Chapter 9: Markov Chain

Section 9.1: Transition Matrices

In Section 4.4, Bernoulli Trails:

The probability of each outcome is independent of the outcome of any previous experiments and the probability stays the same.

Example 1: Flipping a fair coin 30 times, the probability stays the same and does not depend on the previous result

Example 2: Computer chips are manufactured with 5% defective. Fifteen are drawn at random from an assembly line by an inspector, what is the probability that he will find 3 defective chips?

In Section 9.1, Markov Chain:

What happens next is governed by what happened immediately before. (see the next Examples)

Example 3: An independent landscape contractor works in a weekly basis.

Each week he works (**W**), there is a probability of 80% that will be called again to work the following week.
Each week he is not working (**N**), there is a probability of only 60% that he will be called again to work

Draw the tree for all possibilities of 2 weeks from now and show all probabilities.
Note: You need to draw 2 different trees, one if he is working now and the other if he is not. We cannot start from one point covering the initial states with one branch for W and the other for N since it is not given to us and we cannot assume it 50% each.

Use the tree to find

a) The probability that if he is working now, then he will be working in 2 weeks.

b) The probability that if he is not working now, then he will be working in 2 weeks.

Example 4: Use the information of example 3 again

Each week he works (**W**), there is a probability of 80% that will be called again to work the following week.
Each week he is not working (**N**), there is a probability of only 60% that he will be called again to work

and find:

a)The **Transition Matrix**. Show all probabilities and make sure the sum per row = 1
b)The **Transition Diagram.** Show all probabilities (*the sum of probabilities leaving a nod + itself = 1)*

Example 4 Cont. : Use the information of example 3 again and find:

c) The probability that if he is working now, then he will be working in 2 weeks

d)The probability that if he is <u>not</u> working now, then he will be working in 2 week

$$\mathbf{T} = \begin{array}{c|cc|} & W & N \\ W & 0.80 & 0.20 \\ N & 0.60 & 0.40 \end{array}$$

$$\mathbf{T}^2 = \mathbf{T} \cdot \mathbf{T} = \begin{vmatrix} 0.80 & 0.20 \\ 0.60 & 0.40 \end{vmatrix} \cdot \begin{vmatrix} 0.80 & 0.20 \\ 0.60 & 0.40 \end{vmatrix}$$

$$\mathbf{T}^2 = \begin{array}{c|cc|} & W & N \\ W & 0.76 & 0.24 \\ N & 0.72 & 0.28 \end{array}$$

Example 4 Cont. : Use the information of example 3 again and find:

e)The probability that if he is working now, then he will be working in 4 weeks

$$\mathbf{T}^2 = \begin{array}{c|cc|} & W & N \\ W & 0.76 & 0.24 \\ N & 0.72 & 0.28 \end{array}$$

$$\mathbf{T}^4 = \mathbf{T}^2 \cdot \mathbf{T}^2 = \begin{vmatrix} 0.76 & 0.24 \\ 0.72 & 0.28 \end{vmatrix} \cdot \begin{vmatrix} 0.76 & 0.24 \\ 0.72 & 0.28 \end{vmatrix}$$

$$\mathbf{T}^4 = \begin{array}{c|cc|} & W & N \\ W & 0.7504 & 0.2496 \\ N & 0.7488 & 0.2512 \end{array}$$

Example 5: A study by an overseas travel agency reveals that among the airlines: American, Delta and United, traveling habits are as follows:

- If a customer has just traveled on American, there is a 50% chance he will choose American again on his next trip, but if he switches, he is just likely to switch to Delta or United.
- If a customer has just traveled on Delta, there is a 60% chance he will choose Delta again on his next trip, but if he switches, he is three times as likely to switch to American as to United.
- If a customer has just traveled on United, there is a 70% chance he will choose United again on his next trip, but if he switches, he is twice as likely to switch to Delta as to American.

a) Find the probability transition matrix

b) Find the transition diagram

Example 5 Cont.:

c) Find the matrix that describes the customers habits two trips from now, then find the probability that a current Delta ticket holder will <u>not</u> travel on Delta the next time after

$$\mathbf{T} = \begin{array}{c|ccc|} & A & D & U \\ A & 0.50 & 0.25 & 0.25 \\ D & 0.30 & 0.60 & 0.10 \\ U & 0.10 & 0.20 & 0.70 \end{array}$$

$$\mathbf{T}^2 = \mathbf{T} \cdot \mathbf{T} = \left|\begin{matrix} 0.50 & 0.25 & 0.25 \\ 0.30 & 0.60 & 0.10 \\ 0.10 & 0.20 & 0.70 \end{matrix}\right| \cdot \left|\begin{matrix} 0.50 & 0.25 & 0.25 \\ 0.30 & 0.60 & 0.10 \\ 0.10 & 0.20 & 0.70 \end{matrix}\right|$$

$$\mathbf{T}^2 = \begin{array}{c|ccc|} & A & D & U \\ A & 0.350 & 0.325 & 0.325 \\ D & 0.340 & 0.455 & 0.205 \\ U & 0.180 & 0.285 & 0.535 \end{array}$$

Example 5 Cont.:

d) Find the matrix that describes the customers habits three trips from now, then find the probability that a current American ticket holder will switch to United three trips from now.

$$\mathbf{T}^3 = \mathbf{T}^2 \cdot \mathbf{T} = \begin{vmatrix} 0.350 & 0.325 & 0.325 \\ 0.340 & 0.455 & 0.205 \\ 0.180 & 0.285 & 0.535 \end{vmatrix} \cdot \begin{vmatrix} 0.50 & 0.25 & 0.25 \\ 0.30 & 0.60 & 0.10 \\ 0.10 & 0.20 & 0.70 \end{vmatrix}$$

$$\mathbf{T}^3 = \begin{array}{c|ccc|} & \text{A} & \text{D} & \text{U} \\ \text{A} & 0.3050 & 0.3475 & 0.3475 \\ \text{D} & 0.3270 & 0.3990 & 0.2740 \\ \text{U} & 0.2290 & 0.3230 & 0.4480 \end{array}$$

Example 6: A market analyst is interested in whether consumers prefer Dell or Gateway computers. Two market surveys taken one year apart reveals the following:

- 10% of Dell owners had switched to Gateway and the rest continued with Dell.
- 35% of Gateway owners had switched to Dell and the rest continued with Gateway.

At the time of the first market survey, 40% of consumers had Dell computers and 60% had Gateway.

a) Find the probability transition matrix

b) Find the transition diagram

Example 6 Cont.:

c) What percentage will buy their next computer from Dell?

$P_n = P_0 T^n$ (P_0: the initial state vector, T: the transition matrix)

$$\mathbf{P_0} = \begin{array}{cc} D & G \\ \left| 0.40 \right. & \left. 0.60 \right| \end{array} \qquad \mathbf{T}' = \begin{array}{c|cc|} & D & G \\ D & 0.90 & 0.10 \\ G & 0.35 & 0.65 \end{array}$$

$$P_0.T = \begin{vmatrix} 0.40 & 0.60 \end{vmatrix} \cdot \begin{vmatrix} 0.90 & 0.10 \\ 0.35 & 0.65 \end{vmatrix}$$

$$P_0.T = \begin{vmatrix} 0.570 & 0.430 \end{vmatrix}$$

Example 6 Cont.:

d) What percentage will buy their second computer from Dell?

$$\boxed{P_n = P_0\, T^n \quad (P_0\text{: the initial state vector, } T\text{: the transition matrix})}$$

$$T^2 = \begin{vmatrix} 0.90 & 0.10 \\ 0.35 & 0.65 \end{vmatrix} \cdot \begin{vmatrix} 0.90 & 0.10 \\ 0.35 & 0.65 \end{vmatrix}$$

$$T^2 = \begin{vmatrix} 0.845 & 0.155 \\ 0.543 & 0.458 \end{vmatrix}$$

$$P_0.T^2 = \begin{vmatrix} 0.40 & 0.60 \end{vmatrix} \cdot \begin{vmatrix} 0.845 & 0.155 \\ 0.543 & 0.458 \end{vmatrix} = \begin{array}{|cc|} \multicolumn{1}{c}{D} & \multicolumn{1}{c}{G} \\ 0.6635 & 0.3365 \end{array}$$

Example 6 Cont.:

e) Suppose that each consumer buy a new computer each year, what will be the market distribution after 4 years

$$\mathbf{T}^4 = \mathbf{T}^2 . \mathbf{T}^2 = \begin{vmatrix} 0.845 & 0.155 \\ 0.543 & 0.458 \end{vmatrix} \cdot \begin{vmatrix} 0.845 & 0.155 \\ 0.543 & 0.458 \end{vmatrix}$$

$$\mathbf{T}^4 = \begin{vmatrix} 0.7981 & 0.2019 \\ 0.7066 & 0.2934 \end{vmatrix}$$

$$P_0.\mathbf{T}^4 = \begin{vmatrix} 0.40 & 0.60 \end{vmatrix} \cdot \begin{vmatrix} 0.7981 & 0.2019 \\ 0.7066 & 0.2934 \end{vmatrix} = \left|\begin{array}{cc} \mathrm{D} & \mathrm{G} \\ 0.7432 & 0.2568 \end{array}\right|$$

Example 7: Suppose that taxis pick up and deliver passengers in a city which is divided into three zones: *A*, *B* and *C*. Records kept by the drivers show that:

- Of the passengers picked up in zone *A*, 50% are taken to a destination in zone *A*, 40% to zone *B*, and 10% to zone C.
- Of the passengers picked up in zone *B*, 40% go to zone *A*, 30% to zone *B*, and 30% to zone *C*.
- Of the passengers picked up in zone *C*, 20% go to zone *A*, 60% to zone *B*, and 20% to zone *C*.

Suppose that at the beginning of the day 60% of the taxis are in zone *A*, 10% in zone *B*, and 30% in zone *C*.

a) What is the distribution of taxis in the various zones after all have had one rider?

Example 7: Suppose that taxis pick up and deliver passengers in a city which is divided into three zones: *A*, *B* and *C*. Records kept by the drivers show that:

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- Of the passengers picked up in zone *B*, 40% go to zone *A*, 30% to zone *B*, and 30% to zone *C*.
- Of the passengers picked up in zone *C*, 20% go to zone *A*, 60% to zone *B*, and 20% to zone *C*.

Suppose that at the beginning of the day 60% of the taxis are in zone *A*, 10% in zone *B*, and 30% in zone *C*.

a) What is the distribution of taxis in the various zones after all have had one rider?

$$\mathbf{T} = \begin{array}{c|ccc|} & \text{A} & \text{B} & \text{C} \\ \text{A} & 0.50 & 0.40 & 0.10 \\ \text{B} & 0.40 & 0.30 & 0.30 \\ \text{C} & 0.20 & 0.60 & 0.20 \end{array} \qquad \mathbf{P_0} = \left| \begin{matrix} 0.60 & 0.10 & 0.30 \end{matrix} \right|$$

$$P_0.T = \left| \begin{matrix} 0.60 & 0.10 & 0.30 \end{matrix} \right| \cdot \left| \begin{matrix} 0.50 & 0.40 & 0.10 \\ 0.40 & 0.30 & 0.30 \\ 0.20 & 0.60 & 0.20 \end{matrix} \right|$$

$$P_0.T = \left| \begin{matrix} 0.400 & 0.450 & 0.150 \end{matrix} \right|$$

Example 7 Cont.:

b) What is the distribution of taxis in the various zones after all have had two riders?

$$\mathbf{T}^2 = \mathbf{T} \cdot \mathbf{T} = \begin{vmatrix} 0.50 & 0.40 & 0.10 \\ 0.40 & 0.30 & 0.30 \\ 0.20 & 0.60 & 0.20 \end{vmatrix} \cdot \begin{vmatrix} 0.50 & 0.40 & 0.10 \\ 0.40 & 0.30 & 0.30 \\ 0.20 & 0.60 & 0.20 \end{vmatrix}$$

$$= \begin{vmatrix} 0.430 & 0.380 & 0.190 \\ 0.380 & 0.430 & 0.190 \\ 0.380 & 0.380 & 0.240 \end{vmatrix}$$

$$P_0.\mathbf{T}^2 = \begin{vmatrix} 0.60 & 0.10 & 0.30 \end{vmatrix} \cdot \begin{vmatrix} 0.430 & 0.380 & 0.190 \\ 0.380 & 0.430 & 0.190 \\ 0.380 & 0.380 & 0.240 \end{vmatrix}$$

$$P_0.T^2 = \begin{vmatrix} 0.410 & 0.385 & 0.205 \end{vmatrix}$$

Example 7 Cont.:

c) What is the distribution of taxis in the various zones after all have had four riders?

$$\mathbf{T}^4 = \mathbf{T}^2 . \mathbf{T}^2 = \begin{vmatrix} 0.430 & 0.380 & 0.190 \\ 0.380 & 0.430 & 0.190 \\ 0.380 & 0.380 & 0.240 \end{vmatrix} \cdot \begin{vmatrix} 0.430 & 0.380 & 0.190 \\ 0.380 & 0.430 & 0.190 \\ 0.380 & 0.380 & 0.240 \end{vmatrix}$$

$$= \begin{vmatrix} 0.4015 & 0.3990 & 0.1995 \\ 0.3990 & 0.4015 & 0.1995 \\ 0.3990 & 0.3990 & 0.2020 \end{vmatrix}$$

$$\mathrm{P}_0.\mathbf{T}^4 = \begin{vmatrix} 0.60 & 0.10 & 0.30 \end{vmatrix} \cdot \begin{vmatrix} 0.4015 & 0.3990 & 0.1995 \\ 0.3990 & 0.4015 & 0.1995 \\ 0.3990 & 0.3990 & 0.2020 \end{vmatrix}$$

$$\mathrm{P}_0.\mathrm{T}^4 = \begin{vmatrix} 0.401 & 0.399 & 0.200 \end{vmatrix}$$