Natural spectral isomorphisms

Jan Kwiatkowski

Let $(X, T\mu)$ be an ergodic measure preserving transformation of a standard probability space and let $\phi : X \longrightarrow G$ be a cocycle, where G is a compact abelian group with Haar measure λ. It is well known that the space $L^2(X \times G, \mu \times \lambda)$ decomposes into the product of T_ϕ-invariant subspaces $L^2_\gamma = \{f \otimes \gamma,\ f \in L^2(X, \mu)\}$, where γ ranges over $\hat{G}$ and $T_\phi(x, g) = (Tx, g \cdot \phi(x))$.

A spectral isomorphism S between pairs of group extensions defined by cocycles $\phi, \psi : X \longrightarrow G$ is said to be natural if it sends each (L^2_γ, T_ϕ) to $(L^2_{\hat{v}(\gamma)}, T_\psi)$, where $\hat{v}$ is a group automorphism of $\hat{G}$. It has been proved in [N] and [J-L-M] that every spectral isomorphism induced by a metric isomorphism is natural under the assumption that the base is a canonical factor of the group extension). In [D-K-L] is proved that every spectral isomorphism is natural in a class of Morse group extensions.

A natural question arises whether the above property is true in the class of all Morse dynamical systems?

[N] D. Newton, "On canonical factors of ergodic dynamical systems," J. London Math. Soc. 19(1979), 129-136.
[J-L-M] A. del Junco, M. Lemańczyk, M. Mentzen, "Semisimplicity, joinings and group extensions", Studia Math. 112(1995), 141-164.
[D-K-L] T. Downarowicz, J. Kwiatkowski, Y. Lacroix, "Spectral isomorphisms of Morse flows", Fundamenta Math. 163(2000), 193-213.