HOLES, NONCONVEXITY, AND CURVATURE IN METRIC SPACES

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Abstract. We introduce a way of measuring nonconvexity of a metric space. We apply it to define a broad generalization and refinement of the classical curvature of a curve. We also use it to introduce a natural new notion of a fractal set.

1. Introduction

It is natural when investigating properties of a metric space to focus on the points and subsets of this space. But what about what is *not* in the space, what is "missing"? We propose here a way of measuring this missing part. One can think about it as measuring what could be called the nonconvexity of the space. We apply this idea to define a broad generalization and refinement of the classical curvature of a curve, and compute it for a variety of examples. We also use it to introduce a natural new notion of a fractal set.

A metric space is said to have *midpoints* if for every pair of points there is a point whose distance to each of them is half the distance between them. We define a quantity, the *gap* of the space, which is a measure of its failure to have midpoints. The gap can be interpreted as the size of the largest hole, or perhaps more accurately as a measure of the nonconvexity of the space.

While our initial interest in the gap of a metric space was inspired by global considerations, we show here how its local behavior can be used to give a far-reaching generalization of the classical curvature of a curve.

Given a point in our metric space, we can look at how the gap of a ball centered at the point shrinks with its radius r. If the gap scales like r^p, we say that the space has *tryposity* of order p at the point, and the *p-tryposity* gives finer information about its dependence on the radius. In the case that the space is a curve, we also use the term *flexion* for tryposity, because in this case it can be interpreted as a generalized curvature at the point.

In particular, we show that the 3-flexion of a smooth curve at a point is equal to one quarter the square of its classical curvature. However, at a point on a curve for which either the curve is not smooth or the curvature is zero, there may be well defined flexion of order other than 3. In this way flexion both refines and extends the classical notion of curvature.

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We also use the local behavior of the gap to define a new notion of a fractal set in a metric space, called a *trypofractal*, and study the relation between this definition and one based on Hausdorff dimension.

The paper is organized as follows.

In Section 2 we introduce the gap of a metric space and its p-tryposity at a point (or p-flexion if it is a curve) for $p \geq 1$, prove some basic properties, and give examples.

In Section 3 we prove that the 3-flexion of a C^3 curve in $\mathbb{R}^n$ at a point is one quarter the square of the curvature, so that flexion is a generalization of the classical curvature. We make use of the fact that in this case the Menger curvature coincides with the classical curvature.

In Section 4 we show that for every $p > 1$, there is a C^1 planar curve with p-flexion at a point of any positive value. In particular, when $p < 3$, the classical curvature (corresponding to the 3-flexion) is undefined at this point, and when $p > 3$, the classical curvature is zero.

In Section 5 we compute the 1-flexion at the vertex of a V-shaped curve making an angle θ with the horizontal axis, and show that it is maximized when

$$\theta = \sin^{-1}\left(\frac{1+\sqrt{33}}{8}\right) \approx 1.003.$$

In Section 6 we find upper and lower bounds for the 1-flexion of the Koch curve, and define the notions of trypofractal and uniform trypofractal, using the standard Cantor set as an example.

In Section 7 we construct (non-Euclidean) curves with every point having any prescribed p-flexion for each $p \geq 1$.

In Section 8 we end with some open questions.

2. Preliminaries

Let (X, d) be a metric space. We denote by $B(x, r)$ the closed ball in X, centered at $x \in X$ and of radius r. For $x, y \in X$ we define the *hyperdistance* between x and y as

$$hd(x, y) = 2\ \inf\{r : \text{there exists } z \in X \text{ such that } x, y \in B(z, r)\}.$$

Next, we define the *leipodistance*[1] between x and y by

$$ld(x, y) = hd(x, y) - d(x, y).$$

Observe that $d(x, y) \leq hd(x, y) \leq 2d(x, y)$, so $0 \leq ld(x, y) \leq d(x, y)$.

While it is obvious that in general leipodistance is not a metric on X, it is not immediately obvious whether hyperdistance is. Here is an example where hyperdistance does not satisfy the triangle inequality.

Example 2.1. Let X consists of five points, $X = \{A, B, C, D, E\}$. Define the metric in X as follows: $d(A, B) = d(A, C) = d(B, D) = d(B, E) = 4$, $d(C, E) = 3$, $d(A, D) = d(B, C) = d(C, D) = 2$, and $d(A, E) = d(D, E) = 1$. It is easy to check that for all 10 triangles the triangle inequality is satisfied, so indeed d is a metric. However, $hd(A, B) = 8$, while $hd(A, D) = 2$ and $hd(B, D) = 4$. ◊

[1] $\lambda\varepsilon\acute{\iota}\pi\omega$ – to be absent, to be missing

Now we define the *gap* of X as

$$\gamma(X) = \sup\{ld(x, y) : x, y \in X\}.$$

It is immediate that any metric space with midpoints has gap zero. Here a metric space X has midpoints if for all x, y in X there is m in X with $d(x, m) = d(m, y) = d(x, y)/2$. For example, a convex set in a normed linear space has midpoints and therefore gap zero.

Similarly, a metric space with approximate midpoints has gap zero. Here a metric space has approximate midpoints if for all $\varepsilon > 0$ and for all x, y in X there is m in X with $d(x, m), d(m, y) < d(x, y)/2 + \varepsilon$. In particular, every length space has gap zero.

Example 2.2. Let us compute the gap of an $(n-1)$-dimensional sphere of radius r and center x in $\mathbb{R}^n$. If $y, z \in X$ and the angle $\angle yxz = 4\alpha$ ($0 < \alpha \leq \pi/4$) then $d(y, z) = 2r \sin 2\alpha$ and $hd(y, z) = 4r \sin \alpha$, so $ld(y, z) = 4r \sin \alpha - 2r \sin 2\alpha$ (see Figure 1). Differentiating with respect to α we get $4r(\cos \alpha - \cos 2\alpha) > 0$, so the maximum is attained when $\alpha = \pi/4$, that is, when y and z are antipodal points. Therefore, $\gamma(X) = 2\sqrt{2}r - 2r = 2(\sqrt{2} - 1)r$. This is proportional to r, so the gap measures in some sense the size of the hole in the space.

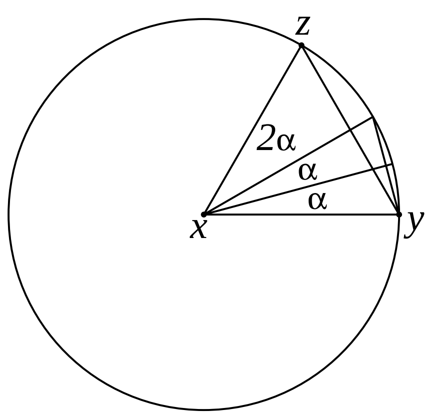

FIGURE 1. Gap of a sphere.

Now consider $Y = X \cup \{x\}$. Take $y \in X$. We have $d(x, y) = r$ and $hd(x, y) = 2r$. Therefore, $ld(x, y) = r$, so $\gamma(Y) \geq r$ (in fact, it is easy to see that $\gamma(Y) = r$). Since $1 > 2(\sqrt{2} - 1)$, this means that adding the center to the sphere increases its gap. While this is may seem counterintuitive, it simply means that the gap measures not only the size of the hole, but also its shape. ◊

Lemma 2.3. *The gap of X is not larger than the diameter of X. Moreover, if X is contained in a ball of radius r then $\gamma(X) \leq r$.*

Proof. The first statement follows from the inequality $ld(x, y) \leq d(x, y)$. Assume now that X is contained in a ball of radius r and take $x, y \in X$. If $d(x, y) \leq r$ then $ld(x, y) \leq d(x, y) \leq r$. If $d(x, y) > r$, then $ld(x, y) = hd(x, y) - d(x, y) < 2r - r = r$. □

Let X, X' be subsets of some larger space Y and let $d_H(X, X')$ be the Hausdorff distance between X and X' (it can be infinite).

Lemma 2.4. *We have*

$$|\gamma(X) - \gamma(X')| \le 6d_H(X, X'). \tag{1}$$

Proof. If $d_H(X, X') = \infty$, there is nothing to prove. Assume that $d_H(X, X')$ is finite. Fix $a > d_H(X, X')$. For any $x, y \in X$ there are $x', y' \in X'$ such that $d(x, x') < a$ and $d(y, y') < a$. Then $d(x', y') < d(x, y) + 2a$, so

$$-d(x, y) < -d(x', y') + 2a. \tag{2}$$

Fix $\varepsilon > 0$. There exists $z' \in X'$ such that $x', y' \in B(z', r)$ for some $2r < hd(x', y') + \varepsilon$. There is $z \in X$ such that $d(z, z') < a$. Then $x, y \in B(z, r + 2a)$, so $hd(x, y) \le 2r + 4a < hd(x', y') + \varepsilon + 4a$. This proves that

$$hd(x, y) < hd(x', y') + 4a. \tag{3}$$

Adding (2) and (3) we get $ld(x, y) < ld(x', y') + 6a$. This means that for any $x, y \in X$ there are $x', y' \in X'$ such that $ld(x', y') > ld(x, y) - 6a$. This proves that $\gamma(X') \ge \gamma(X) - 6a$. Since a was an arbitrary number larger than $d_H(X, X')$, we get $\gamma(X') \ge \gamma(X) - 6d_H(X, X')$, that is, $\gamma(X) - \gamma(X') \le 6d_H(X, X')$. Similarly, $\gamma(X') - \gamma(X) \le 6d_H(X, X')$, so we get (1). □

There is a simple corollary to this lemma.

Corollary 2.5. *If X' is a dense subset of X then $\gamma(X') = \gamma(X)$.*

Lemma 2.6. *If X is a subset of a Hilbert space Y then $\gamma(X) = 0$ if and only if the closure of X is convex.*

Proof. By Corollary 2.5, $\gamma(X) = \gamma(\overline{X})$. If $\overline{X}$ is convex, then clearly $\overline{X}$ has midpoints so $\gamma(\overline{X}) = 0$.

If $\overline{X}$ is not convex, then (since $\overline{X}$ is closed) there exist points $x, y \in \overline{X}$, such that if I is the segment joining x and y, then $I \setminus \{x, y\}$ is disjoint from $\overline{X}$.

By translating X, we may assume that $y = -x$. Suppose that $ld(x, -x) = 0$. Then for every $\varepsilon > 0$ there is $z \in \overline{X}$ such that $\|z - x\| < \|x\| + \varepsilon$ and $\|z + x\| < \|x\| + \varepsilon$. There is a point $z' \in I$ such that $z - z'$ is orthogonal to x. One of the numbers $\|z' - x\|$ and $\|z' + x\|$ is larger than or equal to $\|x\|$. We may assume that it is $\|z' + x\|$. We have $\|x\| \le \|z' + x\| < \|x\| + \varepsilon$ and $0, x, z'$ belong to I, so $\|z'\| < \varepsilon$. We have

$$\|x\|^2 + \|z - z'\|^2 \le \|z' + x\|^2 + \|z - z'\|^2 = \|z + x\|^2 \le (\|x\| + \varepsilon)^2.$$

Therefore, $\|z - z'\|^2 \le 2\varepsilon\|x\| + \varepsilon^2$, so $\|z\|^2 < 2\varepsilon\|x\| + 2\varepsilon^2$. Since $\varepsilon > 0$ was arbitrary, it follows that $0 \in \overline{X}$, a contradiction. □

It is natural to suspect that the above lemma holds for any normed linear space Y. However, this is not the case.

Example 2.7. Let Y be the plane $\mathbb{R}^2$ with the maximum norm, that is, $\|(x, y)\| = \max(|x|, |y|)$. Let $I \subset \mathbb{R}$ be an interval (in the general meaning, bounded or not), and let $f : I \to \mathbb{R}$ be a function, Lipschitz continuous with constant 1. Let $X \subset Y$ be the graph of f. Then, for $x, y \in I$ we have

$$d((x, f(x)), (y, f(y))) = \max(|x - y|, |f(x) - f(y)|) = |x - y|.$$

Thus, the map $h : X \to I$, given by $h(x, f(x)) = x$, is an isometry. Therefore, $\gamma(X) = \gamma(I) = 0$, although $\overline{X}$ is not convex unless f is affine. ♢

Nevertheless, a similar set in the same space can have positive gap.

Example 2.8. Let Y be as in the preceding example, $I = [-1, 1]$, and let X be the union of the graphs of functions $f, g : I \to \mathbb{R}$, given by $f(x) = |x|-1$ and $g(x) = 1-|x|$ (a "diamond"). Take points $a = (1/2, 1/2)$ and $b = (-1/2, -1/2)$. Their distance is 1. If their leipodistance is 0, then their hyperdistance must be 1, so the intersection of the balls $B(a, 1/2)$ and $B(b, 1/2)$ (in X) has to be nonempty. However, those balls are equal to the intersection of X with the first and third quadrants, respectively, so their intersection is empty. This proves that $\gamma(X) > 0$. ♢

Lemma 2.9. *If X is a nonempty subset of $\mathbb{R}$ with the usual metric, then the gap of X is the supremum of lengths of bounded components of $\mathbb{R} \setminus X$.*

Proof. By Corollary 2.5, we can assume that X is closed in $\mathbb{R}$. If (a, b) is a component of $\mathbb{R} \setminus X$, then $a, b \in X$, and $ld(a, b) = b - a$, so $\gamma(X)$ is larger than or equal to the length of (a, b).

On the other hand, if $x, y \in X$, with $x < y$, look at the midpoint z of $[x, y]$. If $z \in X$, then $ld(x, y) = 0$. Otherwise, z belongs to some component (a, b) of $\mathbb{R} \setminus X$. Suppose that $z - a \le b - z$. Then

$$ld(x, y) = 2(y - a) - (y - x) = x + y - 2a = 2(z - a) \le (z - a) + (b - z) = b - a.$$

Similarly, if $z - a \ge b - z$, we also get $ld(x, y) \le b - a$, so in both cases $ld(x, y)$ is not larger than the supremum of lengths of bounded components of $\mathbb{R} \setminus X$. □

In the rest of the paper we will concentrate on the local behavior of the gap in the neighborhood of a point. If (X, d) is a metric space, $x \in X$, and $p \ge 1$, we define the *lower p-tryposity*[2] and the *upper p-tryposity* of X at x by

$$\underline{\Phi}_p(X, x) = \liminf_{r\to 0} \frac{\gamma(B(x, r))}{r^p}, \quad \overline{\Phi}_p(X, x) = \limsup_{r\to 0} \frac{\gamma(B(x, r))}{r^p}.$$

If $\underline{\Phi}_p(X, x) = \overline{\Phi}_p(X, x)$, we will call it the *p-tryposity* of X at x and denote it by $\Phi_p(X, x)$. Observe that by Lemma 2.3, we always have $\overline{\Phi}_1(X, x) \le 1$.

Note that we allow p-tryposities to be 0 or infinity. However, each of $\underline{\Phi}_p(X, x)$ and $\overline{\Phi}_p(X, x)$ is an increasing function of p, and is positive finite for at most one value of p. If there is $p > 1$ such that $\overline{\Phi}_q(X, x) = 0$ for all $q < p$ and $\underline{\Phi}_q(X, x) = \infty$ for all $q > p$, then we will say that the tryposity of X at x has *order p*. Moreover, if $\underline{\Phi}_1(X, x) > 0$, we will say that the tryposity of X at x has order 1, and if $\overline{\Phi}_q(X, x) = 0$ for all $q \ge 1$ we will say that the tryposity has infinite order, or that the *total tryposity* of X at x is zero. In all other cases, we will say that the order of the tryposity of X at x does not exist.

In particular, if there is p such that $0 < \underline{\Phi}_p(X, x) < \overline{\Phi}_p(X, x) < \infty$, then the order is p. If there are $p < q$ such that $\overline{\Phi}_p(X, x) > 0$ and $\underline{\Phi}_q(X, x) < \infty$, then the order of the tryposity of X at x does not exist.

A convex subset or an open subset of a normed linear space are trivial examples of a space whose total tryposity at every point is zero: a ball in such a subset centered at any point and with sufficiently small radius is convex, so the gap of this ball is 0.

It is easy to give an example where the order of the tryposity does not exist.

[2] $\tau\rho\acute{\upsilon}\pi\alpha$ – hole

Example 2.10. Let $X = \{a_n : n = 1, 2, 3, \dots\} \cup \{0\}$, where $a_n = 1/2^{2^n}$. We have $B(0, a_{n+1}) = B(0, a_n/2)$, so by Lemma 2.9, the gap of this ball is $a_{n+1} - a_{n+2}$. Since $\lim_{n\to\infty} a_{n+2}/a_{n+1} = 0$ and $a_{n+1} = a_n^2$, we get

$$\lim_{n\to\infty} \frac{\gamma(B(0, a_{n+1}))}{a_{n+1}} = 1, \quad \text{but} \quad \lim_{n\to\infty} \frac{\gamma(B(0, a_n/2))}{(a_n/2)^2} = 4. \tag{4}$$

Therefore, the order of the tryposity of X at 0 does not exist. ♢

Similar examples can be constructed to show that even when the tryposity has order p, the p-tryposity can be 0 or infinity.

Example 2.11. Let $X = \{a_n : n = 1, 2, 3, \dots\} \cup \{0\}$, where a_n is defined by induction: $a_1 = 1/2$, and $a_{n+1} = a_n + \frac{a_n^2}{\log a_n}$. It is a standard exercise to check that the sequence (a_n) decreases and converges to 0. If $a_n \le r < a_{n-1}$, then by Lemma 2.9 we have

$$\gamma(B(0, r)) = a_n - a_{n+1} = \frac{a_n^2}{-\log a_n}.$$

Hence, the tryposity of X at 0 has order 2, and since

$$\lim_{n\to\infty} \frac{a_n}{a_{n-1}} = \lim_{n\to\infty} \frac{a_{n-1} + \frac{a_{n-1}^2}{\log a_{n-1}}}{a_{n-1}} = 1 + \lim_{n\to\infty} \frac{a_{n-1}}{\log a_{n-1}} = 1,$$

we get

$$\Phi_2(X, 0) = \lim_{n\to\infty} \frac{\frac{a_n^2}{-\log a_n}}{a_n^2} = \lim_{n\to\infty} \frac{1}{-\log a_n} = 0.$$

♢

Example 2.12. We make the same construction as in Example 2.11 with one difference. Instead of setting $a_{n+1} = a_n + \frac{a_n^2}{\log a_n}$, we set $a_{n+1} = a_n + a_n^2 \log a_n$. Then the computations are very similar (in particular, the tryposity of X at 0 has order 2), but we get

$$\Phi_2(X, 0) = \lim_{n\to\infty} \frac{-a_n^2 \log a_n}{a_n^2} = -\lim_{n\to\infty} \log a_n = \infty.$$

♢

We will also use the term *flexion* for tryposity if X is a curve. We will see that flexion gives a substantial generalization of the usual curvature, with 3-flexion essentially recovering the classical curvature, higher order flexion enabling us to distinguish between for example a quartic and a sextic at the origin, and lower order flexion enabling us to deal with cases where the curvature is undefined, for example corners.

3. Curves with nonzero curvature

In this section we will consider a simple regular curve Γ in the Euclidean space $\mathbb{R}^n$, of class C^3. It turns out that we can compute the curvature of Γ using the gap. Note that in such a way we use only the metric on Γ, inherited from $\mathbb{R}^n$, and we can forget about the rest of $\mathbb{R}^n$. In 1930 K. Menger used an idea in the same spirit to define what we call now the *Menger curvature* (see [Me]). In the situation considered above it turns out to be equal to the classical curvature.

Let x, y, z be three distinct points of Γ. Denote by $k_M(x, y, z)$ the reciprocal of the radius of the circle passing through those points (if they are collinear, then $k_M(x, y, z) = 0$). The following theorem is proved in [BM] as Theorem 10.5 there.

Theorem 3.1. *Assume that $n = 3$ and fix $P \in \Gamma$. Then the limit of $k_M(x, y, z)$, as distinct points $x, y, z \in \Gamma$ converge to P, exists and is equal to the classical curvature of Γ at P.*

Since we want to work in $\mathbb{R}^n$ with n possibly larger than 3, we have to generalize this theorem.

Theorem 3.2. *Fix $P \in \Gamma$. Then the limit of $k_M(x, y, z)$, as distinct points $x, y, z \in \Gamma$ converge to P, exists and is equal to the classical curvature of Γ at P.*

Proof. We want to modify the proof from [BM] to avoid using the assumption that $n = 3$. In the statement of the theorem there, the authors choose the parametrization of the curve by arc length. Then the proof starts with choosing the coordinate system in which P is the origin, assuming that the parametrization starts there, and the unit tangent vector is $(1, 0, \dots, 0)$. Thus, if the parametrization is $(x_1, \dots, x_n)$, then $x_i(0) = 0$ for all i, $x_1'(0) = 1$, and $x_i'(0) = 0$ for all $i > 1$.

After that the proof (based on the notion of the Haantjes curvature [Ha], which was earlier shown to be equivalent to the Menger curvature) does not use the assumption that the dimension is 3, until it comes to the proof that $x_1'''(0) = -\kappa^2$ and $\sum_{i=1}^n (x_i''(0))^2 = \kappa^2$, where κ is the classical curvature of Γ at the origin. Here the authors invoke the Frenet-Serret formulas, which require that $n = 3$.

However, those formulas are not necessary. The formula $\sum_{i=1}^n (x_i''(0))^2 = \kappa^2$ is well known, and often is used as the definition of κ. In order to get the formula $x_1'''(0) = -\kappa^2$, observe that $\sum_{i=1}^n (x_i'(t))^2 = 1$ for all t. Differentiating both sides twice, we get

$$\sum_{i=1}^n \left(x_i'''(t) x_i'(t) + (x_i''(t))^2 \right) = 0.$$

Evaluating at $t = 0$ we get $x_1'''(0) + \kappa^2 = 0$, and this completes the proof. □

We have two natural metrics on Γ, one given by arc length, and the other inherited from $\mathbb{R}^n$ (this metric will be denoted d). They may be quite different. It may happen that the points are close to each other one in $\mathbb{R}^n$, but far from each other along the curve. However, the situation is much better locally.

Let us call a subset of a curve, which is connected and consists of more than one point (so it is a curve itself), a *subcurve*. Moreover, let us say that a curve parametrized by a function α has the *monotone distance property (MDP)* if for every t_1, t_2, t_3 such that $t_1 < t_2 < t_3$ we have $d(\alpha(t_1), \alpha(t_2)) < d(\alpha(t_1), \alpha(t_3))$ and $d(\alpha(t_2), \alpha(t_3)) < d(\alpha(t_1), \alpha(t_3))$. Clearly, this property does not depend on the choice of a parametrization. Of course, every subcurve of a curve with MDP also has MDP.

Let us return to our curve Γ.

Lemma 3.3. *Let $\alpha : [a, b] \to \mathbb{R}^n$ be a parametrization of Γ by the arc length. Then for every $t_0 \in (a, b)$ there exists $\varepsilon > 0$ such that $a \le t_0 - \varepsilon$, $t_0 + \varepsilon \le b$, and the subcurve $\alpha([t_0 - \varepsilon, t_0 + \varepsilon])$ has MDP.*

Proof. For every $t \in [a,b]$ there is a unique decomposition $\alpha'(t) = c(t)\alpha'(t_0)+v(t)$ such that $c(t)$ is a scalar and the vector $v(t)$ is orthogonal to $\alpha'(t_0)$. If $\varepsilon > 0$ is sufficiently small then $a \le t_0 - \varepsilon$, $t_0 + \varepsilon \le b$, and $\|v(t)\| < c(t)$ for every $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$. Take t_1, t_2, t_3 such that $t_0 - \varepsilon \le t_1 < t_2 < t_3 \le t_0 + \varepsilon$. Since for every $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$ the vector $\alpha'(t)$ is contained in the (one-sided) cone C consisting of vectors of the form $\hat{c}\alpha'(t_0) + \hat{v}$, where $\|\hat{v}\| < \hat{c}$ and $\hat{v}$ is orthogonal to $\alpha'(t)$, we see that the vectors $\alpha(t_2) - \alpha(t_1)$ and $\alpha(t_3) - \alpha(t_2)$ also belong to C.

If vectors $\hat{c}_i\alpha'(t_0) + \hat{v}_i$, $i = 1, 2$, belong to C, then their scalar product is

$$\hat{c}_1\hat{c}_2 + \langle \hat{v}_1, \hat{v}_2\rangle \ge \hat{c}_1\hat{c}_2 - \|\hat{v}_1\| \cdot \|\hat{v}_2\| > 0.$$

Therefore the scalar product of $\alpha(t_2) - \alpha(t_1)$ and $\alpha(t_3) - \alpha(t_2)$ is positive, so the length of $(\alpha(t_2) - \alpha(t_1)) + (\alpha(t_3) - \alpha(t_2)) = \alpha(t_3) - \alpha(t_1)$ (that is, $d(\alpha(t_1), \alpha(t_3))$) is larger than the length of $(\alpha(t_2) - \alpha(t_1))$ (that is, $d(\alpha(t_1), \alpha(t_2))$).

The proof of the second inequality from the definition of MDP is similar. □

Remember that according to our notation, when we write $B(P, r)$, we mean the (closed) ball in Γ, not in $\mathbb{R}^n$.

Corollary 3.4. *For every point $P \in \Gamma$, which is not an endpoint, if $r > 0$ is sufficiently small then $B(P, r)$ is a subcurve of Γ and has MDP.*

Proof. Use Lemma 3.3 with $\alpha(t_0) = p$. Take r such small that $r < d(P, \alpha(t_0 - \varepsilon))$, $r < d(P, \alpha(t_0 + \varepsilon))$, and r is smaller than the distance from P to $\Gamma \setminus B(P, r)$. Then $B(P, r)$ is a subcurve of $\alpha([t_0-\varepsilon, t_0+\varepsilon])$. Thus, it is a subcurve of Γ and has MDP. □

We want to establish a connection between gap and curvature. Let us start with a simple geometrical lemma.

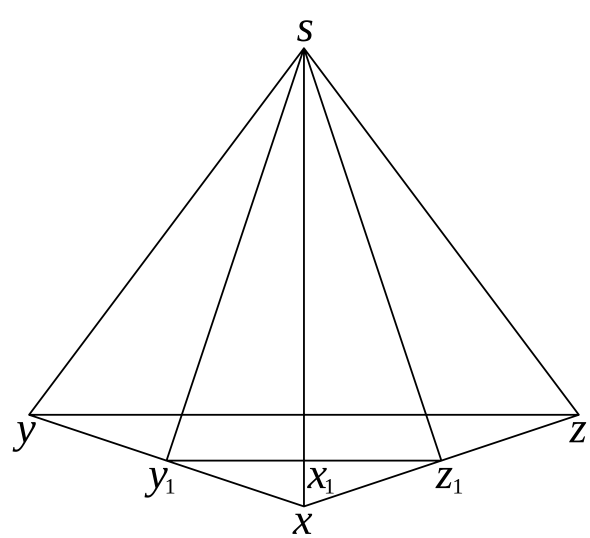

FIGURE 2. Situation from Lemma 3.5.

Lemma 3.5. *Let distinct points x, y, z belong to a circle centered at s with radius $R > 0$, and let $d(x, y) = d(x, z) = \varrho$, where $\varrho < R$. Then*

$$ld(y, z) = \frac{\varrho^3}{2R^2\left(1 + \sqrt{1 - \frac{\varrho^2}{4R^2}}\right)}, \tag{5}$$

where the leipodistance is computed in the space $\{x, y, z\}$.

Proof. The situation is depicted in Figure 2. Point y_1 is the midpoint of the segment xy, and z_1 is the midpoint of xz. Point x_1 is the point of intersection of the segments sx and y_1z_1.

Since our space consists of three points, x, y, z, we have $ld(y, z) = 2\varrho - d(y, z)$. From the right triangle sxz_1 we get

$$d(x_1, z_1) = \frac{d(x, z_1)d(s, z_1)}{d(x, s)} = \frac{(\varrho/2)\sqrt{R^2 - (\varrho/2)^2}}{R}.$$

However, $d(x_1, z_1) = d(y_1, z_1)/2 = d(y, z)/4$, so

$$ld(y, z) = 2\varrho - \frac{2\varrho}{R}\sqrt{R^2 - \frac{\varrho^2}{4}} = 2\varrho\left(1 - \sqrt{1 - \frac{\varrho^2}{4R^2}}\right) = \frac{\varrho^3}{2R^2\left(1 + \sqrt{1 - \frac{\varrho^2}{4R^2}}\right)}. \qquad \square$$

Remark 3.6. The assumption that $\varrho < R$ is made only to avoid discussion of cases which are unnecessary for us. We will use this lemma only for R bounded away from zero and ϱ converging to zero. Note that then the expression in the parenthesis in (5) converges to 2. Moreover, note that (5) is valid also in the case of $R = \infty$, that is, when the points x, y, z collinear.

Theorem 3.7. *Let Γ be a simple curve of class C^3 in $\mathbb{R}^n$, and let $P \in \Gamma$ be a point that is not an endpoint of Γ. Then*

$$\Phi_3(\Gamma, P) = \frac{\kappa^2}{4}, \tag{6}$$

where κ is the curvature of Γ at P. In particular, if $\kappa \neq 0$, then the flexion of Γ at P has order 3.

Proof. For simplicity, denote

$$\xi(\varrho, R) = \frac{\varrho^3}{2R^2\left(1 + \sqrt{1 - \frac{\varrho^2}{4R^2}}\right)}.$$

By Corollary 3.4, if r is sufficiently small, then $I_r = B(P, r))$ is a subcurve of Γ and had MDP. Let y_r and z_r be the endpoints of I_r. By MDP, the ball in I_r containing both y_r and z_r with the smallest radius is $B(P, r)$. Thus, by Lemma 3.5, $ld(y_r, z_r) = \xi(r, R_r)$, so $\gamma(I_r) \geq \xi(r, R_r)$, where $R_r = 1/k_M(P, y_r, z_r)$. As r approaches 0, by Theorem 3.2 and Remark 3.6, $\xi(r, R_r)/r^3$ approaches $\kappa^2/4$. This proves that

$$\liminf_{r\to 0} \frac{\gamma(B(P, r))}{r^3} \geq \frac{\kappa^2}{4}.$$

Now fix $\varepsilon > 0$. If r is sufficiently small, then by Theorem 3.2 for every three distinct points $x, y, z \in I_r$ the radius R of the circle passing through x, y, z is larger than $1/(\kappa+\varepsilon)$. When computing the hyperdistance between y and z in I_r, we minimize the radius of the ball containing both x and y when its center x is equidistant from y and z (that is, $d(x, y) = d(x, z)$). Thus, it suffices to consider only such triples. But then we can use Lemma 3.5, and we see that $ld(y, z) \le \xi(d(x, y), R)$. The function ξ is increasing with respect to the first argument and decreasing with respect to the second argument. Since $R > 1/(\kappa + \varepsilon)$, we get $ld(y, z) \le \xi(d(x, y), 1/(\kappa + \varepsilon))$.

Either both x, y or both x, z are on the same side of P. Since I_r has MDP, it follows that either $d(x, y) \le r$ or $d(x, z) \le r$. Since $d(x, y) = d(x, z)$, in both cases $d(x, y) \le r$. Then $ld(y, z) \le \xi(r, 1/(\kappa + \varepsilon))$, and hence

$$\limsup_{r\to 0} \frac{\gamma(B(P, r))}{r^3} \le \frac{(\kappa + \varepsilon)^2}{4}.$$

Since ε can be taken arbitrarily small, this completes the proof of (6). □

4. Zero curvature or undefined curvature

The assumption of the second part of Theorem 3.7 was that the curvature of Γ at P is nonzero. The question remains, what happens when it is equal to zero. In this section we will investigate a family of standard examples including such situations, as well as situations in which the usual curvature is undefined. Namely, we assume the curve Γ in the plane is given by $y = K|x|^q$, where $q > 1$ and $K > 0$ are real numbers, and the point P is the origin O.

When we try to compute the leipodistance between two points on this curve (we really need only to find out what happens when they both approach the origin), we have to consider two possibilities: they are on different sides of the origin (including the case when one of them is at the origin), and they are on the same side of the origin. Let us start with the first case, which turns out to be simpler.

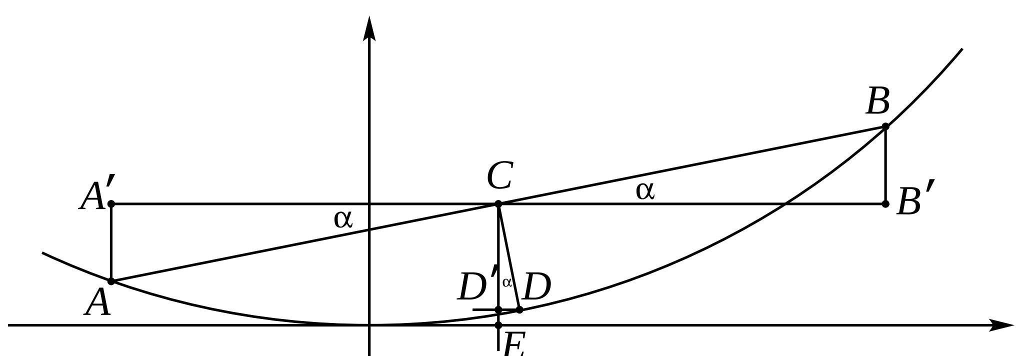

Figure 3. The case when A and B are on opposite sides of the origin.

Consider the situation from Figure 3, where the curve is $y = K|x|^q$, with $q > 1$ and $K > 0$. We assume that $0 \le a \le b$. The points are $A = (-a, Ka^q)$, $B = (b, Kb^q)$,

$C = \left(\frac{b-a}{2}, K\frac{b^q+a^q}{2}\right)$ is the midpoint of AB, $D = (d, Kd^q)$ and CD is perpendicular to AB, $A' = \left(-a, K\frac{b^q+a^q}{2}\right)$, $B' = \left(b, K\frac{b^q+a^q}{2}\right)$, $D' = \left(\frac{b-a}{2}, Kd^q\right)$, $E = \left(\frac{b-a}{2}, 0\right)$.

Let us look first at the simple case when $a = b$. Then $d(A,B) = 2b$ and $hd(A,B) = 2\sqrt{b^2 + K^2 b^{2q}}$. Therefore

$$ld(A,B) = 2b\left(\sqrt{1 + K^2 b^{2q-2}} - 1\right) = \frac{2K^2 b^{2q-1}}{\sqrt{1 + K^2 b^{2q-2}} + 1}.$$

This function divided by b^{2q-1} goes to K^2 as b goes to 0. Moreover, if b (and consequently, a) lies on the sphere centered at the origin O with radius r, then $r = \sqrt{1 + K^2 b^{2q-2}}$, so the ratio b/r goes to 1. Therefore,

$$\liminf_{r\to 0} \frac{\gamma(B(P,r))}{r^{2q-1}} \geq K^2. \tag{7}$$

In the general case, we have

$$\frac{d(C,A')}{d(C,A)} = \frac{d(C,B')}{d(C,B)} = \frac{d(C,D')}{d(C,D)} = \cos\alpha.$$

Hence, $d(C,B') = d(C,B)\cos\alpha$. Since the triangles CBD and $CB'D'$ are similar, we get also $d(D',B') = d(D,B)\cos\alpha$, so

$$d(D',B') - d(C,B') = \big(d(D,B) - d(C,B)\big)\cos\alpha.$$

Moreover, $d(D',B') \leq d(E,B')$. Therefore (using the inequality $\sqrt{1+x} \leq 1 + x/2$), we get

$$\begin{aligned}
ld(A,B) &= 2\big(d(D,B) - d(C,B)\big) = 2\big(d(D',B') - d(C,B')\big)\sec\alpha \\
&\leq 2\big(d(E,B') - d(C,B')\big)\sec\alpha = 2\left(\sqrt{\left(\frac{b+a}{2}\right)^2 + K^2\left(\frac{b^q+a^q}{2}\right)^2} - \frac{b+a}{2}\right)\sec\alpha \\
&= \left(\sqrt{(b+a)^2 + K^2(b^q+a^q)^2} - (b+a)\right)\sec\alpha \\
&= (b+a)\left(\sqrt{1 + K^2\left(\frac{b^q+a^q}{b+a}\right)^2} - 1\right)\sec\alpha \\
&\leq \frac{1}{2}(b+a)K^2\left(\frac{b^q+a^q}{b+a}\right)^2\sec\alpha = \frac{K^2}{2}\frac{(b^q+a^q)^2}{b+a}\sec\alpha.
\end{aligned}$$

Set $t = a/b$. We assume that $0 \leq t \leq 1$. We have $a = bt$, so

$$ld(A,B) \leq \frac{K^2}{2}\frac{(1+t^q)^2}{1+t} b^{2q-1}\sec\alpha.$$

We claim that $(1+t^q)^2/(1+t) \leq 2$. This is equivalent to $Q(t) \leq 0$, where $Q(t) = t^{2q} + 2t^q - 2t - 1$. The second derivative of Q is nonnegative on $[0,1]$, so Q attains its maximal value on $[0,1]$ at one of the endpoints. However, $Q(0) = -1$ and $Q(1) = 0$, so Q is nonpositive on $[0,1]$.

In such a way we get $ld(A,B) \leq K^2 b^{2q-1}\sec\alpha$. Since α goes to 0 as b goes to zero, using this inequality shows that if for the computation of the tryposity of Γ at O we use only the case of A and B on opposite sides of the origin, we would get

$$\limsup_{r\to 0} \frac{\gamma(B(O,r))}{r^{2q-1}} \le K^2.$$

Then, together with (7), we would get $\Phi_{2q-1}(\Gamma, O) = K^2$.

However, we still have to consider the case of both A and B on the same side of the origin (see Figure 4). The natural guess would be that we will not get anything new, but a surprise is awaiting us.

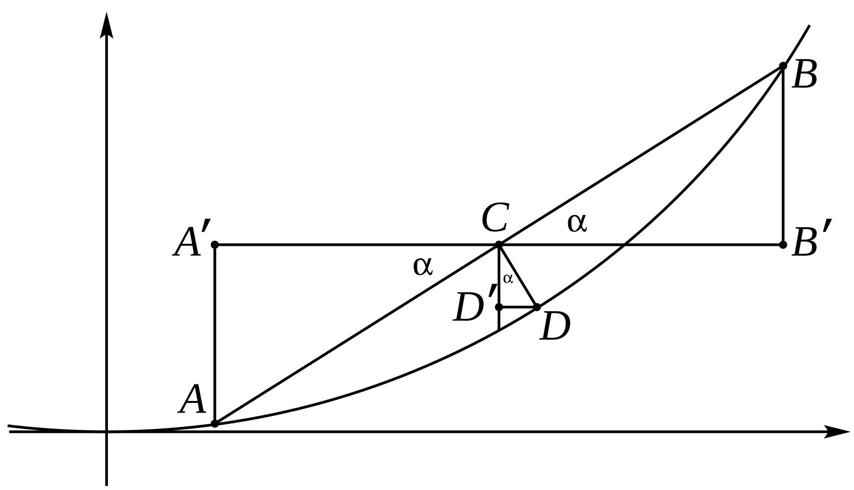

FIGURE 4. The case when A and B are on the same side of the origin.

We have now $0 \le a \le b$ and the points are $A = (a, Ka^q)$, $B = (b, Kb^q)$, $C = (c, \widehat{c}) = \left(\frac{b+a}{2}, K\frac{b^q+a^q}{2}\right)$ is the midpoint of AB, $D = (d, Kd^q)$ and CD is perpendicular to AB, $A' = (a, \widehat{c})$, $B' = (b, \widehat{c})$, $D' = (c, Kd^q)$.

As in the first case, we have

$$d(D', B') - d(C, B') = \big(d(D, B) - d(C, B)\big) \cos\alpha.$$

However, now estimating leipodistance from above does not suffice, so we have to make estimates from both sides. Fortunately for us, we only need to consider the limit as b goes to 0, and in the limit some of those estimates become equalities. For two expressions, ζ and ξ, that depend on a and b, we will write $\zeta \equiv \xi$ if the limit of ζ/ξ is 1 uniformly in a, when b goes to 0.

In particular, as b goes to 0, α also goes to zero (uniformly in a), so we get

$$d(D', B') - d(C, B') \equiv d(D, B) - d(C, B).$$

From the triangle CDD' we get $d - c = (\widehat{c} - Kd^q) \tan\alpha$. Moreover,

$$\frac{d^q - c^q}{d - c} < qd^{q-1} < qb^{q-1}.$$

Therefore,

$$1 < \frac{\widehat{c} - Kc^q}{\widehat{c} - Kd^q} = 1 + \frac{Kd^q - Kc^q}{\widehat{c} - Kd^q} = 1 + K\frac{d^q - c^q}{d - c} \tan\alpha < 1 + Kqb^{q-1} \tan\alpha,$$

so $\widehat{c} - Kc^q \equiv \widehat{c} - Kd^q$.

Now,

$$ld(A,B) = 2\big(d(D,B) - d(C,B)\big) \equiv 2\big(d(D',B') - d(C,B')\big)$$
$$= 2\left(\sqrt{\left(\frac{b-a}{2}\right)^2 + (\widehat{c} - Kd^q)^2} - \frac{b-a}{2}\right).$$

Let $a = tb$. We have

$$\frac{\widehat{c} - Kd^q}{b-a} \equiv \frac{\widehat{c} - Kc^q}{b-a} = \frac{K\frac{b^q+a^q}{2} - K\left(\frac{b+a}{2}\right)^q}{b-a} = Kb^{q-1}\frac{\frac{1+t^q}{2} - \left(\frac{1+t}{2}\right)^q}{1-t}.$$

By L'Hospital's rule,

$$\lim_{t\to 1}\frac{\frac{1+t^q}{2} - \left(\frac{1+t}{2}\right)^q}{1-t} = \lim_{t\to 1}\frac{\frac{qt^{q-1}}{2} - \frac{1}{2}q\left(\frac{1+t}{2}\right)^{q-1}}{-1} = -\frac{q}{2} + \frac{q}{2} = 0.$$

Therefore $\frac{\widehat{c}-Kd^q}{b-a}$ goes to 0 uniformly in a as b goes to zero. Thus,

$$\left(\sqrt{\left(\frac{b-a}{2}\right)^2 + (\widehat{c} - Kd^q)^2} - \frac{b-a}{2}\right) \Bigg/ \frac{(\widehat{c} - Kd^q)^2}{b-a} = \frac{2}{\sqrt{1 + \left(\frac{2(\widehat{c}-Kd^q)}{b-a}\right)^2} + 1}$$

goes to 1 uniformly in a as b goes to zero. Therefore

$$ld(A,B) \equiv 2\frac{(\widehat{c} - Kd^q)^2}{b-a} \equiv 2\frac{(\widehat{c} - Kc^q)^2}{b-a} = \frac{2\left(K\frac{b^q+a^q}{2} - K\left(\frac{b+a}{2}\right)^q\right)^2}{b-a}.$$

Set again $a = tb$. Then $ld(A,B) \equiv K^2 f_q(t) b^{2q-1}$, where

$$f_q(t) = \frac{2\left(\frac{1+t^q}{2} - \left(\frac{1+t}{2}\right)^q\right)^2}{1-t}.$$

Together with our earlier results from this section, and taking into account again that if $d(O,B) = r$ then $r \equiv b$, we get the following result. Denote by m_q the supremum of f_q on the interval $[0,1)$ (it is finite, since the limit of $f_q(t)$ at $t = 1$ is 0).

Theorem 4.1. *If the curve Γ in the plane is the graph of the function $y = K|x|^q$ for some real $q > 1$ and $K > 0$, then its flexion at the origin O has order $2q-1$ and $\Phi_{2q-1}(\Gamma, O) = K^2\max(1, m_q)$.*

Note that this agrees with Theorem 3.7, since $m_2 = 1/8$ and the curvature of the graph of the function $y = Kx^2$ at the origin is $2K$.

We get as a corollary to Theorem 4.1:

Corollary 4.2. *For every real $p > 1$ and $\Psi > 0$ there is a C^1 planar curve Γ and a point $P \in \Gamma$ such that the flexion of Γ at P has order p and $\Phi_p(\Gamma, P) = \Psi$.*

To get more information about m_q, let us look more closely at $f_q(t)$ as a function of $q > 1$ and $0 \le t < 1$. If we fix t and let q go to infinity, we get the limit $1/(2(1-t))$. If $t > 1/2$, this is larger than 1, and therefore, if q is large, then $\max(1, m_q) = m_q > 1$.

Let us show that for $0 \le t < 1$, the function $q \mapsto f_q(t)$ is increasing on $(1, \infty)$. This is clear if $t = 0$, so consider $t \in (0, 1)$. Set

$$g(t) = t \log t - (1+t) \log \frac{1+t}{2}$$

for $0 < t \le 1$. We have

$$g'(t) = \log t - \log \frac{1+t}{2} < 0$$

and $g(1) = 0$. Therefore $g(t) > 0$ for $t \in (0, 1)$. Thus,

$$\frac{\log t}{2 \log \frac{1+t}{2}} < \frac{1+t}{2t} < \left(\frac{1+t}{2t}\right)^q,$$

so

$$\frac{\partial}{\partial q}\left(\frac{1+t^q}{2} - \left(\frac{1+t}{2}\right)^q\right) = \frac{t^q}{2}\log t - \left(\frac{1+t}{2}\right)^q \log\frac{1+t}{2} > 0.$$

Moreover,

$$\frac{1+t^q}{2} > \left(\frac{1+t}{2}\right)^q,$$

because the function $t \mapsto t^q$ is strictly convex. This proves that the function $q \mapsto f_q(t)$ is increasing. In particular, the function $q \mapsto m_q$ is increasing.

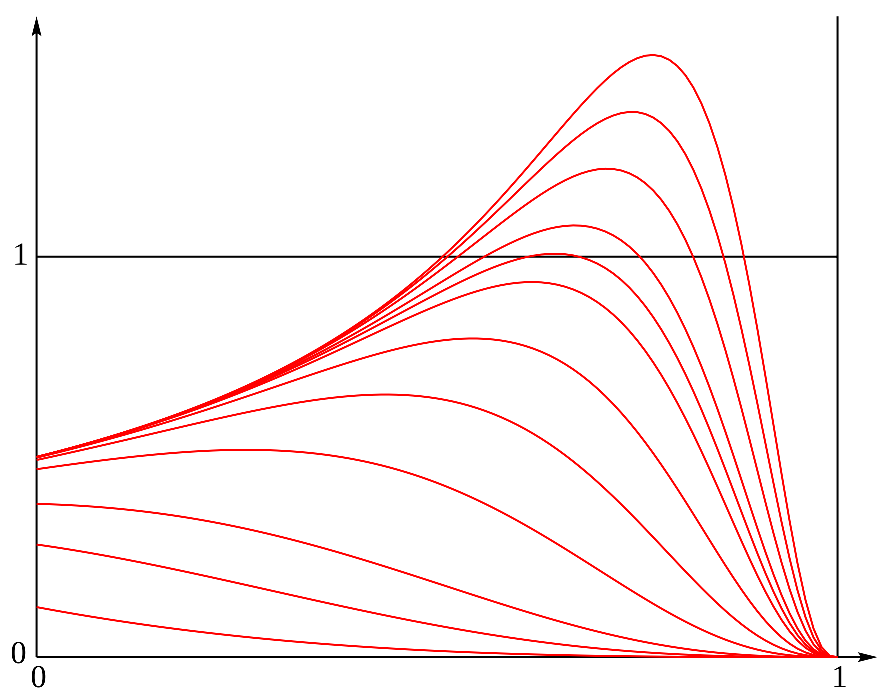

FIGURE 5. Graphs of functions f_q.

Numerical investigation of the functions f_q gives interesting results. In Figure 5 we see the graphs of the functions f_q for $q = 2, 3, 4, 6, 8, 10, 12, 13, 14, 16, 18, 20$ (from the bottom up). We see that $m_q < 1$ for $q \le 12$, but $m_q > 1$ for $q \ge 13$ (we have

$f_{13}(0.65) \approx 1.0071941657$). Since we consider all real $q > 1$, there is some critical $q \in (12, 13)$ for which $m_q = 1$.

5. V-curves

Now we consider the simplest family of nonsmooth curves, namely two rays joined at their endpoints. Then for any point other than the singularity, any sufficiently small ball around such a point is a segment and thus has gap zero. In contrast, at the singularity, the gap is positive and proportional to the ball's radius.

Theorem 5.1. *For $c > 0$, let V_θ be the graph of $y = c|x|$ in the plane, where θ is the angle in the first quadrant between V_θ and the horizontal axis, so that $c = \tan\theta$ (see Figure 6). Then the flexion of V_θ at the origin O has order 1 and the 1-flexion there is*

$$2\left(1 - \frac{1}{\sqrt{1+c^2}}\right) = 2(1-\cos\theta) \quad \text{if} \quad 0 < c \leq 1,$$

and

$$2\left(\frac{2c}{1+c^2} - \frac{1}{\sqrt{1+c^2}}\right) = 2(2\sin\theta - 1)\cos\theta \quad \text{if} \quad c > 1$$

(see Figure 7).

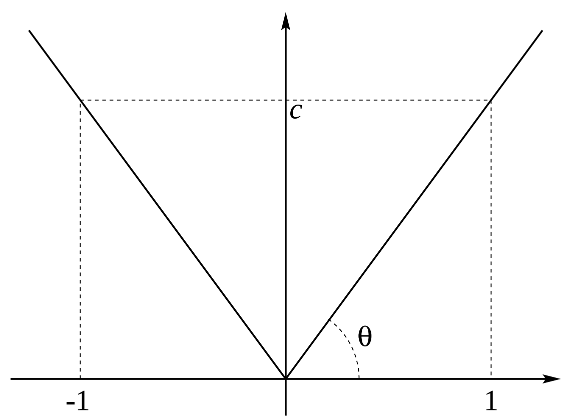

Figure 6. The curve V_θ.

We write V for V_θ, then V^+ for the part of V in the first quadrant (its right arm), and V^- for the part in the second quadrant (its left arm), so that $V = V^+ \cup V^-$.

Let $P_\pm$ be the points in $B(O, r)$ at distance r from the origin, with $P_+ \in V^+$ and $P_- \in V^-$. Thus, $P_\pm = (\pm a, c|a|)$, where $a = r\cos\theta$.

We will prove that the gap of $B(O, r)$ is witnessed at the endpoints P_- and P_+, that is,

$$\gamma(B(O, r)) = ld(P_-, P_+). \tag{8}$$

Note that a homothety scales leipodistances accordingly. Thus, in order to prove (8) it suffices to prove it in the case that $a = 1$, i.e. $r = \sec\theta$, so $P_\pm = (\pm 1, c)$. For the same reason, at least one of any pair of points with maximal leipodistance in $B(O, r)$ is an endpoint. Thus, by symmetry we can take one of our two points to be P_-.

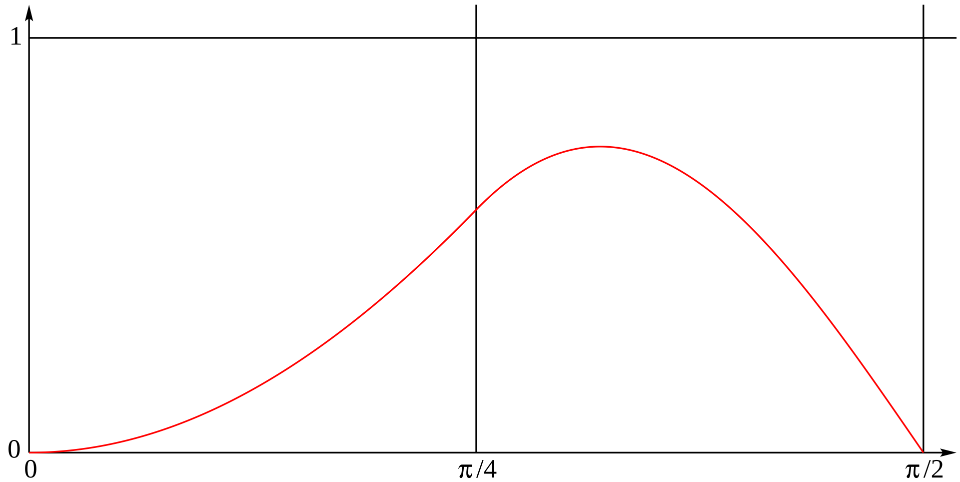

FIGURE 7. Flexion as a function of the angle θ.

Then clearly the second point giving maximal leipodistance cannot also be on V^-. Let $Q = (x, cx)$, with $0 \leq x \leq 1$, be a point in V^+. We want to show that $ld(P_-, Q)$ is maximized when $Q = P_+$, i.e. when $x = 1$.

Consider first the case of $c > 1$. For a given point $Q = (x, cx)$ on the right arm let $Q^- = (-y, cy)$ be the point on the left arm such that the segment joining Q with Q^- is perpendicular to the left arm (see Figure 8). Then the vectors $(x+y, c(x-y))$ and $(-1, c)$ are perpendicular, so $-x - y + c^2x - c^2y = 0$. Therefore,

$$y = \frac{c^2 - 1}{c^2 + 1}x. \tag{9}$$

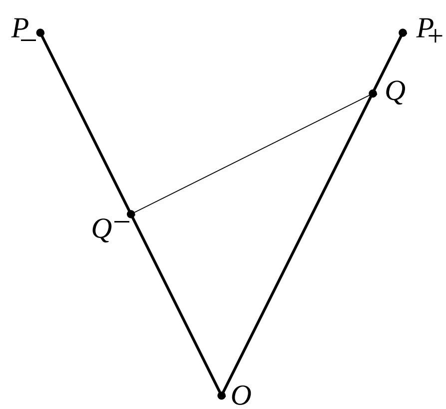

FIGURE 8. The case $c > 1$, $x > x_0$.

There is a unique point $Q_0 = (x_0, cx_0)$ for which $d(P_-, Q_0^-) = d(Q_0, Q_0^-)$, where $Q_0^- = (-y_0, cy_0)$ is the point Q^- as above corresponding to Q_0. We have $d(P_-, Q_0^-) = (1-y_0)\sqrt{c^2+1}$ and $d(Q_0, Q_0^-) = \sqrt{(x_0 + y_0)^2 + c^2(x_0 - y_0)^2}$. By (9), we get $d(Q_0, Q_0^-) =$

$2cx_0/\sqrt{c^2+1}$, so

$$x_0 = \frac{c^2+1}{c^2+2c-1}.$$

If $x \ge x_0$ then $hd(P_-,Q) = 2d(Q,Q^-)$; if $x \le x_0$ then $hd(P_-,Q) = 2d(Q,S)$, where S is the point on the left arm for which $d(Q,S) = d(P_-,S)$ (see Figure 9). Observe that if $c \in [0,1]$ then for all x we have $hd(P_-,Q) = 2d(Q,S)$.

Lemma 5.2. *Assume that $c > 1$ and $x_0 \le x < 1$. Then $ld(P_-,Q) < ld(P_-,P_+)$.*

Proof. We compute $d(Q,Q^-)$ in the same way as $d(Q_0,Q_0^-)$, so $d(Q,Q^-) = 2cx/\sqrt{c^2+1}$. Thus,

$$ld(P_-,Q) = \frac{4cx}{\sqrt{c^2+1}} - \sqrt{(x+1)^2 + c^2(1-x)^2}.$$

For $Q = P_+$ we have $x = 1$, so

$$ld(P_-,P_+) = \frac{4c}{\sqrt{c^2+1}} - 2. \tag{10}$$

Therefore, we have to prove that

$$2 - \frac{4c(1-x)}{\sqrt{c^2+1}} < \sqrt{(x+1)^2 + c^2(1-x)^2}. \tag{11}$$

We square both sides of (11), rearrange it and divide by $(1-x)$ (which, as we know, is positive), and get an equivalent (under our assumptions) inequality

$$3 + x < \frac{16c}{\sqrt{c^2+1}} + \left(c^2 - \frac{16c^2}{c^2+1}\right)(1-x). \tag{12}$$

At $x = 0$, (12) becomes $4 < 16c/\sqrt{c^2+1}$, which holds because $c \ge 1$. At $x = 0$, (12) is equivalent to (11) for $x = 0$, that is,

$$2 - \frac{4c}{\sqrt{c^2+1}} < \sqrt{c^2+1}.$$

This inequality also holds because $c \ge 1$ (in fact, its left-hand side is negative).

Since inequality (12) is affine in x and is satisfied for $x = 0$ and $x = 1$, it is satisfied for all $x \in [0,1]$. □

As we mentioned, if $x \le x_0$ (and also for all x if $c \le 1$) we have $hd(P_-,Q) = 2d(Q,S)$. Let us express this quantity in geometric terms. Let $\alpha \in (0,\pi/2)$ be the angle between the left arm and the segment L joining P_- with Q, and let U be the point on L for which the segment from S to U is perpendicular to L. Then

$$d(Q,S) = d(P_-,S) = d(P_-,U)\sec\alpha = \frac{d(P_-,Q)}{2}\sec\alpha$$

(see Figure 9). Therefore,

$$ld(P_-,Q) = d(P_-,Q)(\sec\alpha - 1). \tag{13}$$

Lemma 5.3. *If $c > 1$ and $0 < x < x_0$, then $ld(P_-,Q) < ld(P_-,Q_0)$. Similarly, if $0 < c \le 1$ and $0 < x < 1$, then $ld(P_-,Q) < ld(P_-,P_+)$.*

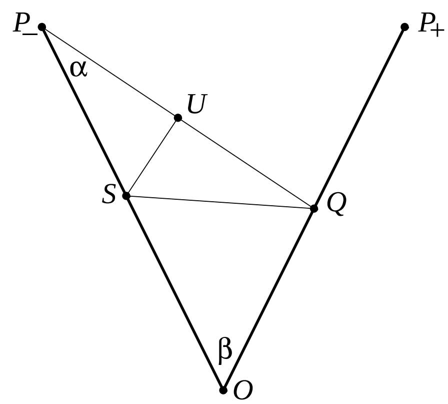

FIGURE 9. The case $c > 1$, $x < x_0$.

Proof. Let β be the angle between the arms of V. By the Law of Sines, $d(P_-, Q) = \sin\beta d(O, Q)/\sin\alpha$, so

$$ld(P_-, Q) = \sin\beta \frac{\sec\alpha - 1}{\sin\alpha} d(O, Q).$$

Observe that (for a fixed β) $d(O, Q)$ is a strictly increasing function of α. Moreover, if $\alpha \in (0, \pi/2)$, then $(\sec\alpha - 1)/\sin\alpha$ is positive. The product of two positive strictly increasing functions is strictly increasing. Thus, in order to prove both parts of the lemma, it is enough to show that the function $g(\alpha) := (\sec\alpha - 1)/\sin\alpha$ is strictly increasing on $(0, \pi/2)$.

We have

$$g'(\alpha) = \frac{\tan^2\alpha + \cos\alpha - 1}{\sin^2\alpha}.$$

The function $h(\alpha) := \tan^2\alpha + \cos\alpha - 1$ has value 0 at $\alpha = 0$, and

$$h'(\alpha) = \sin\alpha(2\sec^3\alpha - 1) > 0$$

for all $\alpha \in (0, \pi/2)$. Thus, $g'(\alpha) > 0$ for all $\alpha \in (0, \pi/2)$, so g is strictly increasing on $(0, \pi/2)$. □

From Lemmas 5.2 and 5.3 we get easily (8).

Proposition 5.4. *We have $\gamma(B(O, r)) = ld(P_-, P_+)$.*

Proof. As we have observed, it has to be proved only for the case of $r = \sec\theta$. If $c > 1$ and $x_0 \le x < 1$, it follows from Lemma 5.2. If $0 < c \le 1$ and $0 < x < 1$, it follows from Lemma 5.3. If $c > 1$ and $0 < x < x_0$ then by Lemma 5.3 $ld(P_-, Q) < ld(P_-, Q_0)$. However, $ld(P_-, Q_0) < ld(P_-, P_+)$ by Lemma 5.2, and the desired inequality follows. □

Now we can prove Theorem 5.1.

Proof of Theorem 5.1. First suppose $0 < c \le 1$. We have $Q = P_+$, so $S = O$. Therefore, $hd(P_-, P_+) = 2r$, and we get

$$\gamma(B(0,r)) = ld(P_-, P_+) = hd(P_-, P_+) - d(P_-, P_+) = 2r - 2a$$
$$= 2\left(1 - \frac{1}{\sqrt{1+c^2}}\right) r = 2(1-\cos\theta)r.$$

Now suppose $c \ge 1$. Then we can use (10), remembering that it was obtained under the assumption that $r = \sqrt{1+c^2}$. Thus, in the general case we have

$$ld(P_-, P_+) = \frac{r}{\sqrt{1+c^2}}\left(\frac{4c}{\sqrt{1+c^2}} - 2\right) = 2\left(\frac{2c}{1+c^2} - \frac{1}{\sqrt{1+c^2}}\right) r.$$

Therefore,

$$\gamma(B(0,r)) = 2\left(\frac{2c}{1+c^2} - \frac{1}{\sqrt{1+c^2}}\right) r = 2\big((2\sin\theta - 1)\cos\theta\big) r.$$

□

Elementary computations give us an unexpected result.

Corollary 5.5. *The 1-flexion of V_θ at the origin is maximized when*

$$\theta = \sin^{-1}\left(\frac{1+\sqrt{33}}{8}\right) \approx 1.00296695386625,$$

i.e. when c is the largest zero of the polynomial $3c^4 - 9c^2 + 4$:

$$c = \sqrt{\frac{9+\sqrt{33}}{6}} \approx 1.5676182914716.$$

Thus, as θ varies, the 1-flexion of V_θ at the origin takes all values between 0 and

$$\frac{\sqrt{414 - 66\sqrt{33}}}{8} \approx 0.738017456956381,$$

but no larger values (see Figure 7).

6. Fractals

Let us now look at curves that are very non-smooth, namely fractal ones [Fa]. If we understand "fractal" as "self-similar", when we take smaller and smaller scales, we see the same picture scaled down. This means that we should expect that the tryposity (flexion) of our curve at every point will have order 1. In fact, this should be true for all self-similar sets, not only for curves. However, an exact statement of such result and a proof may depend heavily on what we really understand as "self-similarity". Therefore, here we will work with a concrete curve, namely with the quite popular *Koch curve*. Let us define it.

We construct it in steps. We start with the interval $[0, 1]$, then we remove its middle 1/3 and replace it by the two other sides of the equilateral triangle whose one side was the remove interval. Next we repeat the same operation with every segment of our curve, and we continue by induction (see Figure 10). In the limit we get a curve which we call the Koch curve and denote by $\mathcal{K}$.

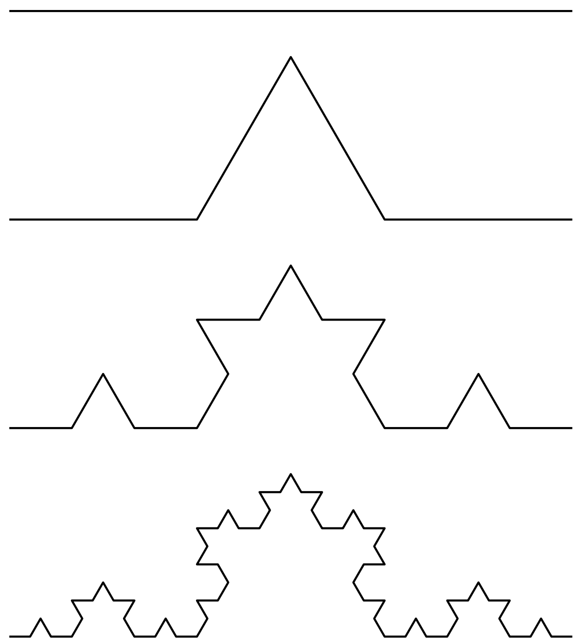

FIGURE 10. Construction of the Koch curve.

We have to estimate the gap of balls in the Koch curve from both sides. One estimate, namely

$$\gamma(B(x,r)) \le r, \tag{14}$$

follows immediately from Lemma 2.3.

The other estimate is more complicated.

Lemma 6.1. *If* $0 < r < 1$ *and* $x \in \mathcal{K}$ *then* $\gamma(B(x,r)) > \frac{\sqrt{3}-1}{9} r$.

Proof. Let $\mathcal{K}_n$ be the curve that we get in the nth step of the construction of $\mathcal{K}$ (so in Figure 10 we see $\mathcal{K}_n$ for $n = 0, 1, 2, 3$). It is clear that the diameter of $\mathcal{K}$ is 1. Therefore, the diameter of any part of $\mathcal{K}$ between two consecutive vertices of $\mathcal{K}_n$ is 3^{-n}. It follows that if $x \in \mathcal{K}$ and $r \ge 3^{-n}$ then $B(x,r)$ contains at least one part of the curve $\mathcal{K}$ between two consecutive vertices of $\mathcal{K}_n$. Call this part X. It looks like all of $\mathcal{K}$, scaled down by the factor 3^n. Let $y, z \in \mathcal{K}$ be the two vertices of $\mathcal{K}_{n+1}$, adjacent to the endpoints of X (see Figure 11). Their distance is $3^{-(n+1)}$, while their hyperdistance is larger than 2 times the altitude of the equilateral triangle whose one side is the segment joining y with z. Thus, $ld(y,z) > (\sqrt{3}-1)3^{-(n+1)}$.

In such a way we see that if $3^{-n} \le r < 3^{-(n-1)}$ then

$$\gamma(B(x,r)) > (\sqrt{3}-1)3^{-(n+1)} \ge \frac{\sqrt{3}-1}{9} r.$$

□

From the inequality (14) and Lemma 6.1 we get immediately the following theorem.

Theorem 6.2. *The flexion of the Koch curve* $\mathcal{K}$ *at any point has order* 1, *and at each point the lower* 1*-flexion is at least* $\frac{\sqrt{3}-1}{9}$, *while the upper* 1*-flexion is at most* 1.

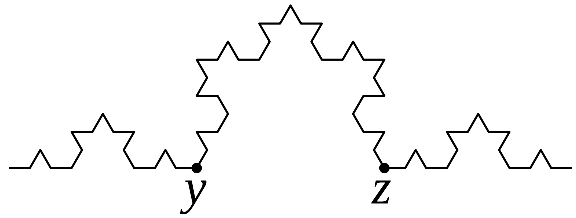

FIGURE 11. Points y, z, where the visible curve is an approximation to X.

As we said, the idea that the tryposity order should be 1 for fractals should apply not only to curves, but also to other spaces. Let us investigate what happens to various Cantor sets (subsets of the real line homeomorphic to the standard middle third Cantor set).

Let X be our space. As always, when we write $B(x, r)$, we mean a closed ball in X. By the definition, in order to check whether the order of the tryposity at all points is 1, we only have to check whether

$$\liminf_{r\to 0} \frac{\gamma(B(x,r))}{r} > 0 \tag{15}$$

for every $x \in X$. If this property is satisfied (that is, if the tryposity order is 1 at all points), we will say that X is a *trypofractal*.

Sometimes we can expect an even stronger property:

$$\liminf_{r\to 0} \inf_{x\in X} \frac{\gamma(B(x,r))}{r} > 0. \tag{16}$$

Then we will say that X is a *uniform trypofractal*.

Proposition 6.3. *Let X be the standard middle third Cantor set. Then, for every $x \in X$ and $r \in (0, 1]$ we have $\gamma(B(x, r)) \geq r/9$, so X is a uniform trypofractal.*

Proof. When $3^{-n} \leq r \leq 3^{1-n}$ for some $n \geq 1$ and $x \in X$, then $B(x, r)$ is the intersection of X with a closed interval, and this interval contains an interval of nth generation (of length 3^{-n}) from the construction of X. In this interval there is a gap[3] of $(n + 1)$st generation. This gap has length 3^{-n-1}, so by Lemma 2.9, $\gamma(B(x, r)) \geq 3^{-n-1} = 3^{1-n}/9 \geq r/9$. □

Example 6.4. Let us modify slightly Example 2.10, by replacing each point a_n by a standard Cantor set containing a_n, scaled down linearly is such a way that its diameter is smaller than a_n^2. Then the limits (4) will not change and we see that again the tryposity at 0 does not exist. Thus, it is not a trypofractal. However, our set has no isolated points and has a countable base consisting of clopen sets, so by Brouwer's Theorem [Br] it is a Cantor set. ♢

We will say that X is a fractal if for every nonempty open subset $U \subset X$ the Hausdorff dimension of U is larger than its topological dimension. However, readers are welcome to use their favorite definition of a fractal.

[3] Here we are talking about the gaps in the Cantor set, not about the gap of any space.

Proposition 6.3 gives us an example of a set that is both fractal and trypofractal. Example 6.4 is an example of a fractal that is not a trypofractal. A remaining question is whether every trypofractal is a fractal. This question can be asked for arbitrary compact metric spaces, or only for Cantor sets.

7. Non-Euclidean curves

Let us now consider some curves that cannot be embedded into Euclidean spaces. Take an interval and define a metric on it that makes it homeomorphic to the same interval with the usual metric. We will make this interval locally homogeneous. That is, any two points have small neighborhoods which are isometric to each other. This means that the flexion will be the same at each point. This is in contrast to the results of Sections 4 and 5, where we obtained special flexion at one point.

Our metric will be of the form $d(x,y) = f(|x-y|)$, and the space will be $(0,\varepsilon)$.

Lemma 7.1. *Assume that $f : (0,\varepsilon) \to (0,\infty)$ is a strictly increasing function of class C^1, such that $\lim_{x\to 0} f(x) = 0$ and f' is decreasing. Then the function $d(x,y) = f(|x-y|)$ if $x \neq y$ and $d(x,x) = 0$ is a metric on $(0,\varepsilon)$ and $(0,\varepsilon)$ with this metric is homeomorphic to an interval.*

Proof. The only property of the metric that is not obvious is the triangle inequality. Suppose $0 < x < y < z < \varepsilon$. Then $d(x,z) > d(x,y)$ and $d(x,z) > d(y,z)$, so we only have to check that $d(x,z) \le d(x,y) + d(y,z)$. Let $y - x = a$ and $z - y = b$. Then we have to prove that $\Psi(a) \ge 0$, where $\Psi(a) = f(a) + f(b) - f(a+b)$, under the assumptions that $a, b > 0$ and $a + b < \varepsilon$. We have $\Psi'(a) = f'(a) - f'(a+b) \ge 0$, because f' is decreasing. Moreover, $\lim_{a\to 0} \Psi(a) = f(b) - f(b) = 0$. Thus, $\Psi(a) \ge 0$.

The property that our space is homeomorphic to an interval follows immediately from the fact that f is strictly increasing and $\lim_{x\to 0} f(x) = 0$. □

First we will construct a metric in which the flexion has order 1 and 1-flexion takes any prescribed value from $(0,1)$.

Example 7.2. Fix $q \in (0,1)$ and set $\varepsilon = 1$ and $f(x) = x^q$. Then the assumptions of Lemma 7.1 are satisfied.

If $0 < x < y < z < 1$ and $d(x,y) = d(y,z) = r$, we get $y - x = z - y = r^{1/q}$, so $z - x = 2r^{1/q}$. Hence, $d(x,z) = 2^q r$, and it is easy to see that $hd(x,z) = 2r$. Therefore, $ld(x,z) = (2 - 2^q)r$, so it is clear that the flexion at y has order 1 and 1-flexion is $2 - 2^q$. In particular, our interval is a trypofractal (in fact, it is even a uniform trypofractal). This is not surprising, because its nonempty open sets have Hausdorff dimension $1/q$ and topological dimension 1, so it is a fractal. ♢

Next we construct a metric in which the flexion has order 1 and 1-flexion is 1.

Example 7.3. Set $\varepsilon = e^{-2}$ and $f(x) = -\frac{1}{\log x}$. We have $f'(x) = \frac{1}{x(\log x)^2} > 0$. Moreover,

$$f''(x) = -\frac{2 + \log x}{x^2(\log x)^3}$$

is negative if $0 < x < e^{-2}$. Therefore, the assumptions of Lemma 7.1 are satisfied.

If $0 < x < y < z < 1$ and $d(x,y) = d(y,z) = r$, we get $y - x = z - y = e^{-1/r}$, so $z - x = 2e^{-1/r}$. Hence,

$$d(x,z) = \frac{-1}{\log(2e^{-1/r})} = \frac{-1}{\log 2 - \frac{1}{r}},$$

and it is easy to see that $hd(x,z) = 2r$. Therefore,

$$ld(x,z) = 2r - \frac{-1}{\log 2 - \frac{1}{r}} = \left(2 + \frac{1}{r\log 2 - 1}\right) r,$$

and since

$$\lim_{r\to 0}\left(2 + \frac{1}{r\log 2 - 1}\right) = 1,$$

it is clear that the flexion at y has order 1 and 1-flexion is 1.

Observe that here nonempty open subsets of our space have infinite Hausdorff dimension. ♢

Finally, we get flexions of any prescribed order $p > 1$ and p-flexion of any prescribed positive value.

Example 7.4. Fix $p > 1$ and $K > 0$. Set $\varepsilon = (Kp)^{\frac{-1}{p-1}}$ and $f(x) = x - Kx^p$. We have $f'(x) = 1 - Kpx^{p-1} > 0$ for $x \in (0,\varepsilon)$. Therefore, the assumptions of Lemma 7.1 are satisfied.

If $0 < x < y < z < \varepsilon$ and $y - x = z - y = a$, then $d(x,z) = 2a - K(2a)^p$ and it is easy to see that $hd(x,z) = 2(a - Ka^p)$. Thus, $ld(x,z) = K(2^p - 2)a^p$, and since the limit of $(a - Ka^p)/a$ is 1 as a goes to 0, we get flexion at y of order p, and p-flexion equal to $K(2^p - 2)$. ♢

8. Questions

Let us look at our results and try to ask some natural questions.

In Section 3 we assume that the curve is of class C^3. The third derivative is used in the proof of Theorem 3.2. But maybe this theorem is also true if we assume only that the curve is of class C^2? Or maybe, even if it is not true, the main results are true under this weaker assumption?

In Section 4, for all orders $p > 1$ we get all positive p-flexions for one point of a C^1 planar curve. However, we got this by considering the concrete examples $y = K|x|^q$. Can one get this for a much more general class of examples?

In Section 5 it is notable that the construction does not yield 1-flexions close to 1. Can we get them arbitrarily close to 1 for planar curves (with order 1)? Or at least closer to 1 than this strange number $\frac{\sqrt{414-66\sqrt{33}}}{8}$? And can we get 1-flexion 1? It is far from clear how to try to construct such curves.

In Section 6 we get estimates for the upper and lower 1-flexion for the Koch curve. But are there points where the upper and lower 1-flexion are equal? Or maybe this happens for every point? And how do those 1-flexions depend on the point? In fact, similar questions can be asked even for the standard Cantor set, but there they should be relatively simple.

References

[BM] L. M. Blumenthal and K. Menger, "Studies in Geometry," Freeman and Co., San Francisco, 1970.

[Br] L. E. J. Brouwer, *On the structure of perfect sets of points*, Proc. Kon. Ned. Akad. Wet., **12** (1910), 785–794.

[Fa] K. Falconer, "Fractal Geometry: Mathematical Foundations and Applications," J. Wiley and Sons, Chichester, 2014.

[Ha] J. Haantjes, *Distance geometry. Curvature in abstract metric spaces.*, Proc. Kon. Ned. Akad. Wet., **50** (1947), 496–508.

[Me] K. Menger, *Untersuchungen uber allgemeine Metrik. Vierte Untersuchung. Zur Metrik der Kurven*, Math. Annalen, **103** (1930), 466–501.

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