Spectral theory of Jacobi matrices on trees whose coefficients are generated by multiple orthogonality

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Jacobi Operators and Spectral Theory
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References

This talk is mostly based on

S. Denisov and M.Y., Spectral theory of Jacobi matrices on trees whose coefficients are generated by multiple orthogonality, *Adv. Math.*, 2022

which is a third paper in a sequence

A. Aptekarev, S. Denisov, and M.Y., Jacobi matrices on trees generated by Angelesco systems: asymptotics of coefficients and essential spectrum, *J. Spectr. Theory*, 2021

A. Aptekarev, S. Denisov, and M.Y., Self-adjoint Jacobi matrices on trees and multiple orthogonal polynomials, *Trans. Amer. Math. Soc.*, 2020

Padé Approximants

Let $F(z)$ be a formal power series at infinity with no positive powers of z and Q_n, P_n be polynomials of degree at most n defined by

$$(P_n F - Q_n)(z) = \mathcal{O}\left(z^{-n-1}\right)$$

Such a pair of polynomials may not be unique, but their ratio always is. Indeed,

$$(P_n^* Q_n - P_n Q_n^*)(z) = P_n(z)(P_n^* F - Q_n^*)(z) - P_n^*(z)(P_n F - Q_n)(z) = \mathcal{O}\left(z^{-1}\right)$$

which means that this difference must be identically zero. We let $P_n(z)$ to be the monic polynomial of smallest degree. The rational function $(Q_n/P_n)(z)$ is called the diagonal Padé approximant to $F(z)$ of order n.

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If the power series for $F(z)$ is convergent and Γ encircles infinity within the disk of convergence, then

$$0 = \oint_\Gamma z^k (P_n F - Q_n)(z) \mathrm{d}z = \oint_\Gamma z^k P_n(z) F(z) \mathrm{d}z$$

for $k = \overline{0, n-1}$ and z belonging to the exterior of Γ.

In particular, if μ is a compactly supported measure on the real line, and

$$F(z) = \int \frac{\mathrm{d}\mu(x)}{z-x}$$

is the Markov function of μ, then

$$0 = \oint_\Gamma z^k P_n(z) F(z) \mathrm{d}z = \int \oint_\Gamma \frac{z^k P_n(z)}{z-x} \mathrm{d}z \mathrm{d}\mu(x)$$

for $k = \overline{0, n-1}$. Hence,

$$0 = \int x^k P_n(x) \mathrm{d}\mu(x), \quad k = \overline{0, n-1}.$$

That is, $P_n(x)$ is the n-th monic orthogonal polynomial w.r.t. μ.

Orthogonal Polynomials

One can readily verified that up to normalization $P_n(x)$ is equal to

$$\det \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n} \\ 1 & x & \cdots & x^n \end{bmatrix},$$

where $\mu_k := \int x^k \mathrm{d}\mu(x)$. In particular, all the coefficients of $P_n(x)$ are real.

Let Δ be the convex hull of the support μ. Write $P_n(x) = P(x)Q(x)$, where all the zeros of $Q(x)$ either do not lie on Δ or have even multiplicity ($Q(x) \equiv 1$ if there are no such zeros). Then $(PP_n)(x)$ has constant sign on Δ. However, if $\deg P < n$, then

$$\int (PP_n)(x)\mathrm{d}\mu(x) = 0,$$

which is impossible. Hence, $P_n(x)$ has degree n and all its zeros are simple and contained in Δ.

Recurrence Relations

Since $(P_n(x))_n$ is a complete sequence,

$$xP_n(x) = P_{n+1}(x) + c_{n,n}P_n(x) + \ldots + c_{n,0}P_0(x).$$

Observe that for each $k < n-1$, it must hold that

$$c_{n,k}m_k = \int xP_k(x)P_n(x)\mathrm{d}\mu(x) = 0,$$

where $m_k := \int P_k^2(x)\mathrm{d}\mu(x)$. Hence, it holds that

$$\boxed{xP_n(x) = P_{n+1}(x) + b_nP_n(x) + a_{n-1}P_{n-1}(x)}$$

where $P_{-1} := 0$, $P_0 = 1$, $b_n := c_{n,n} = m_n^{-1}\int xP_n^2(x)\mathrm{d}\mu(x)$, and

$$a_{n-1} := c_{n,n-1} = m_{n-1}^{-1}\int xP_{n-1}(x)P_n(x)\mathrm{d}\mu(x) = m_n/m_{n-1}.$$

These three-term recurrence relations can be symmetrized:

$$\boxed{xp_n(x) = \sqrt{a_n}\,p_{n+1}(x) + b_np_n(x) + \sqrt{a_{n-1}}\,p_{n-1}(x),}$$

where $p_n(x) := (1/\sqrt{m_n})P_n(x)$ is the n-th orthonormal polynomial.

Finite Jacobi Matrices

Let

$$\mathcal{J}_n := \begin{pmatrix} b_0 & \sqrt{a_0} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \sqrt{a_0} & b_1 & \sqrt{a_1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \sqrt{a_1} & b_2 & \sqrt{a_2} & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \sqrt{a_{n-2}} & b_{n-1} & \sqrt{a_{n-1}} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \sqrt{a_{n-1}} & b_n \end{pmatrix}.$$

Recurrence relations $x p_n(x) = \sqrt{a_n} p_{n+1}(x) + b_n p_n(x) + \sqrt{a_{n-1}} p_{n-1}(x)$ imply that

$$\mathcal{J}_n \begin{pmatrix} p_0(x) \\ p_1(x) \\ \vdots \\ p_{n-1}(x) \\ p_n(x) \end{pmatrix} = x \begin{pmatrix} p_0(x) \\ p_1(x) \\ \vdots \\ p_{n-1}(x) \\ p_n(x) \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \sqrt{a_n} p_{n+1}(x) \end{pmatrix}.$$

Hence, if $\mathcal{J}_n$ is defined with some b_n and $a_n > 0$ while $(p_k(x))_{k=0}^{n+1}$ are defined via the recurrence relations, then the eigenvalues of $\mathcal{J}_n$ are precisely the zeros of $p_{n+1}(x)$ and the eigenvector corresponding to the eigenvalue λ is $(p_0(\lambda)\ p_1(\lambda)\ \cdots\ p_n(\lambda))^\mathsf{T}$.

Hermite-Padé Approximants and Multiple Orthogonal Polynomials

Let $F_1(z)$ and $F_2(z)$ be two formal power series at infinity with no positive powers of z and $\vec{n} = (n_1, n_2) \in \mathbb{N}^2$ be a multi-index. If there exist polynomials $Q_{\vec{n},i}(z)$ and $P_{\vec{n}}(z)$ of degrees at most $|\vec{n}| := n_1 + n_2$ such that

$$(P_{\vec{n}} F_i - Q_{\vec{n},i})(z) = \mathcal{O}\left(z^{-n_i-1}\right)$$

then the pair of rational functions $(Q_{\vec{n},1}/P_{\vec{n}})(z)$ and $(Q_{\vec{n},2}/P_{\vec{n}})(z)$ is called type II Hermite-Padé approximant to the pair of functions $F_1(z)$ and $F_2(z)$.

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If μ_1, μ_2 are compactly supported measures on the real line and $F_i(z) = \int \frac{\mathrm{d}\mu_i(x)}{z-x}$, then

$$0 = \int x^k P_{\vec{n}}(x)\mathrm{d}\mu_i(x), \quad k = \overline{0, n_i - 1}.$$

If $P_{\vec{n}}(x)$ is unique up to normalization and $\deg(P_{\vec{n}}) = |\vec{n}|$, then the multi-index $\vec{n}$ is called normal. If every multi-index is normal, the system μ_1, μ_2 is called perfect.

Nearest Neighbor Recurrence Relations

Let $\vec{e}_1 = (1, 0)$ and $\vec{e}_2 = (0, 1)$. Assume that $\vec{n}$ and $\vec{n} + \vec{e}_k$ are normal. Then

$$xP_{\vec{n}}(x) - P_{\vec{n}+\vec{e}_k}(x) - b_{\vec{n},k}P_{\vec{n}}(x)$$

is a polynomial of degree at most $|\vec{n}| - 1$ that is orthogonal to polynomials of degree at most $n_i - 2$ w.r.t. μ_i. Linear algebra and normality of $\vec{n}$ and $\vec{n} + \vec{e}_k$ show that it must belong to a 2D subspace and that this subspace is spanned by $P_{\vec{n}-\vec{e}_1}(x)$ and $P_{\vec{n}-\vec{e}_2}(x)$.

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That is,

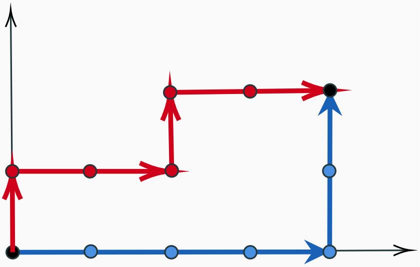
$$xP_{\vec{n}}(x) = P_{\vec{n}+\vec{e}_k}(x) + b_{\vec{n},k}P_{\vec{n}}(x) + a_{\vec{n},1}P_{\vec{n}-\vec{e}_1}(x) + a_{\vec{n},2}P_{\vec{n}-\vec{e}_2}(x)$$

where

$$a_{\vec{n},i} = \frac{\int x^{n_i} P_{\vec{n}}(x) \mathrm{d}\mu_i(x)}{\int x^{n_i-1} P_{\vec{n}-\vec{e}_i}(x) \mathrm{d}\mu_i(x)}.$$

Consistency Conditions

Recurrence relations imply that $P_{\vec{n}}(x)$ can be build in many different ways:



This, in particular, means that the recurrence coefficients cannot be arbitrary. It can be shown that they must satisfy

$$b_{\vec{n}+\vec{e}_1,2} - b_{\vec{n}+\vec{e}_2,1} = b_{\vec{n},2} - b_{\vec{n},1},$$

$$\sum_{k=1}^{2} a_{\vec{n}+\vec{e}_j,k} - \sum_{k=1}^{2} a_{\vec{n}+\vec{e}_i,k} = b_{\vec{n}+\vec{e}_j,i} b_{\vec{n},j} - b_{\vec{n}+\vec{e}_i,j} b_{\vec{n},i},$$

$$a_{\vec{n},i}(b_{\vec{n},j} - b_{\vec{n},i}) = a_{\vec{n}+\vec{e}_j,i}(b_{\vec{n}-\vec{e}_i,j} - b_{\vec{n}-\vec{e}_i,i}).$$

Jacobi Operators on a Lattice

Recurrence relations

$$xP_{\vec{n}}(x) = P_{\vec{n}+\vec{e}_k}(x) + b_{\vec{n},k}P_{\vec{n}}(x) + a_{\vec{n},1}P_{\vec{n}-\vec{e}_1}(x) + a_{\vec{n},2}P_{\vec{n}-\vec{e}_2}(x)$$

naturally lead to two Jacobi operators on the lattice $\mathbb{N}^2$:

$$(\mathcal{J}_k f)_{\vec{n}} := f_{\vec{n}+\vec{e}_k} + b_{\vec{n},k}f_{\vec{n}} + a_{\vec{n},1}f_{\vec{n}-\vec{e}_1} + a_{\vec{n},2}f_{\vec{n}-\vec{e}_2}$$

where f is a function on $\mathbb{N}^2$ (we call it a Jacobi operator because only the values of f at the nearest neighbors of $\vec{n}$ are used to compute the value $\mathcal{J}_k f$ at $\vec{n}$). Notice that

$$\mathcal{J}_k P(x) = xP(x),$$

where $P(x) = (P_{\vec{n}}(x))_{\vec{n}}$. Aptekarev, Derevyagin, and Van Assche investigated these operators and showed that to symmetrize their average:

$$(\mathcal{J}f)_{\vec{n}} := \frac{1}{2}f_{\vec{n}+\vec{e}_1} + \frac{1}{2}f_{\vec{n}+\vec{e}_2} + \frac{1}{2}(b_{\vec{n},1} + b_{\vec{n},2})f_{\vec{n}} + a_{\vec{n},1}f_{\vec{n}-\vec{e}_1} + a_{\vec{n},2}f_{\vec{n}-\vec{e}_2}$$

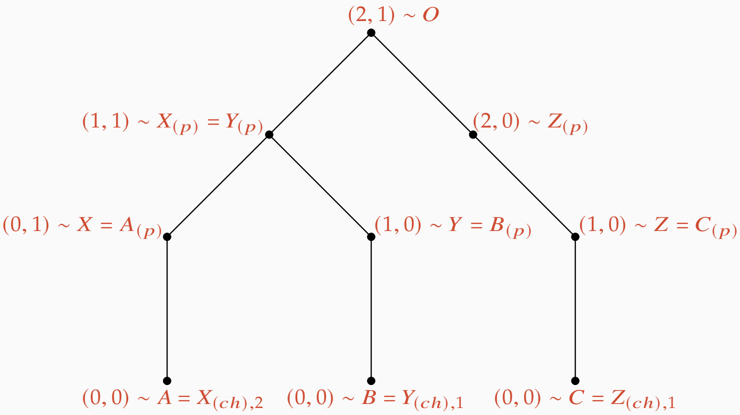
one needs to additionally require

$$b_{\vec{n}+\vec{e}_1,1} - b_{\vec{n}+\vec{e}_1,2} = b_{\vec{n}+\vec{e}_2,1} - b_{\vec{n}+\vec{e}_2,2}.$$

Besides the average, they could have also considered $\mathcal{J}_\kappa := \kappa\mathcal{J}_1 + (1-\kappa)\mathcal{J}_2$.

Finite Trees

Fix $\vec{N} = (N_1, N_2) \in \mathbb{N}^2$ and untwine all paths within connecting $\vec{N}$ and $(0,0)$ within $\{\vec{n} : n_1 \le N_1, n_2 \le N_2\}$ into a tree $\mathcal{T}_{\vec{N}}$ with the set of vertices $\mathcal{V}_{\vec{N}}$.



We denote by $\Pi : \mathcal{V}_{\vec{N}} \to \mathbb{N}^2$ the natural projection and by $\iota : \mathcal{V}_{\vec{N}} \to \{1,2\}$ the child function, i.e., $\iota_Y = i$ iff $\Pi(Y_{(p)}) = \Pi(Y) + \vec{e}_i$ (we also write $Y = Z_{ch,i}$, $Z = Y_{(p)}$).

Define functions V, W and σ by

$$V_O := \kappa_1 b_{\vec{N},1} + \kappa_2 b_{\vec{N},2} \quad \text{and} \quad V_Y := b_{\Pi(Y),\iota_Y},$$

where $\vec{\kappa} = (\kappa_1, \kappa_2)$ is such that $\kappa_1 + \kappa_2 = 1$,

$$W_O := 1 \quad \text{and} \quad W_Y := \left| a_{\Pi(Y_{(p)}),\iota_Y} \right|, \ \ Y \neq O,$$

and $\sigma_Y \in \{0, 1\}$ is such that

$$\sigma_O := 0 \quad \text{and} \quad (-1)^{\sigma_Y} W_Y = a_{\Pi(Y_{(p)}),\iota_Y}, \ \ Y \neq O.$$

Jacobi matrix $\mathcal{J}_{\vec{\kappa},\vec{N}}$ on $\mathcal{T}_{\vec{N}}$ corresponding to (μ_1, μ_2) is defined by

$$(\mathcal{J}_{\vec{\kappa},\vec{N}} f)_Y := V_Y f_Y + W_Y^{1/2} f_{Y_{(p)}} + \sum_{l \in ch(Y)} (-1)^{\sigma_{Y_{(ch),l}}} W_{Y_{(ch),l}}^{1/2} f_{Y_{(ch),l}}$$

If $\sigma \equiv 0$, this operator is self-adjoint, and, in general, it is $\mathfrak{S}$-self-adjoint with respect to an indefinite inner product that depends on σ.

Set

$$p_Y(z) := m_Y^{-1} P_{\Pi(Y)}(z), \quad m_Y := \prod_{Z \in \text{path}(Y,O)} W_Z^{-1/2}$$

If $Y \neq O$, $\Pi(Y) = \vec{n}$, and $\iota_Y = k$, then

$$\begin{aligned}
\left(\mathcal{J}_{\vec{\kappa},\vec{N}}\, p(x)\right)_Y &= V_Y\, p_Y(x) + W_Y^{1/2} p_{Y_{(p)}}(x) + \sum (-1)^{\sigma_{Y_{(ch),l}}} W_{Y_{(ch),l}}^{1/2} p_{Y_{(ch),l}}(x) \\
&= \frac{1}{m_Y} \left(V_Y P_Y(x) + P_{Y_{(p)}}(x) + \sum (-1)^{\sigma_{Y_{(ch),l}}} W_{Y_{(ch),l}} P_{Y_{(ch),l}}(x) \right) \\
&= \frac{1}{m_Y} \left(b_{\vec{n},k} P_{\vec{n}}(x) + P_{\vec{n}+\vec{e}_k}(x) + \sum a_{\vec{n},i} P_{\vec{n}-\vec{e}_i}(x) \right) = x\, p_Y(x).
\end{aligned}$$

Similarly, if we set $P_{\Pi(O_{(p)})}(z) := \kappa_1 P_{\vec{N}+\vec{e}_1}(z) + \kappa_2 P_{\vec{N}+\vec{e}_2}(z)$, then

$$\begin{aligned}
\left(\mathcal{J}_{\vec{\kappa},\vec{N}}\, p(x)\right)_O &= \frac{1}{m_O} \left(V_O P_O(x) + \sum (-1)^{\sigma_{O_{(ch),l}}} W_{O_{(ch),l}} P_{O_{(ch),l}}(x) \right) \\
&= \frac{1}{m_O} \left((\kappa_1 b_{\vec{N},1} + \kappa_2 b_{\vec{N},2}) P_{\vec{N}}(x) + \sum a_{\vec{N},i} P_{\vec{N}-\vec{e}_i}(x) \right) \\
&= x\, p_O(x) - \frac{1}{m_O} P_{\Pi(O_{(p)})}(x).
\end{aligned}$$

Main Identity

Let $Z \in \mathcal{V}_{\vec{N}}$ be a vertex with two children, Z_1 and Z_2. Denote by $b_i(x)$ the function that is equal to the restriction of $p(x)$ to the subtree with the root at Z_i and to zero everywhere else. Then

$$\left(\mathcal{J}_{\vec{\kappa},\vec{N}} b_i(x)\right)_{Z_i} = x p_{Z_i}(x) - W_{Z_i}^{1/2} p_Z(x)$$

and

$$\left(\mathcal{J}_{\vec{\kappa},\vec{N}} b_i(x)\right)_{Z} = (-1)^{\sigma_{Z^i}} W_{Z_i}^{1/2} p_{Z_i}(x).$$

Hence, one can take a linear concatenation $b(x) = v_1 b_1(x) + v_2 b_2(x)$ such that

$$\left(\mathcal{J}_{\vec{\kappa},\vec{N}} b(x)\right)_{Z_i} = x b_{Z_i}(x) - v_i W_{Z_i}^{1/2} p_Z(x)$$

and

$$\left(\mathcal{J}_{\vec{\kappa},\vec{N}} b(x)\right)_{Z} = 0 = x b_Z(x)$$

(here its is important that Z has two and not one child).

Assumptions

Denote by E_Y the set of zeros of $P_{\Pi(Y)}(x)$, $Y \in \mathcal{V}^*_{\vec{N}} := \mathcal{V}_{\vec{N}} \cup \{O_{(p)}\}$. We assume that

$$\begin{aligned} E_Y \subset \mathbb{R}, \ \ \#E_Y = |\Pi(Y)|, \quad Y \in \mathcal{V}^*_{\vec{N}}, \\ E_Y \cap E_{Y_{(p)}} = \varnothing, \quad Y \in \mathcal{V}_{\vec{N}}. \end{aligned}$$

These conditions are satisfied by multiple Hermite polynomials, multiple Laguerre polynomials of the second kind, multiple Charlier polynomials, multiple Meixner polynomials of the first kind (WVA), and Angelesco systems. Moreover, in all these examples $a_{\vec{n},i} > 0$, $\vec{n} \in \mathbb{N}$.

They are also satisfied by multiple Laguerre polynomials of the first kind, Jacobi-Piñeiro polynomials, and multiple Meixner polynomials of the second kind (WVA), and Nikishin systems, but with coefficients $a_{\vec{n},i}$ changing sign.

Main Theorem

Theorem (S. Denisov and M.Y.)

Let E_Y be the set of zeros of $P_{\Pi(Y)}(x), Y \in \mathcal{V}^*_{\vec{N}}$. Then

$$\sigma(\mathcal{J}_{\vec{\kappa},\vec{N}}) = \cup_{Y \in \mathcal{V}^*_{\vec{N}}:\, \#ch(Y)=2} E_Y .$$

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Given $E \in \sigma(\mathcal{J}_{\vec{\kappa},\vec{N}})$, the set $\{b(E,X) :\ X \in \text{Joint}^*(E)\}$ forms a basis of E-eigenspace, where $\text{Joint}^*(E)$ is the collection of all the vertices $Y \in \mathcal{V}^*_{\vec{N}}$ with two children such that $P_{\Pi(Y)}(E) = 0$,

$$b(E, O_{(p)}) := p(E) \quad \text{and} \quad b(E,X) := p(E) \sum \upsilon_i \chi_{|\mathcal{T}_{[X_{(ch)},i]}}$$

with constants υ_i chosen so $\mathcal{J}_{\vec{\kappa},\vec{N}} b(E,X)$ at X is 0.

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with constants υ_i chosen so $\mathcal{J}_{\vec{\kappa},\vec{N}} b(E,X)$ at X is 0.

Totality of these vectors forms a basis for $\ell^2(\mathcal{V}_{\vec{N}})$.

Spectral Theorem

Let A be a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$.

Spectral Theorem (version I)

There exists a resolution of identity E_t such that $A = \int t \, \mathrm{d}E_t$.

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Spectral Theorem (version II)

There exists a homomorphism Φ_A that maps continuous functions into bounded operators on $\mathcal{H}$ so that

$$\Phi_A(1) = I, \quad \Phi_A(t) = A, \quad \Phi_A(\overline{f}) = \Phi_A(f)^*, \quad \|\Phi_A(f)\| \leq \|f\|_\infty.$$

This homomorphism extends to bounded Borel functions.

$$\Phi_A(f) = \int f(t) \mathrm{d}E_t \quad \Leftrightarrow \quad E_t = \Phi_A(\chi_{(-\infty,t]}).$$

Spectral Theorem

Given $h \in \mathcal{H}$, the spectral measure of h w.r.t. A is the unique measure μ_h, supported on $\sigma(A)$, the spectrum of A, such that

$$\left\langle (A - z)^{-1} h, h \right\rangle = \int \frac{d\mu_h(x)}{x - z}.$$

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Spectral Theorem (version III)

Let (h_n) be an orthogonal family in $\mathcal{H}$ such that $\mathcal{H} = \bigoplus \mathcal{H}_n$, where $\mathcal{H}_n$ is the cyclic subspace for A generated by h_n. There exist unitary operators

$$U_n : \mathcal{H}_n \to L^2(\mu_n) : \quad (U_n A_{|\mathcal{H}_n} h)(t) = t(U_n h)(t), \quad (U_n h_n) \equiv 1,$$

where μ_n is the spectral measure of h_n. Moreover, $\sigma(A) = \overline{\cup \text{supp}(\mu_n)}$.

To define U_n, set $U_n p(A) h_n = p(t)$ for a polynomial $p(t)$ and then use density.

Let $U : \mathcal{H} \to \bigoplus L^2(\mu_n)$ be the induced unitary operator. Then $\Phi_A(f) = U^{-1} M_f U$, where M_f is the multiplication by $f(t)$ in each $L^2(\mu_n)$.

Back to Orthogonal Polynomials

Let μ be a probability Borel measure supported on an interval $[c-L, c+L]$ and $(p_n(x))_n$ be the sequence of orthonormal polynomials:

$$\int p_m(x)p_n(x)\mathrm{d}\mu(x) = \delta_{mn}.$$

Then it holds that

$$xp_n(x) = \sqrt{a_n}p_{n+1}(x) + b_np_n(x) + \sqrt{a_{n-1}}p_{n-1}(x),$$

where

$$\begin{aligned} 0 < \sqrt{a_{n-1}} &= \int xp_{n-1}(x)p_n(x)\mathrm{d}\mu(x) = \int (x-c)p_{n-1}(x)p_n(x)\mathrm{d}\mu(x) \\ &\leq L\int |p_{n-1}(x)p_n(x)|\mathrm{d}\mu(x) \leq L \end{aligned}$$

by orthogonality and Cauhcy-Schwarz inequality while

$$|b_n| = \left|\int xp_n^2(x)\mathrm{d}\mu(x)\right| \leq \max\left\{|c-L|, |c+L|\right\}.$$

Jacobi Operator

Boundedness of (a_n, b_n) means that

$$\mathcal{J} := \begin{pmatrix} b_0 & \sqrt{a_0} & 0 & 0 & 0 & 0 & \cdots \\ \sqrt{a_0} & b_1 & \sqrt{a_1} & 0 & 0 & 0 & \cdots \\ 0 & \sqrt{a_1} & b_2 & \sqrt{a_2} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{a_2} & b_3 & \sqrt{a_3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is a bounded self-adjoint operator from $\ell^2(\mathbb{N})$ into itself. Moreover, the cyclic subspace for $\mathcal{J}$ generated by $\delta^{(0)} := (1\ 0\ 0\ \cdots)$ is the whole space.

Spectral Measure

Let $r_n(z)$ be the function of the second kind:

$$r_n(z) := \int \frac{p_n(x)}{x-z} \mathrm{d}\mu(x) = \int \left(\frac{x}{z}\right)^n \frac{p_n(x)}{x-z} \mathrm{d}\mu(x).$$

Put $r := (r_0, r_1, \ldots)$. One can check that

$$(\mathcal{J} - z) r(z) = \delta^{(0)}$$

Since $r(z) \in \ell^2(\mathbb{N})$ for all z large,

$$r(z) = (\mathcal{J} - z)^{-1} \delta^{(0)}, \quad z \notin \sigma(\mathcal{J}).$$

Therefore μ is the spectral measure of $\delta^{(0)}$ wr.t. $\mathcal{J}$ as

$$\left\langle (\mathcal{J} - z)^{-1} \delta^{(0)}, \delta^{(0)} \right\rangle = \int \frac{\mathrm{d}\mu(x)}{x-z}$$

Hence, $\sigma(\mathcal{J}) = \operatorname{supp}(\mu)$.

Unitary Map

Recall $\mathcal{J}p(x) = xp(x)$, $p(x) = (p_n(x))_n$. The unitary map $U : \ell^2(\mathbb{N}) \to L^2(\mu)$ is explicitly defined via

$$\widehat{\alpha} = U^{-1}\alpha := \int \alpha(x)p(x)\mathrm{d}\mu(x)$$

i.e., $\widehat{\alpha} = \{\widehat{\alpha}(n)\}_n$, where $\widehat{\alpha}(n) := \int \alpha(x)p_n(x)\mathrm{d}\mu(x)$. As expected,

$$\mathcal{J}\widehat{\alpha} = \int \alpha(x)\mathcal{J}p(x)\mathrm{d}\mu(x) = \int x\alpha(x)p(x)\mathrm{d}\mu(x).$$

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For orthogonal polynomials the following cyclic relation holds:

$$\mu \;\to\; (p_n(x))_n \;\to\; (a_n, b_n)_n \;\to\; \mathcal{J} \;\to\; \mu_{\delta(0)} = \mu.$$

Type I Multiple Orthogonal Polynomials

Let μ_1, μ_2 be compactly supported measures and $\vec{n} \in \mathbb{N}^2$ be a multi-index. Type I multiple orthogonal polynomials corresponding to $\vec{n}$ are defined by

$$\int x^k Q_{\vec{n}}(x) = 0, \quad k = \overline{0, |\vec{n}| - 2},$$

where $|\vec{n}| = n_1 + n_2$ and the form $Q_{\vec{n}}(x)$ is given by

$$Q_{\vec{n}}(x) := A_{\vec{n}}^{(1)}(x)\mathrm{d}\mu_1(x) + A_{\vec{n}}^{(2)}(x)\mathrm{d}\mu_2(x), \quad \deg A_{\vec{n}}^{(i)} \leq n_i - 1.$$

If the multi-index $\vec{n}$ is normal, $Q_{\vec{n}}(x)$ is unique up to multiplication by a constant and is normalized so that $\int x^{|\vec{n}|-1} Q_{\vec{n}}(x) = 1$.

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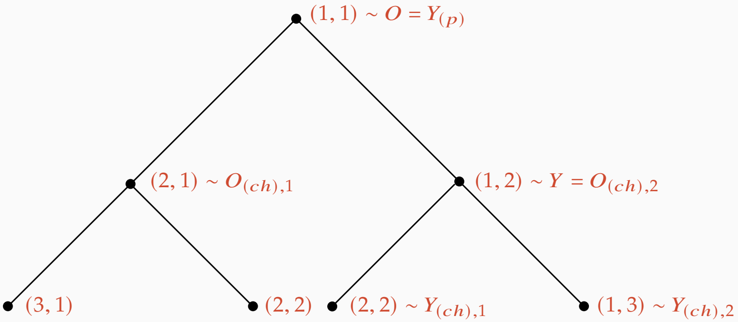
It is known that

$$xQ_{\vec{n}}(x) = Q_{\vec{n}-\vec{e}_i}(x) + b_{\vec{n}-\vec{e}_i,i} Q_{\vec{n}}(x) + a_{\vec{n},1} Q_{\vec{n}+\vec{e}_1}(x) + a_{\vec{n},2} Q_{\vec{n}+\vec{e}_2}(x)$$

where the recurrence coefficients $a_{\vec{n},i}, b_{\vec{n},i}$ are the same as for type II polynomials.

Homogeneous Rooted Tree

Let $\mathcal{T}$ be the rooted tree of all possible increasing paths on $\mathbb{N}^2$ starting at $(1,1)$.



We let $\mathcal{V}$ be the set of its vertices and $\Pi : \mathcal{V} \to \mathbb{N}^2$ be the natural projection and by $\iota : \mathcal{V} \to \{1,2\}$ the child function, i.e., $\iota_Y = i$ iff $\Pi(Y) = \Pi(Y_{(p)}) + \vec{e}_i$ (we also write $Y = Z_{ch,i}$, $Z = Y_{(p)}$).

Jacobi Matrices on $\mathcal{T}$ corresponding to MOPs

Assume that

$$\sup |a_{\vec{n},i}| < \infty \quad \text{and} \quad \sup |b_{\vec{n},i}| < \infty\,.$$

Define functions V, W and σ by

$$V_O := \kappa_1 b_{(0,1),1} + \kappa_2 b_{(1,0),2} \quad \text{and} \quad V_Y := b_{\Pi(Y_{(p)}),\iota_Y}\,,$$

where $\vec{\kappa} = (\kappa_1, \kappa_2)$ is such that $\kappa_1 + \kappa_2 = 1$,

$$W_O := 1 \quad \text{and} \quad W_Y := \left|a_{\Pi(Y_{(p)}),\iota_Y}\right|, \;\; Y \neq O,$$

and $\sigma_Y \in \{0, 1\}$ is such that

$$\sigma_O := 0 \quad \text{and} \quad (-1)^{\sigma_Y} W_Y = a_{\Pi(Y_{(p)}),\iota_Y}\,, \;\; Y \neq O.$$

Jacobi matrix $\mathcal{J}_{\vec{\kappa}}$ on $\mathcal{T}$ corresponding to (μ_1, μ_2) is defined by

$$(\mathcal{J}_{\vec{\kappa}} f)_Y := V_Y f_Y + W_Y^{1/2} f_{Y_{(p)}} + \sum_{l \in \{1,2\}} (-1)^{\sigma_{Y_{(ch),l}}} W_{Y_{(ch),l}}^{1/2} f_{Y_{(ch),l}}$$

Main Identity

Let $Q_{\vec{n}}(x)$ be the type I forms for (μ_1, μ_2) and $L_{\vec{n}}(z) := \int (z-x)^{-1} Q_{\vec{n}}(x)$. Set

$$l_Y(z) := m_Y^{-1} L_{\Pi(Y)}(z), \quad m_Y := \prod_{Z \in \mathrm{path}(Y,O)} W_Z^{-1/2}$$

We further put $L_{\Pi(O_{(p)})}(z) := \kappa_1 L_{\vec{e}_2}(z) + \kappa_2 L_{\vec{e}_1}(z)$. Then

$$(\mathcal{J}_{\vec{\kappa}} - z)\, l(z) = -L_{\Pi(O_{(p)})}(z)\, \delta^{(O)}$$

where $\delta^{(Y)}$ is the delta-function of Y on $\mathcal{V}$.

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where $\delta^{(Y)}$ is the delta-function of Y on $\mathcal{V}$. Moreover, also holds that

$$(\mathcal{J}_{[X]} - z)\, l_{[X]}(z) = -m_X^{-1} L_{\Pi(X_{(p)})}(z)\, \delta^{(X)}$$

where $[X]$ denotes the restriction to a subtree with root at X.

Boundedness Assumption

Theorem (A. Aptekarev, S. Denisov, and M.Y.)

Boundedness assumption is satisfied by Angelesco systems (μ_1, μ_2):

$$\Delta_1 \cap \Delta_2 = \varnothing, \quad \Delta_i := \mathrm{ch}(\operatorname{supp} \mu_i).$$

Moreover, it holds that $a_{\vec{n},i} > 0$ for $\vec{n} \in \mathbb{N}^2$.

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Theorem (S. Denisov and M.Y.)

For a Nikishin system of Szegő measures it holds that

$$\lim_{n\to\infty} a_{(n,n+1),1} = -\infty \quad \text{and} \quad \lim_{n\to\infty} a_{(n,n+1),2} = \infty.$$

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In the rest of this talk it is assumed that (μ_1, μ_2) is an Angelesco system and therefore the operator $\mathcal{J}_{\vec{\kappa}}$ is self-adjoint.

Green's functions

Given $X \in \mathcal{V}$ and $Y \in \mathcal{V}_{[X]}$, the corresponding Green's function is defined by

$$G(Y,X;z) := \left\langle (\mathcal{J}_{[X]} - z)^{-1} \delta^{(X)}, \delta^{(Y)} \right\rangle$$

The limit $\mathrm{Im} G(X,Y;x+\mathrm{i}\epsilon)$ as $\epsilon \to 0^+$ exists in the weak*-sense and we denote the corresponding generally signed measure by $\mathrm{Im} G(Y,X)^+$.

Of course, $\rho_{[X]} = \pi^{-1} \mathrm{Im} G(X,X)^+$, the spectral measure $\delta^{(X)}$ restricted to $\mathcal{V}_{[X]}$ w.r.t. $\mathcal{J}_{[X]}$, is positive.

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Proposition

For all $z \notin \Delta_1 \cup \Delta_2$ it holds that

$$G(Y, X; z) = -\frac{m_X}{m_Y} \frac{L_{\Pi(Y)}(z)}{L_{\Pi(X_{(p)})}(z)}.$$

Proposition (S. Denisov and M.Y.)

Measure $\rho_{[X]}$ has a semi-explicit expression.

It holds that

$$\mathrm{dIm}G(Y,X)^{+}(x) = \pi\Psi_{Y}(X;x)\mathrm{d}\rho_{[X]}(x),$$

for every $Y \in \mathcal{V}_{[X]}$, where $\Psi(X;x)$ is such that

$$\mathcal{J}_{[X]}\Psi(X;x) = x\Psi(X;x) \quad \text{and} \quad \delta_{Y}^{(X)} = \int \Psi_{Y}(X;x)\mathrm{d}\rho_{[X]}(x).$$

Trivial Cyclic Subspaces

Let $\mathfrak{C}^{(X)}$ be the cyclic subspace of $\ell^2(\mathcal{V}_{[X]})$ generated by $\delta^{(X)}$, that is,

$$\mathfrak{C}^{(X)} := \overline{\operatorname{span}\left\{\mathcal{J}_{[X]}^n \delta^{(X)} : n \in \mathbb{Z}_+\right\}}.$$

Proposition (S. Denisov and M.Y.)

Fix $X \in \mathcal{V}$. The map

$$\alpha(x) \mapsto \widehat{\alpha} = \{\widehat{\alpha}_Y\}_{Y \in \mathcal{V}_{[X]}}, \quad \widehat{\alpha}_Y := \int \alpha(x)\Psi_Y(X;x)\mathrm{d}\rho_{[X]}(x),$$

is a unitary map from $L^2(\rho_{[X]})$ onto $\mathfrak{C}^{(X)}$. In particular, it holds that

$$\|\alpha\|^2_{L^2(\rho_{[X]})} = \|\widehat{\alpha}\|^2_{\ell^2(\mathcal{V}_{[X]})} \quad \text{and} \quad \mathfrak{C}^{(X)} = \left\{\widehat{\alpha} : \alpha \in L^2(\rho_{[X]})\right\}.$$

We also have that

$$x\alpha(x) \mapsto \mathcal{J}_{[X]}\widehat{\alpha}, \quad \alpha \in L^2(\rho_{[X]}).$$

Non-Trivial Cyclic Subspaces

Fix $X \in \mathcal{V}$ and let $X_i = X_{(ch),i}$, $i \in \{1,2\}$. There exists measure $\widetilde{\rho}_X$ such that

$$d\rho_{[X_i]}(x) = \nu_{X_i}(x) d\widetilde{\rho}_X(x),$$

where it holds that $c_X^{-1} \leq \nu_{X_i}(x) \leq c_X$, $x \in \Delta_1 \cup \Delta_2$. Let

$$\widehat{\Psi}_Y(X;x) := (-1)^i W_{X_i}^{-1/2} \Psi_Y(X_i;x), \quad Y \in \mathcal{V}_{[X_i]},$$

and $\widehat{\Psi}_Y(X;x) := 0$ otherwise. Define

$$\widehat{\mathfrak{C}}^{(X)} := \left\{ \int \alpha(x) \widehat{\Psi}(X;x) d\widetilde{\rho}_X(x) : \quad \alpha \in L^2(\widetilde{\rho}_X) \right\}$$

Non-Trivial Cyclic Subspaces

Proposition (S. Denisov and M.Y.)

It holds that $\mathcal{J}_{\vec{\kappa}}\widehat{\Psi}(X;x) = x\widehat{\Psi}(X;x)$. Let $g_i^{(X)} \in \widehat{\mathfrak{C}}^{(X)}$ be given by

$$g_i^{(X)} := (-1)^i W_{X_i}^{1/2} \int \widehat{\Psi}(X;x)\mathrm{d}\rho_{[X_i]}(x).$$

Then, it holds that

$$\widehat{\mathfrak{C}}^{(X)} = \overline{\operatorname{span}\left\{\mathcal{J}_{\vec{\kappa}}^n g_i^{(X)} \,:\, n \in \mathbb{Z}_+\right\}}.$$

Furthermore, it holds that

$$\mathrm{d}\rho_{X,i}(x) = \sum_{k=1}^{2} \frac{W_{X_i}}{W_{X_k}} \frac{\nu_{X_i}^2(x)}{\nu_{X_k}(x)} \mathrm{d}\widetilde{\rho}_X(x),$$

where $\rho_{X,i}$ is the spectral measure of $g_i^{(X)}$ with respect to $\mathcal{J}_{\vec{\kappa}}$.

Spectral Analysis

Theorem (S. Denisov and M.Y.)

$$\ell^2(\mathcal{V}) = \mathfrak{C}^{(O)} \bigoplus \mathcal{L}, \quad \mathcal{L} = \bigoplus_{X \in \mathcal{V}} \widehat{\mathfrak{C}}^{(X)} .$$

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The set $E_{\vec{\kappa}} := \{E : \kappa_1 L_{\vec{e}_2}(E) + \kappa_2 L_{\vec{e}_1}(E) = 0, \ E \in \mathbb{R} \setminus (\Delta_1 \cup \Delta_2)\}$ is either empty or has exactly one element in it. It is empty when $\vec{\kappa} = \vec{e}_i$, $i \in \{1, 2\}$. It holds that

$$\sigma(\mathcal{J}_{\vec{\kappa}}) \subseteq \Delta_1 \cup \Delta_2 \cup E_{\vec{\kappa}}.$$

If $\operatorname{supp} \mu_k = \Delta_k$, $k \in \{1, 2\}$, then inclusion becomes equality.

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If $d\mu_k(x) = \mu_k'(x)\mathrm{d}x$ and $(\mu_k')^{-1} \in L^\infty(\Delta_k)$, $k \in \{1, 2\}$, then the spectrum of $\mathcal{J}_{\vec{e}_i}$ is purely absolutely continuous.