto g(t)f(t)$$

is equal to $\pi_{A_g,\mathbf{0}}$, where

$$A_g = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \dots \\ 0 & a_0 & a_1 & a_2 & \dots \\ 0 & 0 & a_0 & a_1 & \dots \\ 0 & 0 & 0 & a_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Theorem (S., Sidki, 2015)

$$N_{\mathrm{Aut}(X^*)}(\mathrm{Aff}_I(X^*)) = \mathrm{Aff}(X^*)$$

Corollary

In the case $|X| = 2$,

$$N_{\mathrm{Aut}(X^*)}(\mathrm{SHAut}(X^*)) = \mathrm{Aff}(X^*)$$

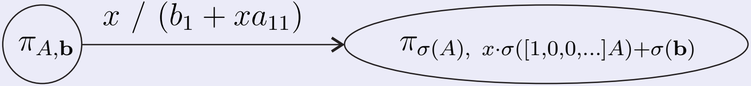
Moreover

Theorem

Each faithful automaton representations of $L_2 \cong \mathbb{Z}_2 \wr \mathbb{Z}$ *on the binary tree is conjugate to the one with the base group inside* $\mathrm{SHAut}(X^*)$.

Theorem (S., Sidki)

The group $\mathrm{Aff}(X^*)$ *is generated by an automaton with transitions:*

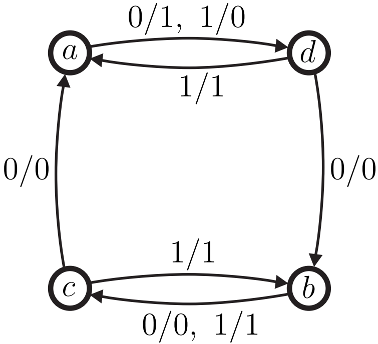


Corollary

An automorphism $\pi_{A,\mathbf{b}}$ *of* X^* *is defined by finite automaton* $\Leftrightarrow$ *matrix* A*, its rows, and vector* $\mathbf{b}$ *are eventually periodic.*

[Note: Similar result was obtained also by Oliynyk and Sushchansky]

Principal Example



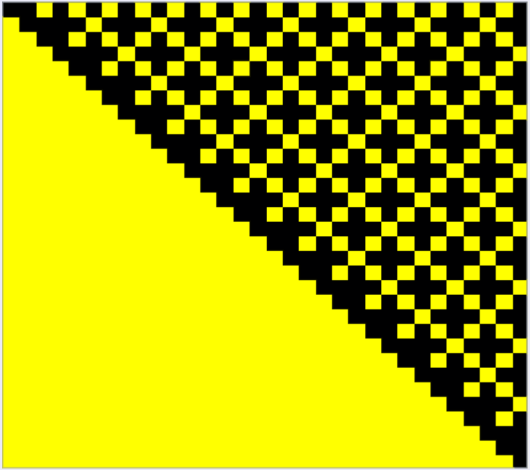
Theorem (S., Sidki)

$$G \cong \left(\mathbb{Z}_2^2 \wr \mathbb{Z}\right) \rtimes \mathbb{Z}_2 = \left(\langle x, y\rangle \wr \langle t\rangle\right) \rtimes \langle a\rangle,$$

where the action of a on x, y, t is defined as follows: $x^a = x$, $y^a = y^{t^{-1}}$, $t^a = t^{-1}$.

Proposition

The automorphism $t = ac$ *normalizes* $\mathrm{SHAut}(X^*)$ *and is equal to* $\pi_{A,\mathbf{b}}$ *for the matrix* A



and $\mathbf{b} = [(1,0,0,1,1,1,0,0)^\infty]$.

Idea of the proof

Conjugates of any $z \in \mathrm{SHAut}(X^*)$ by powers of t are also in $\mathrm{SHAut}(X^*)$, so we can define

$$z^{t^{i_1}+t^{i_2}+\cdots+t^{i_n}} := z^{t^{i_1}} z^{t^{i_2}} \cdots z^{t^{i_n}}$$

Proposition

Elements $x := ab$ and $y = cd$ are in $\mathrm{SHAut}(X^*)$, so $x^{p(t)}$ and $y^{p(t)}$ are defined for each Laurent polynomial $p(t) \in \mathbb{Z}_d[t, t^{-1}]$.

Proposition

$\langle x, y, t\rangle \cong \mathbb{Z}_2^2 \wr \mathbb{Z}$.

To prove this we need to show $x^{p(t)} y^{q(t)}$ is not trivial for all $p(t), q(t) \in \mathbb{Z}_d[[t]]$.

Define

$$\phi_n(t) := 1 + t + t^2 + \cdots + t^{n-1}.$$

For each polynomial $p(t) = \sum_{i=0}^{k} a_i t^i \in \mathbb{Z}_2[t]$ define also

$$\psi_p(t) = \sum_{i=1}^{k} a_i \phi_i(t).$$

Lemma

For all pairs of polynomials $p(t), q(t) \in \mathbb{Z}_2[t]$

- *the state of $x^{p(t)} y^{q(t)}$ at each vertex of the first level is $x^{\psi_p(t^{-1}) + q(t^{-1})} y^{p(t^{-1})}$.*
- *the state of $x^{p(t^{-1})} y^{q(t^{-1})}$ at each vertex of the first level is $x^{t\psi_p(t) + q(t)} y^{p(t)}$.*

This defines a dynamical system on $(\mathbb{Z}_2[t])^2$ whose analysis yields the result.

Thank You!