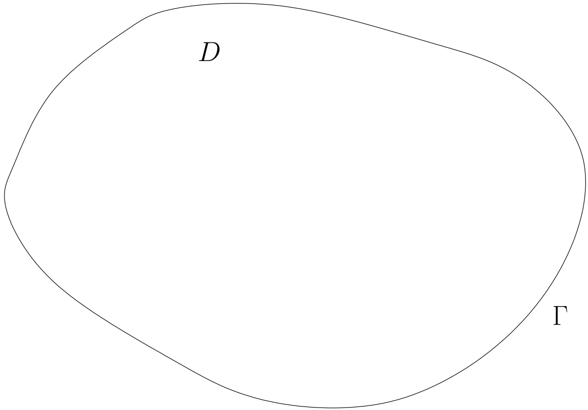
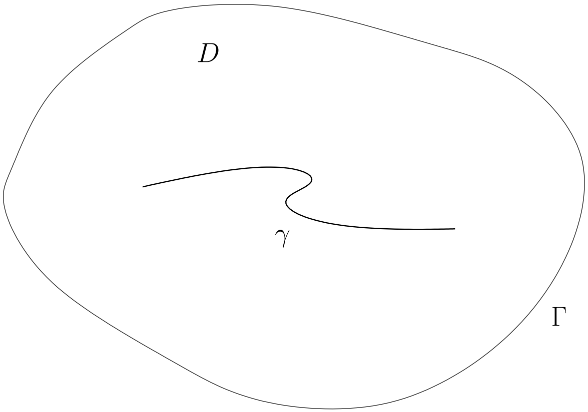
Asymptotics Uniqueness of Best Rational Approximants in $L^2(\mathbb{T})$ to Cauchy Transforms

M. Yattselev
Project APICS, INRIA, Sophia Antipolis, France

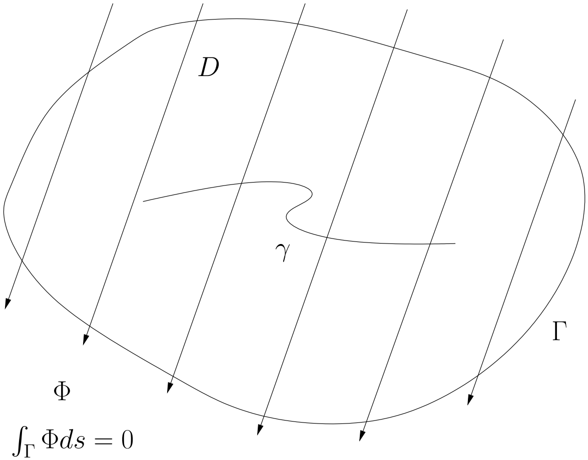
joint work with

L. Baratchart
Project APICS, INRIA, Sophia Antipolis, France

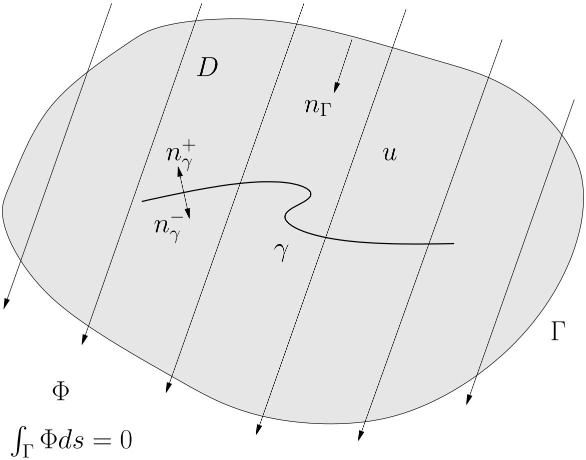

D
Γ


D
γ
Γ

“Crack” Problem



"Crack" Problem



Let u be the equilibrium distribution of heat or current. Then

$$\begin{cases} \Delta u = 0 & \text{in } D \setminus \gamma \\ \dfrac{\partial u}{\partial n_\Gamma} = \Phi & \text{on } \Gamma := \partial D \\ \dfrac{\partial u^\pm}{\partial n_\gamma^\pm} = 0 & \text{on } \gamma \setminus \{\gamma_0, \gamma_1\} \end{cases},$$

where Δu is the Laplacian of u.

Methods of crack identification:

- iterative methods: solve direct problem, use some minimizing criteria, crack needs to be localized in advance;

Methods of crack identification:

- iterative methods: solve direct problem, use some minimizing criteria, crack needs to be localized in advance;
- semi-explicit methods: localization through approximation of u in the whole domain D;

Methods of crack identification:

- iterative methods: solve direct problem, use some minimizing criteria, crack needs to be localized in advance;
- semi-explicit methods: localization through approximation of u in the whole domain D;
- method of meromorphic approximants introduced by L. Baratchart and E. B. Saff.

It can be shown that u has well-defined conjugate in $D \setminus \gamma$ and

$$\mathcal{F}(\xi) = u(\xi) - i\int_{\xi_0}^{\xi} \Phi ds, \quad \xi \in \Gamma.$$

It can be shown that u has well-defined conjugate in $D \setminus \gamma$ and

$$\mathcal{F}(\xi) = u(\xi) - i\int_{\xi_0}^{\xi} \Phi ds, \quad \xi \in \Gamma.$$

Further,

$$\mathcal{F}(z) = h(z) + \frac{1}{2\pi i}\int_{\gamma} \frac{(\mathcal{F}^- - \mathcal{F}^+)(t)}{z-t} dt, \quad z \in D \setminus \gamma,$$

where h is analytic in D and continuous in $\overline{D}$.

It can be shown that u has well-defined conjugate in $D \setminus \gamma$ and

$$\mathcal{F}(\xi) = u(\xi) - i\int_{\xi_0}^{\xi} \Phi ds, \quad \xi \in \Gamma.$$

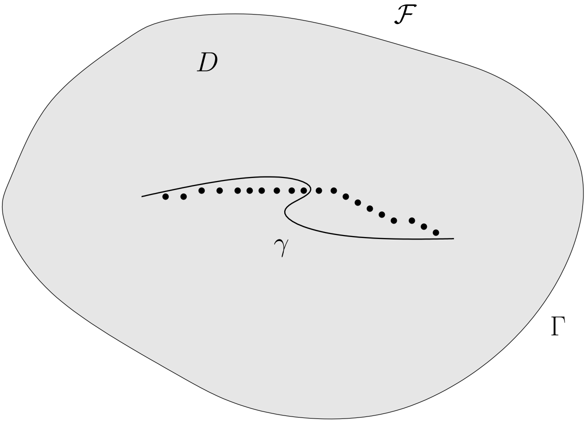
Further,

$$\mathcal{F}(z) = h(z) + \frac{1}{2\pi i}\int_{\gamma} \frac{(\mathcal{F}^- - \mathcal{F}^+)(t)}{z-t} dt, \quad z \in D \setminus \gamma,$$

where h is analytic in D and continuous in $\overline{D}$.

One approximates $\mathcal{F}$ on Γ by meromorphic in D functions and observes the asymptotic behavior of their poles as the number of poles growth large.

Idea of the Method



Let μ be a complex measure whose support, S_μ, is a subset of the unit disk, $\mathbb{D}$.

Define the Cauchy transform of μ by

$$\mathcal{F}(z) = \mathcal{F}(\mu; z) := \int \frac{d\mu(t)}{z - t}$$

and denote

$$D_{\mathcal{F}} := \overline{\mathbb{C}} \setminus S_\mu.$$

Let h be a complex-valued function on the unit circle, $\mathbb{T}$. Then

$$h \in L^p \quad \text{iff} \quad \|h\|_p^p := \sum |h_j|^p < \infty, \; h_j := \frac{1}{2\pi}\int_{\mathbb{T}} \xi^{-j} h(\xi)|d\xi|,$$

$$h \in L^\infty \quad \text{iff} \quad \|h\|_\infty := \operatorname*{ess.sup}_{\mathbb{T}} |h| < \infty.$$

Let h be a complex-valued function on the unit circle, $\mathbb{T}$. Then

$$h \in L^p \quad \text{iff} \quad \|h\|_p^p := \sum |h_j|^p < \infty, \; h_j := \frac{1}{2\pi}\int_{\mathbb{T}} \xi^{-j} h(\xi)|d\xi|,$$
$$h \in L^\infty \quad \text{iff} \quad \|h\|_\infty := \operatorname*{ess.\,sup}_{\mathbb{T}} |h| < \infty.$$

Let $p \in [2, \infty]$. The Hardy spaces are defined by

$$H^p := \left\{ h \in L^p : \; h_j = 0, \; j < 0 \right\},$$
$$\bar{H}_0^p := \left\{ h \in L^p : \; h_j = 0, \; j > -1 \right\}.$$

Fix $p \in [2, \infty]$ and $n \in \mathbb{N}$. The space of meromorphic functions of the degree n is defined as

$$H^p_n := H^p + \mathcal{R}_n,$$

where $\mathcal{R}_n$ is the set of rational functions of type $(n-1, n)$ with all their poles in $\mathbb{D}$.

Meromorphic approximation problem:

$$\|\mathcal{F} - g_n\|_p = \inf_{g \in H_n^p} \|\mathcal{F} - g\|_p.$$

Meromorphic approximation problem:

$$\|\mathcal{F} - g_n\|_p = \inf_{g \in H_n^p} \|\mathcal{F} - g\|_p.$$

This problem always admits a solution:

- Adamjan, Arov, and Krein[a], $p = \infty$;
- Baratchart and Seyfert[b] & Prokhorov[c], $p \in [1, \infty)$.

[a] Analytic properties of Schmidt pairs for a Hankel operator on the generalized Schur-Takagi problem. *Math. USSR Sb.*, 15: 31-73, 1971

[b] An L^p analog of AAK theory for $p \geq 2$. *J. Funct. Anal.*, 191(1): 52-122, 2002

[c] On L^p-generalization of a theorem of Adamyan, Arov, and Krein. *Comput. Methods Funct. Theory*, 1(2): 501-520, 2001

Let $g_n = h_n + r_n$, $h_n \in H^2$ and $r_n \in R_n$, be a best approximant for $\mathcal{F}$ in MAP with $p = 2$. Then

$$\|\mathcal{F} - g_n\|_2^2 = \|h_n\|_2^2 + \|\mathcal{F} - r_n\|_2^2.$$

Let $g_n = h_n + r_n$, $h_n \in H^2$ and $r_n \in R_n$, be a best approximant for $\mathcal{F}$ in MAP with $p = 2$. Then

$$\|\mathcal{F} - g_n\|_2^2 = \|h_n\|_2^2 + \|\mathcal{F} - r_n\|_2^2.$$

Therefore, we arrive at

Rational Approximation Problem

$$\|\mathcal{F} - r_n\|_2 = \inf_{r \in \mathcal{R}_n} \|\mathcal{F} - r\|_2.$$

Definitions

- We say that $r \in \mathcal{R}_n$ is a critical point in RAP for $\mathcal{F}$ if

$$D\Theta(r) = 0,$$

where $\Theta(r) := \Theta_{\mathcal{F},n}(r) = \|\mathcal{F} - r\|_2^2$.

Definitions

- We say that $r \in \mathcal{R}_n$ is a critical point in RAP for $\mathcal{F}$ if

$$D\Theta(r) = 0,$$

where $\Theta(r) := \Theta_{\mathcal{F},n}(r) = \|\mathcal{F} - r\|_2^2$.

- We say that r_n is irreducible critical point if r_n has exactly n poles. (It is known that all best and locally best rational approximants are always irreducible critical points.)

Let $r_n = p_{n-1}/q_n$ be a critical point in RAP to $\mathcal{F}$. Then

Rational function r_n interpolates $\mathcal{F}$ at the reflections of the zeros of q_n with order 2 in the Hermite sense.

Let $r_n = p_{n-1}/q_n$ be a critical point in RAP to $\mathcal{F}$. Then

Rational function r_n interpolates $\mathcal{F}$ at the reflections of the zeros of q_n with order 2 in the Hermite sense.

In other words, r_n is a multipoint Padé approximant with the implicitly defined interpolation set. Furthermore,

$$\int t^j q_n(t) \frac{d\mu(t)}{\widetilde{q}_n^2(t)} = 0, \quad j = 0, \ldots, n-1,$$

where $\widetilde{q}_n(z) = z^n \overline{q_n(1/\bar{z})}$ is the reciprocal polynomial.

Let F be an interval contained in $(-1,1)$ with the endpoints a and b. Set

- $w(z) = w(F,z) := \sqrt{(z-a)(z-b)}$ to be a holomorphic outside of F function such that $w(z)/z \to 1$ as $z \to \infty$. Then $w^+ = -w^-$ on F;

Let F be an interval contained in $(-1, 1)$ with the endpoints a and b. Set

- $w(z) = w(F, z) := \sqrt{(z-a)(z-b)}$ to be a holomorphic outside of F function such that $w(z)/z \to 1$ as $z \to \infty$. Then $w^+ = -w^-$ on F;
- ϕ to be the conformal map $\overline{\mathbb{C}} \setminus (F \cup F^{-1})$ onto an annulus $\{\rho \leq |z| \leq 1/\rho\}$ such that $\phi(\mathbb{T}) = \mathbb{T}$ and $\phi(\pm 1) = \pm 1$;

Let F be an interval contained in $(-1,1)$ with the endpoints a and b. Set

- $w(z) = w(F,z) := \sqrt{(z-a)(z-b)}$ to be a holomorphic outside of F function such that $w(z)/z \to 1$ as $z \to \infty$. Then $w^+ = -w^-$ on F;
- ϕ to be the conformal map $\overline{\mathbb{C}} \setminus (F \cup F^{-1})$ onto an annulus $\{\rho \leq |z| \leq 1/\rho\}$ such that $\phi(\mathbb{T}) = \mathbb{T}$ and $\phi(\pm 1) = \pm 1$;
- μ to be of the form $d\mu(t) = \dfrac{h(t)dt}{w^+(t)}$, where h is a non-vanishing Dini-continuous function on F.

Then the following theorem takes place.

Theorem 1 (Baratchart and Y.)

Let $\{r_n\}$ be a sequence of irreducible critical points in RAT for $\mathcal{F}$ with μ as described. Then

$$(\mathcal{F} - r_n)(z) = (\mathcal{D} + o(1)) \frac{w^*(z)}{w(z)} \left(\frac{\rho}{\phi(z)} \right)^{2n} D_n(z)$$

locally uniformly in $D_{\mathcal{F}}$, where

- $w^*(z) = z\overline{w(1/\bar{z})}$;
- $\mathcal{D}$ is some constant;
- $\{D_n\}$ is a sequence of outer functions in $\overline{\mathbb{C}} \setminus (F \cup F^{-1})$;
- $|D_n|$ are uniformly bounded away from zero and infinity.

The proof of the above stated result utilizes:

- a priori knowledge of the behavior of the arguments of q_n on F (B, Küstner, Totik[a]);

The proof of the above stated result utilizes:

- a priori knowledge of the behavior of the arguments of q_n on F (B, Küstner, Totik[a]);
- formulae of strong asymptotics for polynomials satisfying non-Hermitian orthogonality relations with varying measures on arcs (last section and almost Aptekarev[b]);

The proof of the above stated result utilizes:

- a priori knowledge of the behavior of the arguments of q_n on F (B, Küstner, Totik[a]);
- formulae of strong asymptotics for polynomials satisfying non-Hermitian orthogonality relations with varying measures on arcs (last section and almost Aptekarev[b]);
- special connection (reciprocity) between the polynomial part of the weight, $\widetilde{q}_n^2$, and the orthogonal polynomials q_n (B, Stahl, Wielonsky[c]).

---

[a] Zero distribution via orthogonality. *Ann. Inst. Fourier.*, 55(5): 1455-1499, 2005

[b] Sharp constants for rational approximations of analytic functions. *Sb. Math.*, 193(1-2): 1-72, 2002

[c] Asymptotic error estimates for L^2 best rational approximants to Markov functions. *J. Approx. Theory.*, 108: 53-96, 2001

Numerical search of best rational approximants is a nonconvex optimization problem and therefore it often gets trapped in local minima. However, if there is only one local minimum, the descent algorithms converge.

Definitions

- A critical point r is called nondegenerate if $D^2\Theta(r)$ is a nonsingular quadratic form.

Definitions

- A critical point r is called nondegenerate if $D^2\Theta(r)$ is a nonsingular quadratic form.
- The Morse index of a nondegenerate critical point r, $M(r)$, is the number of negative eigenvalues of $D^2\Theta_{\mathcal{F}}(r)$.

Definitions

- A critical point r is called nondegenerate if $D^2\Theta(r)$ is a nonsingular quadratic form.
- The Morse index of a nondegenerate critical point r, $M(r)$, is the number of negative eigenvalues of $D^2\Theta_{\mathcal{F}}(r)$.

Theorem (Baratchart and Olivi)[a]

If all the critical points are nondegenerate and neither of them interpolates $\mathcal{F}$ on $\mathbb{T}$, then there are only finitely many such points and

$$\sum(-1)^{M(r_c)} = 1.$$

[a] Index of critical points in l^2-approximation. *Systems Control Lett.*, 10: 167-174, 1988

Theorem (Adopted from Baratchart, Stahl, Wielonsky)[a]

Let r_n be an irreducible critical point of order n that does not interpolate $\mathcal{F}$ on $\mathbb{T}$. If there exists a meromorphic function Π with at most of $n-1$ poles in $\mathbb{D}$, continuous on $\mathbb{T}$, such that

$$2|\mathcal{F} - r_n| \leq |\Pi - r_n| \quad \text{on} \quad \mathbb{T},$$

and the winding number

$$\mathbf{w}_{\mathbb{T}}(\mathcal{F} - \Pi) \leq 1 - 2n,$$

then r_n is a local minimum, i.e. $D^2\Theta(r)$ is positive definite.

[a] Asymptotic uniqueness of best rational approximants of given degree to Markov functions in L^2 of the circle. *Constr. Approx.*, 17: 103-138, 2001

Set

- $\varphi_i(z) = z - w(z)$;
- $\varphi(z) = z + w(z)$;
- E_n to be a set of $2n$ points in $D := \overline{\mathbb{C}} \setminus F$;
- $\Psi_n(z) := \prod_{e \in E_n} \dfrac{\varphi(z) - \varphi(e)}{1 - \varphi(z)\varphi(e)}$;

Definition

A system of sets $\{E_n\}$ is called admissible if, to each $n \in \mathbb{N}$, there is a one-to-one correspondence $\Delta_n : E_n \to E_n$ such that

$$\sup_{n\in\mathbb{N}} \sum_{e\in E_n} \frac{|\bar{\varphi}_i(e) - \Delta_n(\varphi_i(e))|}{(1-|\varphi_i(e)|)(1-|\Delta_n(\varphi_i(e))|)} < \infty$$

and

Definition

A system of sets $\{E_n\}$ is called admissible if, to each $n \in \mathbb{N}$, there is a one-to-one correspondence $\Delta_n : E_n \to E_n$ such that

$$\sup_{n\in\mathbb{N}} \sum_{e\in E_n} \frac{|\bar{\varphi}_i(e) - \Delta_n(\varphi_i(e))|}{(1-|\varphi_i(e)|)(1-|\Delta_n(\varphi_i(e))|)} < \infty$$

and

$$\lim_{n\to\infty} \sum_{e\in E_n} (1-|\varphi_i(e)|) = \infty.$$

Note

- Admissibility implies that $\Psi_n = o(1)$ in $\mathbb{C} \setminus F$ and $|\Psi_n^{\pm}|$ are uniformly bounded above on F.

Note

- Admissibility implies that $\Psi_n = o(1)$ in $\mathbb{C} \setminus F$ and $|\Psi_n^{\pm}|$ are uniformly bounded above on F.
- Let r_n be an irreducible critical point in RAP to $\mathcal{F}$ of order n and let $\{\xi_{j,n}\}$ be its poles. Then $E_n^* := \{1/\bar{\xi}_{j,n}\}$ form an admissible sequence of sets. We shall denote associated "rational" functions by Ψ_n^*.

Theorem 2 (Baratchart and Y.)

Let $\{E_n\}$ be an admissible sequence of sets and $\mathcal{F}$ be as in Theorem 1. Further, let Π_n be the diagonal multipoint Padé approximant of order n with the interpolation set E_n. Then

$$(\mathcal{F} - \Pi_n)(z) = (\mathcal{G} + o(1)) \frac{\Psi_n(z)}{w(z)} S_n(z)$$

locally uniformly in $D_{\mathcal{F}}$, where

- $\mathcal{G}$ is some constant;
- $\{S_n\}$ is a sequence of outer functions in $\overline{\mathbb{C}} \setminus F$;
- $|S_n|$ are uniformly bounded away from zero and infinity.

We take $\Pi = \Pi_{n-1}$ for some admissible interpolation scheme $\{E_n\}$. By the previous theorem $\mathbf{w}(\mathcal{F} - \Pi_{n-1}) = 1 - 2n$ whenever $E_n \subset \mathbb{C} \setminus \overline{\mathbb{D}}$. Thus, points $\{E_n\}$ need to be chosen in $\mathbb{C} \setminus \overline{\mathbb{D}}$ so

$$\left| 1 - \frac{\mathcal{F} - \Pi_{n-1}}{\mathcal{F} - r_n} \right| > 2 \quad \text{on} \quad \mathbb{T},$$

We take $\Pi = \Pi_{n-1}$ for some admissible interpolation scheme $\{E_n\}$. By the previous theorem $\mathbf{w}(\mathcal{F} - \Pi_{n-1}) = 1 - 2n$ whenever $E_n \subset \mathbb{C} \setminus \overline{\mathbb{D}}$. Thus, points $\{E_n\}$ need to be chosen in $\mathbb{C} \setminus \overline{\mathbb{D}}$ so

$$\left|1 - \frac{\mathcal{F} - \Pi_{n-1}}{\mathcal{F} - r_n}\right| > 2 \quad \text{on} \quad \mathbb{T},$$

i.e.

$$|\Psi_{n-1}(z)/\Psi_n^*(z)| > 2.$$

Facts (modified Baratchart, Stahl, Wielonsky)

- One can construct $\{E_n\}$ based on $\{E_n^*\}$ so that functions $\log|\Psi_{n-1}/\Psi_n^*|$ approximate the Green potential of any signed measure of total mass 2 supported on F^{-1};

Facts (modified Baratchart, Stahl, Wielonsky)

- One can construct $\{E_n\}$ based on $\{E_n^*\}$ so that functions $\log|\Psi_{n-1}/\Psi_n^*|$ approximate the Green potential of any signed measure of total mass 2 supported on F^{-1};
- there exists a measure on F^{-1} whose Green potential satisfies $|1-G|>2$ everywhere on $\mathbb{T}$.

Facts (modified Baratchart, Stahl, Wielonsky)

- One can construct $\{E_n\}$ based on $\{E_n^*\}$ so that functions $\log|\Psi_{n-1}/\Psi_n^*|$ approximate the Green potential of any signed measure of total mass 2 supported on F^{-1};
- there exists a measure on F^{-1} whose Green potential satisfies $|1-G|>2$ everywhere on $\mathbb{T}$.

Theorem 3 (Baratchart and Y.)

Let $\mathcal{F}$ be as in Theorem 1. Then for all n large enough there exists a unique critical point of order n.

Let F be now any oriented smooth arc connecting ± 1. Set

- $w(z) := w(F, z)$ defined as before;
- $\varphi(z) = z + w(z)$;
- E_n to be a set of $2n$ points in $D := \overline{\mathbb{C}} \setminus F$;
- v_n to be a polynomial with zeros at finite points of E_n;
- $\Psi_n(z) := \prod_{e \in E_n} \dfrac{\varphi(z) - \varphi(e)}{1 - \varphi(z)\varphi(e)}$;
- h to be a Dini-continuous non-vanishing function on F.

For h as above we define geometric mean:

$$G_h := \exp\left\{\int \log h(t) \frac{idt}{\pi w^+(t)}\right\}$$

For h as above we define geometric mean:

$$G_h := \exp\left\{ \int \log h(t) \frac{idt}{\pi w^+(t)} \right\}$$

and Szegő function:

$$S_h(z) := \exp\left\{ \frac{w(z)}{2} \int \frac{\log(h(t)/G_h)}{t-z} \frac{idt}{\pi w^+(t)} \right\}.$$

For h as above we define geometric mean:

$$G_h := \exp\left\{\int \log h(t) \frac{idt}{\pi w^+(t)}\right\}$$

and Szegő function:

$$S_h(z) := \exp\left\{\frac{w(z)}{2}\int \frac{\log(h(t)/G_h)}{t-z}\frac{idt}{\pi w^+(t)}\right\}.$$

Then S_h is an outer function in $\overline{\mathbb{C}} \setminus F$, $S_h(\infty) = 1$, and $S_h^{\pm}$ are continuous functions on F such that

$$h = G_h S_h^+ S_h^-.$$

Orthogonal polynomials:

$$\int_F t^j q_n(t) w_n(t) \frac{dt}{w^+(t)} = 0, \quad j = 0, \ldots, n-1.$$

Orthogonal polynomials:

$$\int_F t^j q_n(t) w_n(t) \frac{dt}{w^+(t)} = 0, \quad j = 0, \ldots, n-1.$$

Functions of second kind:

$$R_n(z) := \frac{1}{\pi i} \int_F \frac{q_n(t) w_n(t)}{t - z} \frac{dt}{w^+(t)}, \quad z \in \overline{\mathbb{C}} \setminus F.$$

Orthogonal polynomials:

$$\int_F t^j q_n(t) w_n(t) \frac{dt}{w^+(t)} = 0, \quad j = 0, \ldots, n-1.$$

Functions of second kind:

$$R_n(z) := \frac{1}{\pi i} \int_F \frac{q_n(t) w_n(t)}{t - z} \frac{dt}{w^+(t)}, \quad z \in \overline{\mathbb{C}} \setminus F.$$

Weights:

$$w_n(t) = \frac{h(t)}{v_n(t)},$$

where E_n (that is v_n) are such that $\Psi_n = o(1)$ locally uniformly in D and $|\Psi_n^\pm| = O(1)$ uniformly on F.

Theorem (Baratchart and Y.)

Let $\{q_n\}_{n\in\mathbb{N}}$ be a sequence of polynomials as above.

Then each polynomials q_n has exact degree n for all n large enough and therefore can be normalized to be monic.

Under such a normalization we have

$$\begin{cases} q_n &= (1+o(1))/S_n \\ R_n w &= (1+o(1))\gamma_n S_n \end{cases} \quad \text{locally uniformly in } D$$

and

$$\frac{q_n^2(t)w_n(t)}{\gamma_n w^+(t)}dt \stackrel{*}{\to} \frac{dt}{w^+(t)},$$

where $S_n := S_{w_n}(2/\varphi)^n$, $\gamma_n := 2^{1-2n}G_{w_n}$, and $\stackrel{*}{\to}$ stands for the weak* converges of measures.

Theorem (BY)

Further,

$$\left\{\begin{array}{lcl} q_n & = & (1+d_n^-)/S_n^+ + (1+d_n^+)/S_n^- \\ (R_n w)^\pm & = & (1+d_n^\pm)\,\gamma_n S_n^\pm \end{array}\right. \quad \text{on } F,$$

where $d_n^\pm$ are continuous on F and satisfy

$$\int_F \frac{|d_n^-(t)|^p + |d_n^+(t)|^p}{\sqrt{|1-t^2|}}|dt| \to 0 \text{ as } n \to \infty$$

for any $p \in [1, \infty)$.

Remarks

- smoothness of F can be reduced. Most likely we can handle quasismooth arcs without twisting points;

Remarks

- smoothness of F can be reduced. Most likely we can handle quasismooth arcs without twisting points;
- function h, in fact, can vanish at a finite number of points in a "controlled manner";

Remarks

- smoothness of F can be reduced. Most likely we can handle quasismooth arcs without twisting points;
- function h, in fact, can vanish at a finite number of points in a "controlled manner";
- we can consider a compact family $\{h_n\}$ instead of h.

For any $\alpha \in \mathbb{R}$ denote

$$F_\alpha := \left\{ \frac{i\alpha + x}{1 + i\alpha x} : \ x \in [-1, 1] \right\}.$$

and for any point $e \in \mathbb{C}$ define

$$e^* = \frac{2i\alpha + (1-\alpha^2)\bar{e}}{(1-\alpha^2) + 2i\alpha\bar{e}}.$$

Then

$$e^* = e \quad \text{for any} \quad e \in F_\alpha^{-1}$$

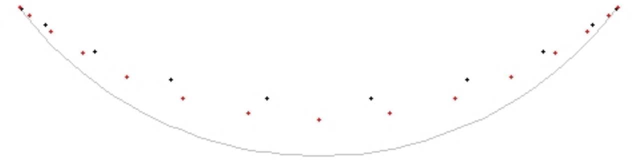
and

$$|(\Psi_e \Psi_{e*})^\pm| = 1 \quad \text{on} \quad F_\alpha,$$

where

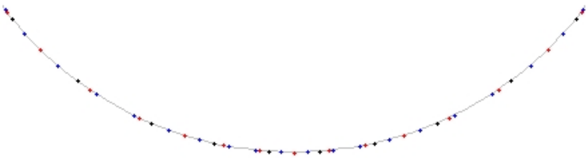
$$\Psi_e(z) := \frac{\varphi(z) - \varphi(e)}{1 - \varphi(z)\varphi(e)}.$$

$$w_n(t) = \exp\left\{\frac{2it-1}{2i-t}\pi\right\}/(t-2i)^{2n}$$



Zeros of q_{10} (black) and q_{15} (red).

$$w_n(t) = t^{-n}(t + 4i/3)^{-n}$$



Zeros of q_{10} (black), q_{15} (red), and q_{20} (blue).