<u>Section 9.2:</u> **Regular Markov Chains**

- **Irreducible Markov Chain:** When all its states **<u>communicate</u>** with each others, or it is easier to think of it as: *connectable*. (*It is strongly recommended to draw the transition diagram*)

Example 1: Determine if the following is irreducible (*connectable*):

$$T = \begin{bmatrix} 0.25 & 0.75 \\ 0.65 & 0.35 \end{bmatrix}$$

Example 2: Determine if the following is irreducible (*connectable*):

$$T = \begin{bmatrix} 0.2 & 0.4 & 0.4 \\ 0.3 & 0 & 0.7 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: Anytime a state is communicating only with <u>itself</u> as in state 3, the matrix is not irreducible (not connectable)

Example 3: Determine if the following is irreducible (*connectable*):

$$T = \begin{bmatrix} 0.6 & 0 & 0.4 \\ 0.2 & 0 & 0.8 \\ 0 & 0.8 & 0.2 \end{bmatrix}$$

Regular Markov Chain: A transition matrix is regular when there is power of T that contains all positive *no zeros* entries**.**

a) If the transition matrix is not irreducible (*not connectable*), then it is not regular

b) If the transition matrix is irreducible (*connectable*) and at least one entry of the main diagonal is nonzero, then it is regular

c) If all entries on the main diagonal are zero, but T^n (after multiplying by itself n times) contain all postive entries, then it is regular.

Example 4: Determine which of the following matrices is regular:

a) $T = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$ b) $T = \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0 \end{bmatrix}$ c) $T = \begin{bmatrix} 1 & 0 & 0 \\ 0.25 & 0.5 & 0.25 \\ 0 & 1 & 0 \end{bmatrix}$

a) yes, all entries are positive

b) yes because has only positive entries. You can also look at it as irreducible matrix with at least one element in the main diagonal not equal to zero.

$$T^2 = \begin{bmatrix} 0.75 & 0.25 \\ 0.5 & 0.5 \end{bmatrix}$$

c) No, because it is not irreducible (*not connectable*). Also, if you multiply it by itself over and over it will still contain zeros

$$T = \begin{vmatrix} 0.000 & 0.100 & 0.900 \\ 0.700 & 0.000 & 0.300 \\ 1.000 & 0.000 & 0.000 \end{vmatrix}$$

$$T^2 = \begin{vmatrix} 0.970 & 0.000 & 0.030 \\ 0.300 & 0.070 & 0.630 \\ 0.000 & 0.100 & 0.900 \end{vmatrix}$$

$$T^3 = \begin{vmatrix} 0.030 & 0.097 & 0.873 \\ 0.679 & 0.030 & 0.291 \\ 0.970 & 0.000 & 0.030 \end{vmatrix}$$

$$T^4 = \begin{vmatrix} 0.941 & 0.003 & 0.056 \\ 0.312 & 0.068 & 0.620 \\ 0.030 & 0.097 & 0.873 \end{vmatrix}$$

Notice that T^4 have all postive entries, so it is regular.

$$T = \begin{vmatrix} 0.000 & 1.000 & 0.000 \\ 0.500 & 0.000 & 0.500 \\ 0.000 & 1.000 & 0.000 \end{vmatrix}$$

$$T^2 = \begin{vmatrix} 0.500 & 0.000 & 0.500 \\ 0.000 & 1.000 & 0.000 \\ 0.500 & 0.000 & 0.500 \end{vmatrix}$$

$$T^3 = \begin{vmatrix} 0.000 & 1.000 & 0.000 \\ 0.500 & 0.000 & 0.500 \\ 0.000 & 1.000 & 0.000 \end{vmatrix}$$

Notice that T^3 is the same as the original matrix, so it cycles back and forth. This is called ***periodic*** **and it is not regular.**

For T= $\begin{bmatrix} 0 & 0.7 & 0.3 \\ 0.8 & 0 & 0.2 \\ 0.4 & 0.6 & 0 \end{bmatrix}$

Draw the transition diagram

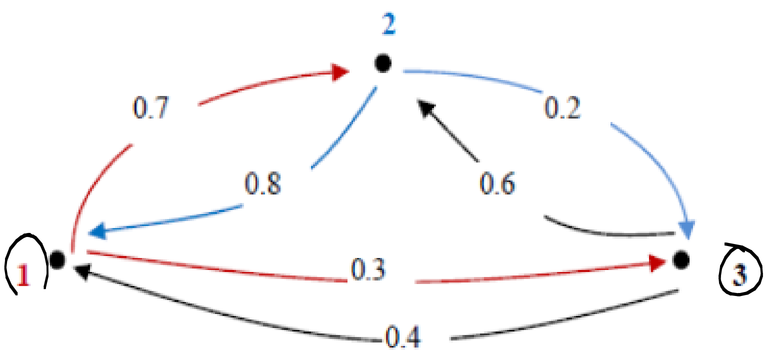
find T^2

a) Irreducible ? Yes____ No ____

b) Regular? Yes_____ No_____

For T= $\begin{bmatrix} 0 & 0.7 & 0.3 \\ 0.8 & 0 & 0.2 \\ 0.4 & 0.6 & 0 \end{bmatrix}$

Draw the transition diagram



a) Irreducible ? Yes____ No ____

find $T^2 = T.\,T$

$$= \begin{bmatrix} 0 & 0.7 & 0.3 \\ 0.8 & 0 & 0.2 \\ 0.4 & 0.6 & 0 \end{bmatrix} . \begin{bmatrix} 0 & 0.7 & 0.3 \\ 0.8 & 0 & 0.2 \\ 0.4 & 0.6 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.680 & 0.180 & 0.140 \\ 0.080 & 0.680 & 0.240 \\ 0.480 & 0.280 & 0.240 \end{bmatrix}$$

b) Regular? Yes_____ No_____

From section 9.1, we had:

$P_n = P_0 T^n$ (P_0: the initial state vector, T: the transition matrix)

$$P_1 = (P_0 . T)$$

$$P_2 = P_1 . T = (P_0 . T) . T = P_0 . T^2$$

$$P_3 = P_2 . T = (P_0 . T^2) . T = P_0 . T^3$$

$$P_4 = P_3 . T = (P_0 . T^3) . T = P_0 . T^4$$

Example 5: Previously in section 9.1, we had the following example:

A market analyst is interested in whether consumers prefer Dell or Gateway computers. two market surveys taken one year apart reveals the following:

- 10% of Dell owners had switched to Gateway and the rest continued with Dell.
- 35% of Gateway owners had switched to Dell and the rest continued with Gateway.

If a consumer has Dell Computer now:	**If a consumer has Gateway Computer now:**
Now: $\begin{bmatrix}1 & 0\end{bmatrix}$	Now: $\begin{bmatrix}0 & 1\end{bmatrix}$
After 1 year: $\begin{bmatrix}1 & 0\end{bmatrix}\cdot\begin{bmatrix}0.9 & 0.1\\0.35 & 0.65\end{bmatrix}=\begin{bmatrix}0.9 & 0.1\end{bmatrix}$	After 1 year: $\begin{bmatrix}0 & 1\end{bmatrix}\cdot\begin{bmatrix}0.9 & 0.1\\0.35 & 0.65\end{bmatrix}=\begin{bmatrix}0.35 & 0.65\end{bmatrix}$
After 2 year: $\begin{bmatrix}0.9 & 0.1\end{bmatrix}\cdot\begin{bmatrix}0.9 & 0.1\\0.35 & 0.65\end{bmatrix}=\begin{bmatrix}0.85 & 0.16\end{bmatrix}$	After 2 year: $\begin{bmatrix}0.35 & 0.65\end{bmatrix}\cdot\begin{bmatrix}0.9 & 0.1\\0.35 & 0.65\end{bmatrix}=\begin{bmatrix}0.54 & 0.46\end{bmatrix}$
After 3 year: $\begin{bmatrix}0.85 & 0.16\end{bmatrix}\cdot\begin{bmatrix}0.9 & 0.1\\0.35 & 0.65\end{bmatrix}=\begin{bmatrix}0.81 & 0.19\end{bmatrix}$	After 3 year: $\begin{bmatrix}0.54 & 0.46\end{bmatrix}\cdot\begin{bmatrix}0.9 & 0.1\\0.35 & 0.65\end{bmatrix}=\begin{bmatrix}0.65 & 0.35\end{bmatrix}$
After 4 year: $\begin{bmatrix}0.81 & 0.19\end{bmatrix}\cdot\begin{bmatrix}0.9 & 0.1\\0.35 & 0.65\end{bmatrix}=\begin{bmatrix}0.8 & 0.2\end{bmatrix}$	After 4 year: $\begin{bmatrix}0.65 & 0.35\end{bmatrix}\cdot\begin{bmatrix}0.9 & 0.1\\0.35 & 0.65\end{bmatrix}=\begin{bmatrix}0.71 & 0.29\end{bmatrix}$
After 5 year: $\begin{bmatrix}0.8 & 0.2\end{bmatrix}\cdot\begin{bmatrix}0.9 & 0.1\\0.35 & 0.65\end{bmatrix}=\begin{bmatrix}0.79 & 0.21\end{bmatrix}$	After 5 year: $\begin{bmatrix}0.71 & 0.29\end{bmatrix}\cdot\begin{bmatrix}0.9 & 0.1\\0.35 & 0.65\end{bmatrix}=\begin{bmatrix}0.74 & 0.26\end{bmatrix}$

If a consumer has Dell Computer now:	If a consumer has Gateway Computer now:
After 6 year: $\begin{bmatrix}0.79 & 0.21\end{bmatrix} \cdot \begin{bmatrix}0.9 & 0.1 \\ 0.35 & 0.65\end{bmatrix} = \begin{bmatrix}0.78 & 0.22\end{bmatrix}$	After 6 year: $\begin{bmatrix}0.74 & 0.26\end{bmatrix} \cdot \begin{bmatrix}0.9 & 0.1 \\ 0.35 & 0.65\end{bmatrix} = \begin{bmatrix}0.76 & 0.24\end{bmatrix}$
After 7 year: $\begin{bmatrix}0.78 & 0.22\end{bmatrix} \cdot \begin{bmatrix}0.9 & 0.1 \\ 0.35 & 0.65\end{bmatrix} = \begin{bmatrix}0.78 & 0.22\end{bmatrix}$	After 7 year: $\begin{bmatrix}0.76 & 0.24\end{bmatrix} \cdot \begin{bmatrix}0.9 & 0.1 \\ 0.35 & 0.65\end{bmatrix} = \begin{bmatrix}0.77 & 0.23\end{bmatrix}$
After 8 year: $\begin{bmatrix}0.78 & 0.22\end{bmatrix} \cdot \begin{bmatrix}0.9 & 0.1 \\ 0.35 & 0.65\end{bmatrix} = \begin{bmatrix}0.78 & 0.22\end{bmatrix}$	After 8 year: $\begin{bmatrix}0.77 & 0.23\end{bmatrix} \cdot \begin{bmatrix}0.9 & 0.1 \\ 0.35 & 0.65\end{bmatrix} = \begin{bmatrix}0.78 & 0.22\end{bmatrix}$
After 9 year: $\begin{bmatrix}0.78 & 0.22\end{bmatrix} \cdot \begin{bmatrix}0.9 & 0.1 \\ 0.35 & 0.65\end{bmatrix} = \begin{bmatrix}0.78 & 0.22\end{bmatrix}$	After 9 year: $\begin{bmatrix}0.78 & 0.22\end{bmatrix} \cdot \begin{bmatrix}0.9 & 0.1 \\ 0.35 & 0.65\end{bmatrix} = \begin{bmatrix}0.78 & 0.22\end{bmatrix}$

* After certain years, the probability stabilizes at 78% for Dell and 22% for Gateway. Notice that whether we start with Gateway or Dell, the result is the same and that is not accidental.

* The state vector of is called the **Steady State Vector** where: $\boldsymbol{P.T = P}$

(multiplying the Steady State Vector by the Transition Matrix = the Steady State Vector.)

* The above can only applied on **Regular** Markov chain

Example 6: The same example again:

A market analyst is interested in whether consumers prefer Dell or Gateway computers. two market surveys taken one year apart reveals the following:

- 10% of Dell owners had switched to Gateway and the rest continued with Dell.
- 35% of Gateway owners had switched to Dell and the rest continued with Gateway.

Find the distribution of the market after "a long period of time" or the **Steady State Vector**.

Solution:

The answer is in finding the **Steady State Vector** P where: $P.T = P$

$$P = \begin{bmatrix} x & y \end{bmatrix} \quad ; \quad T = \begin{bmatrix} 0.9 & 0.1 \\ 0.35 & 0.65 \end{bmatrix}$$

$$P.T = P \text{ then : } \begin{bmatrix} x & y \end{bmatrix} \cdot \begin{bmatrix} 0.9 & 0.1 \\ 0.35 & 0.65 \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}$$

Or:

$$0.9x + 0.35y = x \rightarrow 0.9x - x + 0.35y = 0$$

$$0.1x + 0.65y = y \rightarrow 0.1x + 0.65y - y = 0$$

Simplify the above equations by moving all variable to one side:

$$-0.1x + 0.35y = 0$$

$$0.1x - 0.35y = 0$$

The two equations are dependent and have infinite number of solutions. We must add another equation in order to get the answer: $x + y = 1$

Now, use the Echelon's Method to solve:

$$-0.1x + 0.35y = 0$$
$$0.1x - 0.35y = 0$$
$$x + y = 1$$

Multiply each equation by 100 to remove decimals, except the last equation:

$$-10x + 35y = 0$$
$$10x - 35y = 0$$
$$x + y = 1$$

x	y	
-10*	35	0
10	-35	0
1	1	1
-10	35	0
0	0	0
0	-45	-10
-10	35	0
0	-45*	-10
-45	0	-35
0	-45	-10
1	0	0.78
0	1	0.22

Remove the line with all zeros

The answer is $x = 78\%$ and $y = 22\%$ which is the same answer we got in example 6 when we did it the long way.

Example 7: Suppose that General Motors (GM), Ford (F), and Chrysler (C) each introduce a new SUV vehicle.

- General Motors keeps 85% of its customers but loses 10% to Ford and 5% to Chrysler.
- Ford keeps 80% of its customers but loses 10% to General motors and 10% to Chrysler.
- Chrysler keeps 60% of its customers but loses 25% to General Motors and 15% to Ford..

Find the distribution of the market in the long run or the Steady State Vector.

Solution: Lets assume the probabilities to be x for GM, y for F and z for C just to make it easier to solve

$$P = \begin{bmatrix} x & y & z \end{bmatrix} \quad ; \quad T = \begin{bmatrix} 0.85 & 0.1 & 0.05 \\ 0.1 & 0.8 & 0.1 \\ 0.25 & 0.15 & 0.6 \end{bmatrix}$$

$$\boldsymbol{P.T = P} \text{ then: } \begin{bmatrix} x & y & z \end{bmatrix} \cdot \begin{bmatrix} 0.85 & 0.1 & 0.05 \\ 0.1 & 0.8 & 0.1 \\ 0.25 & 0.15 & 0.6 \end{bmatrix} = \begin{bmatrix} x & y & z \end{bmatrix}$$

Or:

$$0.85x + 0.1y + 0.25z = x \rightarrow 0.85x - x + 0.1y + 0.25z = 0$$
$$0.1x + 0.8y + 0.15z = y \rightarrow 0.1x + 0.8y - y + 0.25z = 0$$
$$0.05x + 0.1y + 0.6z = z \rightarrow 0.05x + 0.1y + 0.6z - z = 0$$

Simplify the above equations by moving all variable to one side:

$$-0.15x + 0.1y + 0.25z = 0$$
$$0.1x - 0.2y + 0.15z = 0$$
$$0.05x + 0.1y - 0.4z = 0$$

and: $x + y + z = 1$

Multiply each equation by 100 to remove decimals, except the last equation:.

$$-0.15x + 0.1y + 0.25z = 0$$
$$0.1x - 0.2y + 0.15z = 0$$
$$0.05x + 0.1y - 0.4z = 0$$
$$x + y + z = 1$$

$$-15x + 10y + 25z = 0$$
$$10x - 20y + 15z = 0$$
$$5x + 10y - 40z = 0$$
and: $$x + y + z = 1$$

It makes it easier if you multiply the first 3 equations by 100 to remove the decimal:

X	y	z	
-15*	10	25	0
10	-20	15	0
5	10	-40	0
1	1	1	1
-15	10	25	0
0	200*	-475	0
0	-200	475	0
0	-25	-40	-15
200	0	-650	0
0	200	-475	0
0	0	0	0
0	0	1325	200
200	0	-650	0
0	200	-475	0
0	0	1325*	200
1325	0	0	650
0	1325	0	475
0	0	1325	200
1	0	0	0.49
0	1	0	0.36
0	0	1	0.15

Remove the line with all zeros

GM = 49%
Ford = 36%
Chrysler = 15%

Example 8: A marketing analysis shows that 63% of the consumers who currently drink Coke will purchase Coke the next time, and 12% of consumers who drink Pepsi will switch to Coke. Find the steady state vector.

Example 9: A an extensive survey of customers of three major cable companies (**A, B and C**) found the following:

Company **A** will keep 71% of its customers, 12% will move to **B** and the rest will move to **C**.

Company B will lose 32% of its customer to **A** and 34% to **C**.

Company **C** will keep 96% of its customers with half of the rest moving to **A** and half to **B**.

Find the steady state vector