An Extension of Brolin's Theorem & Relevant Tools

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Brolin's Theorem

Theorem (Brolin, 1965)

If $f(z) = z^\alpha + \dots$ is a polynomial of degree $\alpha \geq 2$, then there is an exceptional set $\mathcal{E}$ with $\#\mathcal{E} \leq 1$ such that if $a \in \mathbb{C} \setminus \mathcal{E}$, then

$$\frac{1}{\alpha^n} \sum_{f^n(z)=a} \delta_z \to \mu \text{ as } n \to \infty,$$

where μ is harmonic measure on the filled Julia set of f.

- The limit is independent of a.
- $\mathcal{E} = \emptyset$ *or*, if f is affinely conjugate to $z \mapsto z^\alpha$, $\mathcal{E} = \{0\}$.
- This result is specific for polynomials in $\mathbb{C}$.

Q. Can Brolin's Theorem extend to other types of maps or spaces?

Yes, with additional assumptions, to:

- rational maps in $\mathbb{P}^1_{\mathbb{C}}$ by Lyubich & Freire-Lopez-Mañé [1983]
- holomorphic maps in $\mathbb{P}^2_{\mathbb{C}}$ by Favre-Jonsson [2001]

Extending Brolin's Theorem

Theorem (Brolin, 1965)

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Theorem (Favre-Jonsson, 2001)

Let $f = [P : Q : R] : \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$, where P, Q, R are homogeneous polynomials of degree $\alpha \geq 2$ and let $\mathcal{E}$ be a special set.
If S is a positive closed $(1,1)$ current on $\mathbb{P}^2$ with mass 1 that behaves nicely on $\mathcal{E}$, then

$$\frac{1}{\alpha^n} f^{n*} S \to T \text{ as } n \to \infty,$$

where T is the Green current of f.

Overview

1. What is a current?

   Focus on positive closed $(1,1)$*-currents on* $\mathbb{P}^2_{\mathbb{C}}$.

2. Precise statement of extension of Brolin's Theorem to $\mathbb{P}^2_{\mathbb{C}}$

   Focus on Theorem A of "Brolin's Theorem for Curves in Two Complex Dimensions" by Favre-Jonsson from 2001.

3. Some Ingredients in the proof

   Including Hartog's Lemma.

What is a p-current?

Let M be a smooth ($\mathbb{R}$) manifold of dimension m.

Let $D^p(M)$ be the space of smooth p-forms with compact support on M.

Definition

*S is a p-**current** on M if it is a (continuous) linear functional:*

$$S : D^p(M) \to \mathbb{R}.$$

Note: The action of S on $\nu \in D^p(M)$ is often denoted $\langle S, \nu \rangle$.

Let $D'_p(M)$ be the space of p-currents on M.

What is a p-current?

Example 1. p-dimensional submanifolds

Let M be a smooth manifold of dimension m.

Let $Z \subset M$ be a closed oriented submanifold of dim p and class C^1.

Geometrically, a p-current can represent integration over Z.

The current of integration over Z, $[Z]$, is a p-current defined by:

$$\langle [Z], u \rangle = \int_Z u, \text{ for } u \in D^p(M).$$

A p-current $S \in D'_p(M)$ can be expressed as a $(m-p)$-form:

$$S = \sum_{|I|=m-p} S_I dx^I, \text{ where}$$

$I = (i_1, \ldots, i_{m-p})$, $dx^I = dx_{i_1} \wedge \ldots \wedge dx_{i_{m-p}}$, and $i_1 < \ldots < i_{m-p}$.

What is a p-current?

Example 2. $(m-p)$-form

A form $\alpha \in D^{m-p}(M)$ with coefficients in L^1_{loc} defines a p-current:

$$\langle \alpha, \phi \rangle := \int_M \alpha \wedge \phi \text{ for any } \phi \in D^p(M)$$

since $\alpha \wedge \phi \in D^m(M)$ is a volume form.

Consequently, a p-current S acts on p-forms and can act as an $(m-p)$-form.

We say that S has dimension p and degree $m-p$.

Extending from $\mathbb{R}$ to $\mathbb{C}$

Each complex variable, z_j, has 2 corresponding real variables and so we have 2 corresponding differentials. In particular, dz_j and $d\bar{z}_j$.

Note that dz_j is a $(1,0)$-form and $d\bar{z}_j$ is a $(0,1)$-form.

More generally, $\alpha = \sum_{|I|=p,|J|=q} \alpha_{IJ} dz_I \wedge d\bar{z}_J$ is a (p,q)-form and we say that $\alpha \in D^{p,q}$.

Notation: $d = \partial + \bar{\partial}$ and $d^c = \frac{i}{2\pi}(\bar{\partial} - \partial)$, where

$$\partial\alpha = \sum_{k,|I|=p,|J|=q} \frac{\partial \alpha_{IJ}}{dz_k} dz_k \wedge dz_I \wedge d\bar{z}_j \text{ and}$$

$$\bar{\partial}\alpha = \sum_{k,|I|=p,|J|=q} \frac{\partial \alpha_{IJ}}{d\bar{z_k}} d\bar{z}_k \wedge dz_I \wedge d\bar{z}_j.$$

It follows that $dd^c = \frac{i}{\pi}\partial\bar{\partial}$.

What is a $(1,1)$-current?

For simplicity, we now focus on $M = \mathbb{P}^2_{\mathbb{C}}$.

Let z_1 and z_2 be local coordinates on $\mathbb{P}^2_{\mathbb{C}}$.

Let $D^{1,1}(\mathbb{P}^2_{\mathbb{C}})$ be the space of smooth compactly supported $(1,1)$-forms. Any $\nu \in D^{1,1}(\mathbb{P}^2_{\mathbb{C}})$ can be expressed as:

$$\nu = \sum_{1 \leq\ j,k\ \leq 2} a_{jk} dz_j \wedge d\overline{z}_k,$$

Definition (For $\mathbb{P}^2_{\mathbb{C}}$)

*A $(1,1)$-**current** S is a linear functional on $D^{1,1}(\mathbb{P}^2_{\mathbb{C}})$ and can be represented as a $(1,1)$-form with distributional coefficients.*

Closed positive $(1,1)$-currents and why they are special.

Definition

Let S be a $(1,1)$-current and express it as $S = i\sum S_{jk} dz_j \wedge d\bar{z}_k$.
*S is **positive** if the distribution $\sum S_{jk}\zeta_j\bar{\zeta}_k \geq 0$ for all $\zeta \in \mathbb{C}^2$.*

Definition

*A $(1,1)$-current S is **closed** if $dS = 0$ (Recall $dS = (\partial + \bar{\partial})S$).*

Why are closed positive $(1,1)$-currents special?

Proposition (A.4.1, Sibony – some of the proposition)

1. *Every positive $(1,1)$-current is representable by integration.*
   (The distributional coefficients are measurable)
2. *If S is a closed positive $(1,1)$-current, then $\forall z_0 \in M$, $\exists$ an open neighborhood $U \subset M$ of z_0 and a plurisubharmonic function u on U such that $S = dd^c u$ in U.*
   *(Note: u is called a **potential** of S and $dd^c = \frac{i}{\pi}\partial\bar{\partial}$)*

Let S be a positive closed $(1,1)$-current on $\mathbb{P}^2_{\mathbb{C}}$ and ω the standard Kahler form on $\mathbb{P}^2_{\mathbb{C}}$ corresponding to the Fubini-Study metric.

Definition

S has ***unit mass*** *if* $1 = ||S|| = \int_{\mathbb{P}^2_{\mathbb{C}}} S \wedge \omega$.

Let $f : \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ be holomorphism of algebraic degree $\alpha \geq 2$.

$\Rightarrow f = [P : Q : R]$, P, Q, R homogenenous degree α polynomials.

We are now prepared to revisit FJ's extension of Brolin's Theorem using more precise language.

Precise statement of FJ Theorem

Theorem (Favre-Jonsson, 2001)

Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be holomorphism of algebraic degree $\alpha \geq 2$.

Then $\exists$ a set $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$, where:

$\mathcal{E}_1$ is a totally invariant, algebraic set consisting of ≤ 3 $\mathbb{C}$-lines &
$\mathcal{E}_2$ is a totally invariant (i.e., $f^{-1}(\mathcal{E}_2) = \mathcal{E}_2$), finite set,

and $\mathcal{E}$ has the following property:
If S is a positive closed $(1,1)$ current on $\mathbb{P}^2$ of mass 1 such that

1. *S does not change any irreducible component of $\mathcal{E}_1$;*
2. *S has a bounded local potential at each point of $\mathcal{E}_2$;*

then we have the convergence

$$\star \qquad \frac{1}{\alpha^n} f^{n*} S \to T \text{ as } n \to \infty,$$

where T is the Green current of f.

Part 1 of Proof of FJ Theorem

Let ω be the Fubini-Study Kahler form on $\mathbb{P}^2_{\mathbb{C}}$.

$f^*\omega$ and $\alpha\omega$ are cohomologous positive closed $(1,1)$ currents,so there is a continuous function u such that:

$$f^*\omega = \alpha\omega + dd^c u.$$

Then:

$$\begin{aligned} f^{2*}\omega &= \alpha(f^*\omega) + dd^c(f^*u) \\ &= \alpha^2\omega + dd^c\left(\alpha u + u \circ f\right). \end{aligned}$$

Consequently,

$$f^{n*}\omega = \alpha^n\omega + dd^c(\alpha^{n-1}u + \alpha^{n-2}u \circ f + \ldots + u \circ f^{n-1}) \text{ and}$$

$$\frac{1}{\alpha^n} f^{n*}\omega = \omega + dd^c \sum_{j=1}^{n-1} \alpha^{-j} u \circ f^{j-1} \to \omega + dd^c G := T \text{ as } n \to \infty.$$

T is the **Green's current of** f, G is continuous, and $f^*T = \alpha T$.

Part 1 of Proof of FJ Theorem

On the previous slide, we had:

$$\frac{1}{\alpha^n} f^{n*}\omega \to T \text{ as } n \to \infty,$$

where ω was the Kahler-Study form.

When can we can replace ω with a current and have the same limit?

In particular, we consider positive closed $(1,1)$ currents of mass 1.

In their proof, FJ use that such a current may affect the size of forward iterates of a ball in $\mathbb{P}^2_{\mathbb{C}}$ to determine sufficient conditions on a current to attain the above limit.

Part 1 of Proof of FJ Theorem

Suppose that S is a positive closed $(1,1)$-current for which limit $\star$ fails. S can be written as:

$$S = \omega + dd^c u,$$

where $u \leq 0$ is the sum of a psh function and a smooth function.

Then, $\forall n \geq 0$,

$$\alpha^{-n} f^{n*} S = \alpha^{-n} f^{n*} \omega + \alpha^{-n} dd^c (u \circ f^n).$$

By assumption, $\alpha^{-n} f^{n*} S \not\to T$ and we know that $\alpha^{-n} f^{n*} \omega \to T$.

So $\alpha^{-n} dd^c (u \circ f^n) \not\to 0$. Equivalently, $\alpha^{-n} u \circ f^n \not\to 0$ in L^1_{loc} since:

$$\int_{\mathbb{P}^2_{\mathbb{C}}} \alpha^{-n} dd^c (u \circ f^n) \wedge \phi = \int_{\mathbb{P}^2_{\mathbb{C}}} (\alpha^{-n} u \circ f^n) \wedge dd^c \phi.$$

We want to determine for which S, $v_n := \alpha^{-n} u \circ f^n \not\to 0$.

Part 1 of Proof of FJ Theorem

Recall: We want to determine for which S, $v_n := \alpha^{-n} u \circ f^n \not\to 0$.

$\{v_n\}$ is a sequence of subharmonic functions bounded above by 0.

Hartog's Lemma (In Dynamics of Rational Maps on $\mathbb{P}^k$ by Sibony)

Let $\{v_j\}$ be a sequence of subharmonic functions on a domain Ω. Suppose $\{v_j\}$ is bounded above on every compact subset K of Ω.

If $v_j \not\to -\infty$ on K, then there is a subsequence $\{v_{j_k}\}$ converging on L^1_{loc} to a subharmonic function v. In addition,

$$\limsup_{j\to\infty} \sup_K v_j \leq \sup_K v, \text{ for all compact } K.$$

If $v_n \not\to -\infty$ on a ball $B \subset \mathbb{P}^2_{\mathbb{C}}$, then there is a subsequence $\{v_{n_j}\}$ that converges to subharmonic $v < c$, for constant $c < 0$. Then:

$$B \subset \{v_{n_j} = \alpha^{-n_j} u \circ f^{n_j} < c\} \Rightarrow f^{n_j}(B) \subset \{u < \alpha^{n_j} c < 0\}.$$

The rest of [FJ] is spent showing that if S satisfies the properties relating to $\mathcal{E}$ from the theorem, then we cannot have:

$$f^{n_j}(B) \subset \{u < \alpha^{n_j} c < 0\}.$$

That is done in [FJ] by estimating the volume of $f^{n_j}(B)$ from below (using dynamics) and the volume of $\{u < c\alpha^{n_j}\}$ from above (using pluripotential theory).

Conclusion

Brolin's Theorem, originally for monic polynomials in $\mathbb{C}$, has extensions to rational functions in $\mathbb{P}^1_{\mathbb{C}}$ and to holomorphic functions in $\mathbb{P}^2_{\mathbb{C}}$. We focused on the latter extension.