TOPOLOGICAL ENTROPY OF GENERALIZED BUNIMOVICH STADIUM BILLIARDS

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Abstract. We estimate from below the topological entropy of the generalized Bunimovich stadium billiards. We do it for long billiard tables, and find the limit of estimates as the length goes to infinity. We also get estimates for some shorter tables. We repeat this for generalized semistadium billiards, including the mushroom ones.

1. Introduction

In this paper, we generalize the results of [6] to a much larger class of billiards. They are similar to Bunimovich stadium billiards (see [3]), but the semicircles are replaced by almost arbitrary curves. That is, those curves are not completely arbitrary, but the assumptions on them is very mild. An example of such curves is shown in Figure 1. We also consider the case when one of the curves is a vertical line segment. This class includes Bunimovich mushroom billiards.

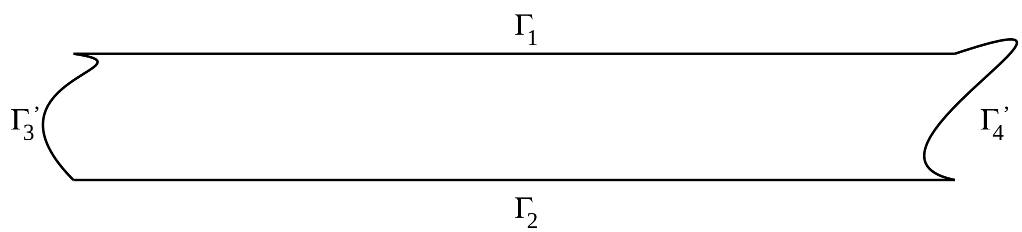

Figure 1. Generalized Bunimovich stadium.

We consider billiard maps (not the flows) for two-dimensional billiard tables. Thus, the phase space of a billiard is the product of the boundary of the billiard table and the interval $[-\pi/2, \pi/2]$ of angles of reflection. This phase space will be denoted as $\mathcal{M}$. We will use the variables (r, φ), where r parametrizes the table boundary by the arc length, and φ is the angle of reflection. Those billiards have the natural measure; it is $c \cos\varphi \, dr \, d\varphi$, where c is the normalizing constant. This measure is invariant for the billiard map.

However, we will not be using this measure, but rather investigate our system as a topological one. The first problem one encounters with this approach is that the map can be discontinuous, or even not defined at certain points. In particular, if we

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want to define topological entropy of the system, we may use one of several methods, but we cannot be sure that all of them will give the same result.

To go around this problem, similarly as in [6], we consider a compact subset of the phase space, invariant for the billiard map, on which the map is continuous. Thus, the topological entropy of the billiard map, no matter how defined, is larger than or equal to the topological entropy of the map restricted to this subset.

Positive topological entropy is recognized as one of the forms of chaos. In fact, topological entropy even measures how large this chaos is. Hence, whenever we prove that the topological entropy is positive, we can claim that the system is chaotic from the topological point of view.

We will be using similar methods as in [6]. However, the class of billiards to which our results can be applied, is much larger. The class of Bunimovich stadium billiards, up to similarities, depends on one positive parameter only. Our class is enormously larger, although we keep the assumption that two parts of the billiard boundary are parallel segments of straight lines. Nevertheless, some of our proofs are simpler than those in [6].

2. Assumptions

We will think about the billiard table positioned as in Figure 1. Thus, we will use the terms *horizontal, vertical, lower, upper, left, right.* While we are working with the billiard map, we will also look at the billiard flow. Namely, we will consider *trajectory lines*, that is, line segments between two consecutive reflections from the table boundary. For such a trajectory line (we consider it really as a line, not a vector) we define its *argument* (as an argument of a complex number), which is the angle between the trajectory line and a horizontal line. For definiteness, we take the angle from $(-\pi/2, \pi/2]$. We will be also speaking about the arguments of lines in the plane. Moreover, for $x \in \mathcal{M}$, we define the argument of x as the argument of of the trajectory line joining x with its image.

We will assume that the boundary of billiard table is the union of four curves, Γ_1, Γ_2, Γ_3' and Γ_4'. The curves Γ_1 and Γ_2 are horizontal segments of straight lines, and Γ_2 is obtained from Γ_1 by a vertical translation. The curve Γ_3' joins the left endpoints of Γ_1 and Γ_2, while Γ_4' joins the right endpoints of Γ_1 and Γ_2 (see Figure 1). We will consider all four curves with endpoints, so they are compact.

For $\varepsilon \geq 0$, we will call a point $p \in \Gamma_i'$ ($i \in \{3, 4\}$) *ε-free* if any forward trajectory of the flow (here we mean the full forward trajectory, not just the trajectory line), beginning at p with a trajectory line with argument whose absolute value is less than or equal to ε, does not collide with Γ_i' before it collides with Γ_{7-i}'.

Further, we will call a subarc $\Gamma_i \subset \Gamma_i'$ *ε-free* (see Figure 2) if:

(a) Γ_i is of class C^1,
(b) every point of Γ_i is ε-free,
(c) there are points $p_{i+}, p_{i-} \in \Gamma_i$ such that the argument of the line normal to Γ_i is larger than or equal to ε at p_{i+} and less than or equal to $-\varepsilon$ at p_{i-} (see Figure 2),
(d) Γ_i is disjoint from $\Gamma_1 \cup \Gamma_2$.

Clearly, if Γ_i is ε-free then it is also δ-free for all $\delta \in (0, \varepsilon)$.

Our last assumption is that there is $\varepsilon > 0$ and ε-free subarcs $\Gamma_i \subset \Gamma_i'$ for $i = 3, 4$, such that $\Gamma_3 \cup \Gamma_4$ is disjoint from $\Gamma_1 \cup \Gamma_2$. We will denote the class of billiard tables satisfying all those assumptions by $\mathcal{H}(\varepsilon)$.

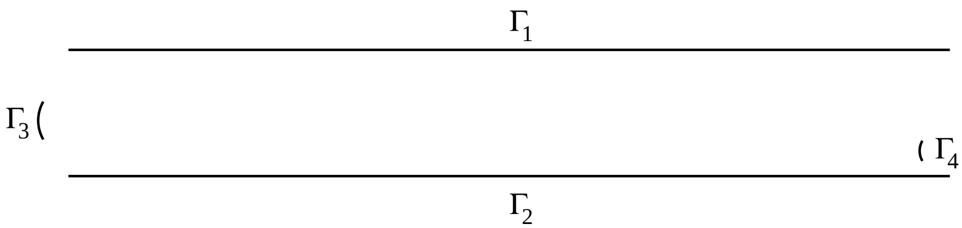

FIGURE 2. Curves Γ_i, $i = 1, 2, 3, 4$.

Observe that there are two simple situations when we know that there is $\varepsilon > 0$ such that Γ_i' has an ε-free subarc. One is when there is a 0-free point $p_i \in \Gamma_i'$ such that there is a neighborhood of p_i where Γ_i is of class C^1 and the curvature of Γ_i at p_i exists and is non-zero (see Figure 3). The other one is when Γ_i' is the graph of a non-constant function $x = f(y)$ of class C^1 (then we take a neighborhood of a point where f attains its extremum; this neighborhood may be large if the extremum is attained on an interval), like Γ_3' (but not Γ_4') in Figure 1.

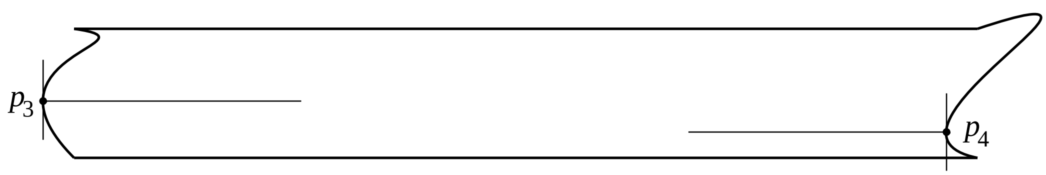

FIGURE 3. Points p_3 and p_4.

We forget about the other parts of the curves Γ_i' and look only at Γ_i, $i = 1, 2, 3, 4$ (see Figure 2).

Let us mention that since we will be using only those four pieces of the boundary of the billiard table, it does not matter whether the rest of the boundary is smooth or not. If it is not smooth, we can include it (times $[-\pi/2, \pi/2]$) into the set of singular points, where the billiard map is not defined.

3. CODING

We consider a billiard table from the class $\mathcal{H}(\varepsilon)$. Since transforming the table by homothety does not change the entropy, we may assume that the distance between Γ_1 and Γ_2 is 1. Now we can introduce a new characteristic of our billiard table. We will say that a billiard table from the class $\mathcal{H}(\varepsilon)$ is in the class $\mathcal{H}(\varepsilon, \ell)$ if the horizontal distance between Γ_3 and Γ_4 is at least ℓ. We can think about ℓ as a big number (it will go to infinity).

We start with a trivial geometrical fact, that follows immediately from the rule of reflection. We include the assumption that the absolute values of the arguments are smaller than $\pi/6$ in order to be sure that the absolute value of the argument of T_2 is smaller than $\pi/2$.

Lemma 3.1. *If T_1 and T_2 are incoming and outgoing parts of a trajectory reflecting at q and the argument of the line normal to the boundary of the billiard at q is α, and $|\alpha|, |\arg(T_1)| < \pi/6$, then* $\arg(T_2) = 2\alpha - \arg(T_1)$.

We consider only trajectories that reflect from the curves Γ_i, $i = 1, 2, 3, 4$. In order to have control over this subsystem, we fix an integer $N > 1$ and denote by $\mathcal{K}_{\ell,N}$ the space of points whose (discrete) trajectories go only through Γ_i, $i = 1, 2, 3, 4$ and have no $N + 1$ consecutive collisions with the straight segments.

We can unfold the billiard table by using reflections from the straight segments (see Figure 4). The liftings of trajectories (of the flow) consist of segments between points of liftings of Γ_3 and Γ_4. In $\mathcal{K}_{\ell,N}$ they go at most N levels up or down.

FIGURE 4. Five levels of the unfolding. Only Γ_3 and Γ_4 are shown instead of Γ_3' and Γ_4'.

Now for a moment we start working on the lifted billiard. That is, we consider only Γ_3 and Γ_4, but at all levels, as pieces of the boundary from which the trajectories of the flow can reflect. We denote those pieces by $\Gamma_{i,k}$, where $i \in \{3, 4\}$ and $k \in \mathbb{Z}$. Clearly, flow trajectories from some points (r, φ) will not have more collisions, so the lifted billiard map F will be not defined at such points. We denote by $\widetilde{\mathcal{M}}$ the product of the union of all sets $\Gamma_{i,k}$ and the interval $[\pi/2, \pi/2]$.

Now we specify how large ℓ should be for given N, ε in order to get nice properties of the billiard map restricted to $\mathcal{K}_{\ell,N}$.

Assume that our billiard table belongs to $\mathcal{H}(\varepsilon,\ell)$ and fix $i\in\{3,4\}$, $k\in\mathbb{Z}$. Call a continuous map $\gamma:[a,b]\to\widetilde{\mathcal{M}}$, given by

$$\gamma(t)=(\gamma_r(t),\gamma_\varphi(t)),$$

an *(i,k,ε)-curve* if $\gamma_r([a,b])=\Gamma_{i,k}$ and for every $t\in[a,b]$ the absolute value of the argument of the trajectory line *incoming to* $\gamma(t)$ is at most ε. We can think of γ as a bundle of trajectories of a flow incoming to $\Gamma_{i,k}$. In order to be able to use Lemma 3.1, we will always assume that $\varepsilon<\pi/6$.

Lemma 3.2. *Assume that the billiard table belongs to $\mathcal{H}(\varepsilon,\ell)$ and fix $N\geq 0$, $i\in\{3,4\}$, $k\in\mathbb{Z}$, and $j\in\{-N,-N+1,\dots,N-1,N\}$. Assume that*

$$\ell\geq\frac{N+1}{\tan\varepsilon}\tag{1}$$

Then every (i,k,ε)-curve γ has a subcurve whose image under F (that is, $F\circ\gamma|_{[a',b']}$ for some subinterval $[a',b']\subset[a,b]$) is a $(7-i,k+j,\varepsilon)$-curve.

Proof. There are points $c_-,c_+\in[a,b]$ such that $\gamma_r(c_-)$ is a lifting of p_{i-} and $\gamma_r(c_+)$ is a lifting of p_{i+}. Then, by Lemma 3.1, the lifted trajectory line outgoing from $\gamma(c_-)$ (respectively, $\gamma(c_+)$) has argument smaller than $-\varepsilon$ (respectively, larger than ε). Since the direction of the line normal to $\Gamma_{i,k}$ at the point $\gamma_r(t)$ varies continuously with t, the argument of the lifted trajectory line outgoing from $\gamma(t)$ also varies continuously with t. Therefore, there is a subinterval $[a'',b'']\subset[a,b]$ such that at one of the points a'',b'' this argument is $-\varepsilon$, at the other one is ε, and in between is in $[-\varepsilon,\varepsilon]$. When the bundle of lifted trajectory lines starting at $\gamma([a'',b''])$ reaches liftings of Γ_{7-i}, it collides with all points of $\Gamma_{7-i,k+j}$ whenever $j+1\leq\ell\tan\varepsilon$. By (1), this includes all j with $j\leq N$. Therefore, there is a subinterval $[a',b']\subset[a'',b'']$ such that $(F\circ\gamma)_r([a',b'])=\Gamma_{7-i,k+j}$. The arguments of the lifted trajectory lines incoming to $(F\circ\gamma)([a',b'])$ are in $[-\varepsilon,\varepsilon]$, so we get a $(7-i,k+j,\varepsilon)$-curve. □

Using this lemma inductively we get immediately the next lemma.

Lemma 3.3. *Assume that the billiard table belongs to $\mathcal{H}(\varepsilon,\ell)$ and fix $N\geq 0$ such that* (1) *is satisfied. Then for every finite sequence*

$$(k_{-j},\dots,k_{-1},k_0,k_1,\dots,k_j)$$

of integers with absolute values at most N there is a trajectory piece in the lifted billiard going between liftings of Γ_3 and Γ_4 with the differences of levels $k_{-j},\dots,k_{-1},k_0,k_1,\dots,k_j$.

Note that in the above lemma we are talking about trajectory pieces of length $2j+1$, without requiring that those pieces can be extended backward or forward to a full trajectory.

Proposition 3.4. *Under the assumption of Lemma 3.3, for every two-sided sequence*

$$(\dots,k_{-2},k_{-1},k_0,k_1,k_2,\dots)$$

of integers with absolute values at most N there is a trajectory in the lifted billiard going between liftings of Γ_3 and Γ_4 with the differences of levels $\dots,k_{-2},k_{-1},k_0,k_1,k_2,\dots$.

Proof. For every finite sequence $(k_{-j}, \dots, k_{-1}, k_0, k_1, \dots, k_j)$ the set of points of $\Gamma_3 \times [-\pi/2, \pi/2]$ or $\Gamma_4 \times [-\pi/2, \pi/2]$ whose trajectories from time $-j$ to j exist and satisfy Lemma 3.3 is compact and nonempty. As j goes to infinity, we get a nested sequence of compact sets. Its intersection is the set of points whose trajectories behave in the way we demand, and it is nonempty. □

Consider the following subshift of finite type $(\Sigma_{\ell,N}, \sigma)$. The states are

$$-N, -N+1, \dots, -1, 0, 1, \dots, N-1, N,$$

and the transitions are: from 0 to 0, 1 and -1, from i to $i+1$ and 0 if $1 \le i \le N-1$, from N to 0, from $-i$ to $-i-1$ and 0 if $1 \le i \le N-1$, and from $-N$ to 0. Each trajectory of a point from $\mathcal{K}_{\ell,N}$ can be coded by assigning the symbol 0 to $\Gamma_3 \cup \Gamma_4$ and for the parts between two zeros either $1, 2, \dots, j$ if the the first point is in Γ_1, or $-1, -2, \dots, -j$ if the first point is in Γ_2. This defines a map from $\mathcal{K}_{\ell,N}$ to $\Sigma_{\ell,N}$. This map is continuous, because the preimage of every cylinder is open (this follows immediately from the fact that the straight pieces of our trajectories of the billiard flow intersect the arcs Γ_i, $i = 1, 2, 3, 4$, only at the endpoints of those pieces, and that the arcs are disjoint). It is a surjection by Proposition 3.4. Therefore it is a semiconjugacy, and therefore, the topological entropy of the billiard map restricted to $\mathcal{K}_{\ell,N}$ is larger than or equal to the topological entropy of $(\Sigma_{\ell,N}, \sigma)$.

4. Computation of topological entropy

In the preceding section we obtained a subshift of finite type. Now we have to compute its topological entropy. If the alphabet of a subshift of finite type is $\{1, 2, \dots, n\}$, then we can write the *transition matrix* $M = (m_{ij})_{i,j=1}^n$, where $m_{ij} = 1$ if there is a transition from i to j and $m_{ij} = 0$ otherwise. Then the topological entropy of our subshift is the logarithm of the spectral radius of M (see [5, 1]).

In the case of large, but not too complicated, matrices, in order to compute the spectral radius one can use the *rome method* (see [2, 1]). For the transition matrices of $(\Sigma_{\ell,N}, \sigma)$ this method is especially simple. Namely, if we look at the paths given by transitions, we see that 0 is a rome: all paths lead to it. Then we only have to identify the lengths of all paths from 0 to 0 that do not go through 0 except at the beginning and the end. The spectral radius of the transition matrix is then the largest zero of the function $\sum x^{-p_i} - 1$, where the sum is over all such paths and p_i is the length of the i-th path.

Lemma 4.1. *Topological entropy of the system $(\Sigma_{\ell,N}, \sigma)$ is the logarithm of the largest root of the equation*

$$x^2 - 2x - 1 = -2x^{-N}. \tag{2}$$

Proof. The paths that we mentioned before the lemma, are: one path of length 1 (from 0 directly to itself), and two paths of length $2, 3, \dots, N+1$ each. Therefore, our entropy is the logarithm of the largest zero of the function $2(x^{-(N+1)} + \dots + x^{-3} + x^{-2}) + x^{-1} - 1$. We have

$$x(1-x)\big(2(x^{-(N+1)} + \dots + x^{-3} + x^{-2}) + x^{-1} - 1\big) = (x^2 - 2x - 1) + 2x^{-N},$$

so our entropy is the logarithm of the largest root of equation (2). □

Corollary 4.2. *Assume that the billiard table belongs to* $\mathcal{H}(\varepsilon, \ell)$ *and fix* $N \geq 0$ *such that* (1) *is satisfied. Then the topological entropy of the billiard map restricted to* $\mathcal{K}_{\ell,N}$ *is larger than or equal to the logarithm of the largest root of equation* (2)*.*

A particular case of this corollary gives us a sufficient condition for positive topological entropy. Namely, notice that the largest root of the equation $x^2 - 2x - 1 = -2x^{-1}$ is 2.

Corollary 4.3. *Assume that the billiard table belongs to* $\mathcal{H}(\varepsilon, \ell)$ *and* $\ell \tan \varepsilon \geq 2$. *Then the topological entropy of the billiard map is at least* $\log 2$*, so the map is chaotic in topological sense.*

It is interesting how this estimate works for the classical Bunimovich stadium billiard. In fact, for the estimate we will improve a little the corollary.

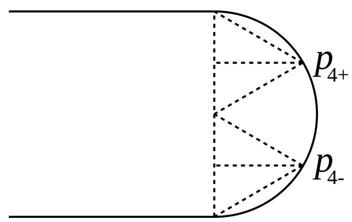

FIGURE 5. Computations for the stadium billiard.

Proposition 4.4. *If the rectangular part of a stadium has the length/width ratio larger than* $\sqrt{3} \approx 1.732$ *(see Figure 6), the billiard map has topological entropy at least* $\log 2$.

Proof. We can take ε as close to $\pi/6$ as we want (see Figure 5), so we get the assumption in the corollary $\ell > 2\sqrt{3}$. However, the factor 2 (in general, $N+1$ in (1)) was taken to get an estimate that works for all possible choices of Γ_i, $i = 3, 4$. For our concrete choice it is possible to replace it by the vertical size of $\Gamma_{i,0} \cup \Gamma_{i,1}$ (or $\Gamma_{i,0} \cup \Gamma_{i,-1}$, bit it is the same in our case). This number is not 2, but $\frac{3}{2}$. Thus, we really get $\ell > \frac{3}{2}\sqrt{3}$. If ℓ' is the length of the rectangular part of the stadium, then $\ell = \ell' + 2 \cdot \frac{\sqrt{3}}{4} = \ell' + \frac{1}{2}\sqrt{3}$. This gives us $\ell' > \sqrt{3}$. $\square$

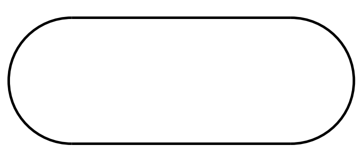

FIGURE 6. Stadium billiard with topological entropy at least $\log 2$.

Now we can prove the main result of this paper.

Theorem 4.5. *For the billiard tables from the class $\mathcal{H}$ with the shapes of* Γ_3 *and* Γ_4 *fixed, the lower limit of the topological entropy of the generalized Bunimovich stadium billiard, as its length ℓ goes to infinity, is at least* $\log(1+\sqrt{2})$.

Proof. In view of Corollary 4.2 and the fact that the largest root of the equation $x^2 - 2x - 1 = 0$ is $1 + \sqrt{2}$, we only have to prove that the largest root of the equation (2) converges to the largest root of the equation $x^2 - 2x - 1 = 0$ as $N \to \infty$. However, this follows from the fact that in the neighborhood of $1+\sqrt{2}$ the right-hand side of (2) goes uniformly to 0 as $N \to \infty$. □

5. Generalized semistadium billiards

In a similar way we can investigate generalized semistadium billiards. They are like generalized stadium billiards, but one of the caps Γ_3', Γ_4' is a vertical straight line segment. The other one contains an ε-free subarc. This class contains, in particular, Bunimovich's Mushroom billiards (see [4]), see Figure 7. We will be talking about the classes $\mathcal{H}_{1/2}$, $\mathcal{H}_{1/2}(\varepsilon)$ and $\mathcal{H}_{1/2}(\varepsilon, \ell)$ of billiard tables.

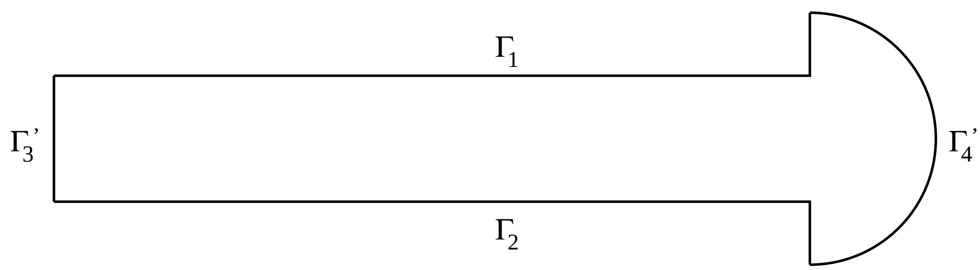

Figure 7. A mushroom.

When we construct a lifting, we add the reflection from the flat vertical cap. In such a way we obtain the same picture as in Section 3, except that there is an additional vertical line through the middle of the picture, and we have to count the flow trajectory crossing it as an additional reflection (see Figure 8). Note that since we will be working with the lifted billiard, in the computations we can take 2ℓ instead of ℓ. In particular, inequality (1) will be now replaced by

$$\ell \geq \frac{N+1}{2\tan\varepsilon} \tag{3}$$

Computation of the topological entropy is this time a little more complicated. We cannot claim that after coding we are obtaining a subshift of finite type. This is due to the fact that if Γ_i' is a vertical segment, we would have to take $\Gamma_i = \Gamma_i'$, and Γ_i would not be disjoint from Γ_1 and Γ_2. The second reason is that the moment when the reflection from the vertical segment occurs depends on the argument of the trajectory line.

The formula for the topological entropy of the subshift of finite type comes from counting of number of cylinders of length n and then taking the exponential growth rate of this number as n goes to infinity. Here we can try do exactly the same, but the problem occurs with the growth rate, since we have additional reflections from the

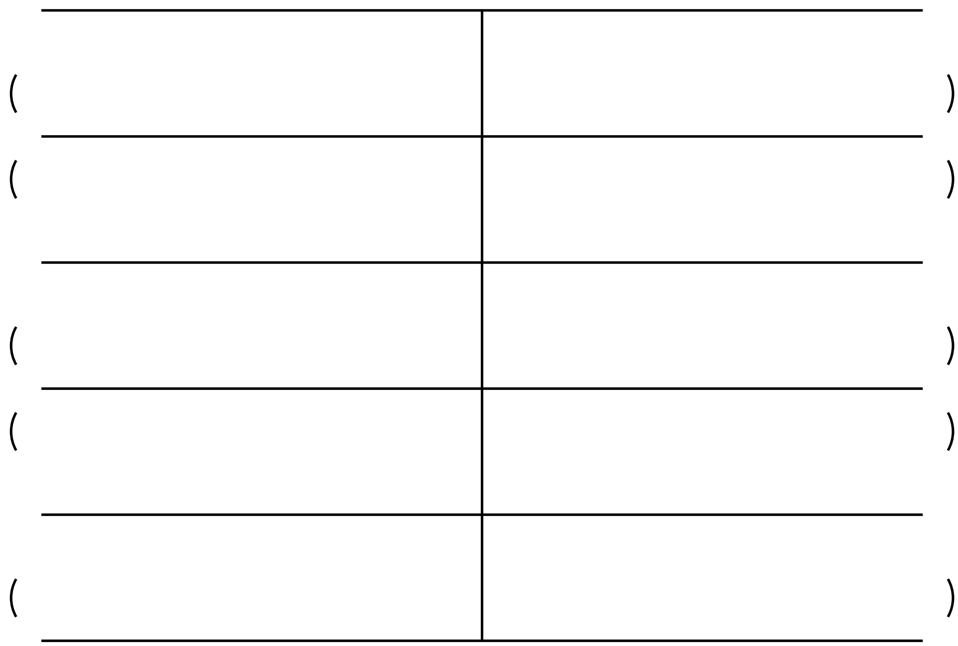

Figure 8. Unfolding.

vertical segment. This means that the cylinders of length n from Section 3 correspond not to time n, but to some larger time. How much larger, depends on the cylinder. However, there cannot be two consecutive reflections from the vertical segment, so this time is not larger than $2n$, and by extending the trajectory we may assume that it is equal to $2n$ (maybe there will be more cylinders, but we need only a lower estimate). Thus, if the number of cylinders (which we count in Section 3) of length n is a_n, instead of taking the limit of $\frac{1}{n}\log a_n$ we take the limit of $\frac{1}{2n}\log a_n$, that is, the half of the limit from Section 3. In such a way we get the following results.

Proposition 5.1. *Assume that the billiard table belongs to $\mathcal{H}_{1/2}(\varepsilon,\ell)$ and fix $N \geq 0$ such that* (3) *is satisfied. Then the topological entropy of the billiard map restricted to $\mathcal{K}_{\ell,N}$ is larger than or equal to one half of the logarithm of the largest root of equation* (2)*.*

Proposition 5.2. *Assume that the billiard table belongs to $\mathcal{H}_{1/2}(\varepsilon,\ell)$ and $\ell\tan\varepsilon \geq 1$. Then the topological entropy of the billiard map is at least $\frac{1}{2}\log 2$, so the map is chaotic in topological sense.*

Theorem 5.3. *For the billiard tables from the class $\mathcal{H}_{1/2}$ with the shape of Γ_3 or Γ_4 (the one that is not the vertical segment) fixed, the lower limit of the topological entropy of the generalized Bunimovich stadium billiard, as its length ℓ goes to infinity, is at least $\frac{1}{2}\log(1+\sqrt{2})$.*

We can apply Proposition 5.2 to the Bunimovich mushroom billiard in order to get entropy at least $\frac{1}{2}\log 2$. As for the stadium, we need to make some computations, and again, we will make a slight improvement in the estimates. The interior of the mushroom billiard consist of a rectangle (the stalk) and a half-disk (the cap).

According to our notation, the stalk is of vertical size 1; denote its horizontal size by ℓ'. Moreover, denote the radius of the cap by t.

Proposition 5.4. *If $\ell' > \frac{1}{2}\sqrt{16t^2 - 1}$ then the topological entropy of the mushroom billiard is at least $\frac{1}{2}\log 2$.*

Proof. Look at Figure 9, where the largest possible ε is used. We have $t \sin\varepsilon = 1/4$.

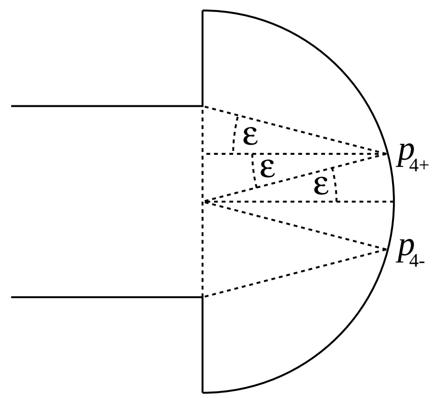

FIGURE 9. Computations for a mushroom.

Therefore, $\tan\varepsilon = 1/\sqrt{16t^2 - 1}$. Similarly as for the stadium, when we use (3) with $N = 1$, we may replace $N + 1$ by $\frac{3}{2}$. Taking into account that we need a strict inequality, we get $\ell > \frac{3}{4}\sqrt{16t^2 - 1}$. However, $\ell = \ell' + t\cos\varepsilon = \ell' + \frac{1}{4}\sqrt{16t^2 - 1}$, so our condition is $\ell' > \frac{1}{2}\sqrt{16t^2 - 1}$. □

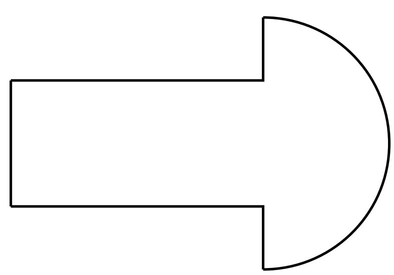

FIGURE 10. A mushroom with topological entropy at least $\frac{1}{2}\log 2$.

Observe that the assumption of Proposition 5.4 is satisfied if the length of the stalk is equal to or larger than the diameter of the cap (see Figure 10).

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