Application of the Compactness Theorem for Subharmonic Functions to Brolin's Theorem

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Subharmonic Functions

Let Ω be a domain in $\mathbb{C}$.

Definition

$u : \Omega \to [-\infty, \infty)$ is *subharmonic* (SH) if:

1. u is upper semicontinuous:
$$\forall z_0 \in \Omega, \limsup_{z \to z_0} u(z) \leq u(z_0)$$
2. u satisfies the submean value property:
$$\forall z \in \Omega, \forall r > 0 \text{ such that } \mathbb{D}(z, r) \subset \Omega,$$
$$u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) d\theta$$
3. $u \not\equiv -\infty$

Example

$u(z) = \log |z|$

Subharmonic Functions

Let $u(z) = u(x + iy)$ have continuous second-order partial derivatives, so $\Delta u = \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2$ is defined.

Definition

$u(z) = u(x + iy)$ is *harmonic* if $\Delta u = 0$ on Ω.

Theorem

A smooth real-valued function $u(z)$ is subharmonic on Ω if and only if $\Delta u \geq 0$ on Ω.

Green Function

Properties for making new subharmonic functions:

1. The max of two SH functions is SH.

   e.g. $\log_+ |z| := \max\{0, \log |z|\}$ is SH
2. If u is SH and $c \geq 0$, then cu is subharmonic.

   e.g. For fixed $n \in \mathbb{N}$, $\frac{1}{2^n} \log_+ |z|$ is SH.
3. If u is SH and p is holomorphic, then $u \circ p$ is SH.

   e.g. Let $p(z) = z^2 + c$, $c \in \mathbb{C} \setminus \{0\}$. For fixed $n \in \mathbb{N}$,

   $\log_+ |z|/2^n$ is SH and $p^n(z)$ is holomorphic, so

$$G_n(z) = \frac{1}{2^n} \log_+ |p^n(z)| \text{ is SH.}$$

Green Function

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Properties for making new subharmonic functions:

4. The uniform limit of SH functions is SH.

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Properties for making new subharmonic functions:

4. The uniform limit of SH functions is SH.

For the following, we restrict attention to $p(z) = z^2 + c$.

Proposition

$\lim_{n\to\infty} G_n(z)$ *converges uniformly on* $\mathbb{C}$.

Green Function

$G(z) := \lim_{n\to\infty} G_n(z)$ is SH.

Escape Rates and the Filled Julia Set

Definition (Filled Julia Set)

$K_p = \{z \in \mathbb{C} : p^n(z) \text{ remains bounded}\}$

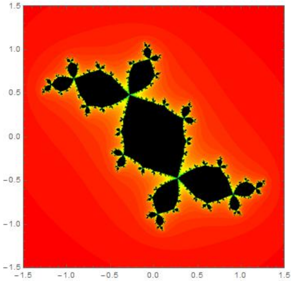


Figure: Escape rate algorithm for $c = -.122 + .745i$.

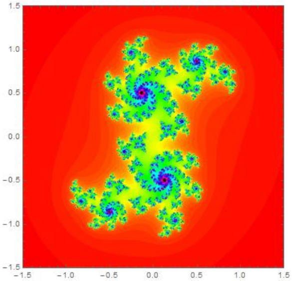


Figure: Escape rate algorithm for $c = 0.365 - 0.37i$.

Distributions

Definition

The Laplacian of a SH function u is a distribution defined by

$$\langle \Delta u, \phi \rangle = \int \Delta\phi \cdot u \, dLeb, \quad \text{for all } \phi \in C_0^\infty(\Omega)$$

Recall:

- $\Delta \log |z| = 2\pi\delta_0$, where δ_0 is the Dirac mass at 0.
- For $p(z) = \prod_{i=1}^{n}(z - z_i)$, we have $\Delta \log |p(z)| = 2\pi \sum_{i=1}^{n} \delta_{z_i}$.

Laplacian of the Green Function

Green Function

$$G(z) := \lim_{n\to\infty} \frac{1}{2^n} \log_+ |p^n(z)|$$

Facts:

- ΔG is zero everywhere except on the Julia set
- $\mu := \frac{1}{2\pi}\Delta G$ is a dynamically important measure that is supported on the Julia set (the unique invariant measure of maximal entropy)

Green Function

$$G(z) := \lim_{n\to\infty} \frac{1}{2^n} \log_+ |p^n(z)|$$

Brolin's Theorem

For all $w \in \mathbb{C}$ except at most two exceptional points,

$$\frac{1}{2^n}(p^n)^*\delta_w \to \frac{1}{2\pi}\Delta G.$$

Note: $\frac{1}{2^n}(p^n)^*\delta_w = \frac{1}{2\pi}\Delta\left(\frac{1}{2^n}\log|p^n(z) - w|\right)$

Brolin's Theorem

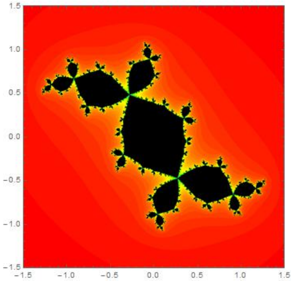


Figure: Escape rate algorithm.

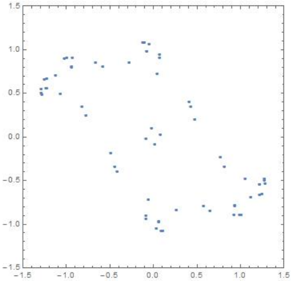


Figure: Sixth preimage of 0.

Brolin's Theorem

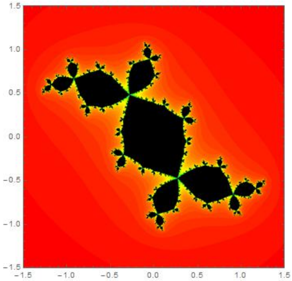


Figure: Escape rate algorithm.

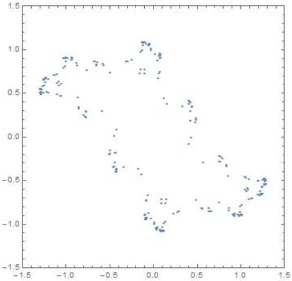


Figure: Eighth preimage of 0.

Brolin's Theorem

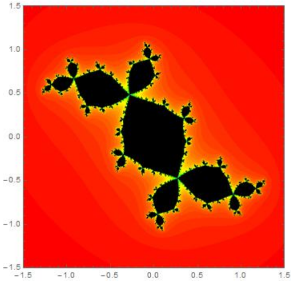


Figure: Escape rate algorithm.

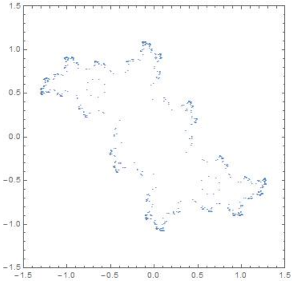
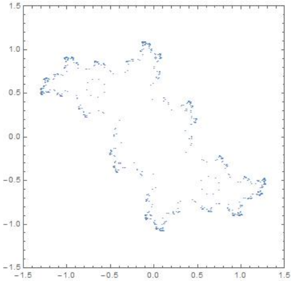


Figure: Tenth preimage of 0.

Brolin's Theorem



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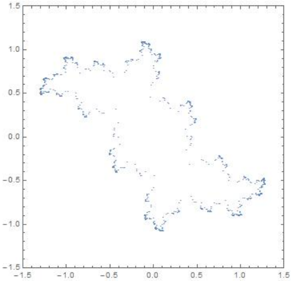

Figure: Tenth preimage of $0.933 + 0.637i$.

Brolin's Theorem

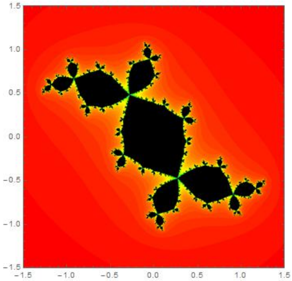

Figure: Escape rate algorithm.

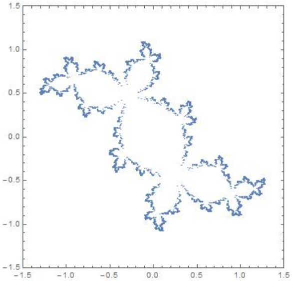

Figure: Random preimages of $0.933 + 0.637i$.

Proof of Brolin's Theorem for $p(z) = z^2 + c$, $c \neq 0$

Notation: $z_n := p^n(z)$ and $u_n(z) := \frac{1}{2^n} \log |z_n - w|$

Green Function

$$G(z) := \lim_{n \to \infty} \frac{1}{2^n} \log_+ |z_n|$$

Goal:

Show that u_n converges to G in L^1_{loc}.

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Compactness Theorem

If v_j is a sequence

- of subharmonic functions on a domain $\Omega \subset \mathbb{C}$
- that has a uniform upper bound on any compact set
- that does not converge to $-\infty$ uniformly on every compact set in Ω,

then there is a subsequence v_{j_k} which converges in $L^1_{\text{loc}}(\Omega)$ to a subharmonic function.

Proof of Brolin's Theorem for $p(z) = z^2 + c$, $c \neq 0$

Notation: $z_n := p^n(z)$ and $u_n(z) := \frac{1}{2^n} \log |z_n - w|$

Step 1: Restrict to a convergent subsequence.

Outside K_p, $u_n \to G$ locally uniformly.

In a disk $\mathbb{D}$ containing K_p, u_n is a sequence

- of subharmonic functions on $\mathbb{D}$
- that has a uniform upper bound on any compact set in $\mathbb{D}$
- that does not converge to $-\infty$ uniformly on every compact set in $\mathbb{D}$.

Thus, there is a subsequence u_{n_k} which converges in $L^1_{\text{loc}}(\mathbb{D})$ to a subharmonic function v.

Proof of Brolin's Theorem for $p(z) = z^2 + c$, $c \neq 0$

Recap:

Goal: Show that u_n converges to G in L^1_{loc}.

Outside K_p, $u_n \to G$ locally uniformly.

In $\mathbb{D}$ containing K_p, (after re-indexing) $u_n \to v$ in L^1_{loc}.

Contradiction hypothesis: Suppose $v \neq G$ (on K_p).

Proof of Brolin's Theorem for $p(z) = z^2 + c$, $c \neq 0$

Hartogs' Lemma

If v_j is a sequence

- of subharmonic functions on a domain $\Omega \subset \mathbb{C}$
- that has a uniform upper bound on any compact set
- that converges in $\mathcal{D}'(\Omega)$ to a subharmonic function v,

then $v_j \to v$ in $L^1_{\text{loc}}(\Omega)$ and

$$\limsup_{j\to\infty} v_j(z) \leq v(z).$$

Proof of Brolin's Theorem for $p(z) = z^2 + c$, $c \neq 0$

$$G(z) := \lim_{n\to\infty} \frac{1}{2^n} \log_+ |z_n| \quad \text{and} \quad u_n(z) := \frac{1}{2^n} \log |z_n - w|$$

Step 2: Upper bound.

Recall: $G \equiv 0$ and $v \not\equiv G$ on K_p.
Hartogs' Lemma implies $\limsup_{n\to\infty} u_n(z) \leq v(z)$.
u_n, v, and G are upper semicontinuous.

There is $\delta > 0$ such that $W := \{v < -2\delta\}$ is nonempty open.

There is a precompact, open $W_0 \subset W$ such that
$\frac{1}{2^n} \log |p^n(z_0) - w| < -\delta$ for all $z_0 \in W_0$.

Solving, $|p^n(z_0) - w| < e^{-\delta 2^n}$ implies $p^n(W_0) \subset \mathbb{D}(w, e^{-\delta 2^n})$

Step 3: Lower bound.

Fix $R < \dfrac{|c|}{2}$. Let $0 < r < R$ and consider $p(\mathbb{D}(z^*, r))$.

1. If $|z^*| \geq R$, then $\mathbb{D}(p(z^*), r \cdot R) \subset p(\mathbb{D}(z^*, r))$.
2. If $r/3 \leq |z^*| \leq R$, then $\mathbb{D}(p(z^*), r^2/9) \subset p(\mathbb{D}(z^*, r/3))$.
3. If $|z^*| < r/3$, then $\mathbb{D}(c, r^2/9) \subset p(\mathbb{D}(z^*, r))$.

Since $r < R < |c|/2$, a disk can only get mapped near the critical point every other iterate.

Hence, we have a constant $A > 0$ and a sequence z_n^* such that

$$\mathbb{D}(z_n^*, A \cdot r^{\sqrt{2}^n}) \subset p^n(W_0).$$

Combining the bounds:

$$\mathbb{D}(z_n^*, A \cdot r^{\sqrt{2}^n}) \subset p^n(W_0) \subset \mathbb{D}(w, e^{-\delta 2^n})$$

For n large enough, $e^{-\delta 2^n} < Ar^{\sqrt{2}^n}$, a contradiction.

Thus, $u_n \to v \equiv G$ in L^1_{loc}, so the Laplacians also converge.

Lyubich, Freire-Lopez-Mañe Theorem

More generally:

Lyubich, Freire-Lopez-Mañe Theorem

Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be rational with degree $d \geq 2$. There exists a set $\mathcal{E}$ (the exceptional set) containing at most two points, such that if $z_0 \notin \mathcal{E}$, then

$$\frac{1}{d^n}(f^n)^*\delta_{z_0} \to \mu,$$

where μ is the measure of maximal entropy.