Symmetric Contours and Convergent Interpolation

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Spaces of An. Functions: Approximation, Interpolation, Sampling

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Euclidean Algorithm

Let $p/q \in \mathbb{Q}$. The Euclidean Algorithm is used to find the gcd of p and q:

$$
\begin{aligned}
p &= a_0 q + r_0 \\
q &= a_1 r_0 + r_1 \\
r_0 &= a_2 r_1 + r_2 \\
&\cdots \\
r_{n-2} &= a_n r_{n-1}.
\end{aligned}
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However, it also has the following consequence:

$$\begin{aligned} \frac{p}{q} &= a_0 + \frac{r_0}{q} = a_0 + \cfrac{1}{a_1 + \cfrac{r_1}{r_0}} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{r_2}{r_1}}} \\ &= a_0 + \cfrac{1}{a_1 + \cfrac{1}{\cdots + \cfrac{1}{a_n}}} = a_0 + \Phi_{k=1}^{n} \frac{1}{a_k}. \end{aligned}$$

Continued Fractions

Let now $x \in \mathbb{R}$. Then

$$\begin{aligned} x &= [x] + \frac{1}{1/\{x\}} = [x] + \cfrac{1}{[1/\{x\}] + \cfrac{1}{1/\{1/\{x\}\}}} = \cdots \\ &=: a_0(x) + \Phi_{k=1}^{\infty} \frac{1}{a_k(x)}, \end{aligned}$$

where $a_k(x) \in \mathbb{Z} \cup \{\infty\}$, which is called a continued fraction representation of x. Set

$$x_n := a_0(x) + \Phi_{k=1}^{n} \frac{1}{a_k(x)} = \frac{p_n}{q_n} \in \mathbb{Q}$$

to be the n-th convergent of the continued fraction.

Fact

Continued fraction

$$a_0(x) + \Phi_{k=1}^{\infty} \frac{1}{a_k(x)}$$

is finite if and only if $x \in \mathbb{Q}$. *Moreover, if* $x \in \mathbb{R} \setminus \mathbb{Q}$, *then*

$$\frac{1}{q_n(q_n + q_{n+1})} \leq |x - x_n| \leq \frac{1}{q_n q_{n+1}}.$$

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Consequence

Convergent x_n *is the optimal rational approximant of the irrational number* x *among all rational numbers with denominators of a fixed size.*

Continued Fraction of a Series

Start with a formal power series at infinity

$$f(z) = \sum_{k=1}^{\infty} f_k z^{-k}$$

such that the Hankel determinants of the coefficients $\{f_k\}$ are non-zero. Then

$$f(z) = \Phi_{k=1}^{\infty} \frac{b_k}{z - a_k}$$

for some well-defined constants $\{a_k, b_k\}$. Denote $[n/n]_f$ the n-th convergent:

$$[n/n]_f(z) := \Phi_{k=1}^{n} \frac{b_k}{z - a_k}.$$

Then it is known that

$$\big(f - [n/n]_f\big)(z) = \mathcal{O}\big(z^{-2n-1}\big)$$

and the above relation uniquely determines $[n/n]_f$. Moreover,

$$\big(q_n f - p_n\big)(z) = \mathcal{O}\big(z^{-n-1}\big), \quad [n/n]_f =: p_n/q_n.$$

Padé Approximants

Equivalently, let p_n, q_n be polynomials of degree at most n defined by

$$(q_n f - p_n)(z) = \mathcal{O}\big(z^{-n-1}\big).$$

Such a pair of polynomials may not be unique, but their ratio always is with no conditions on f. Thus, we normalize q_n to be monic, set

$$p_n/q_n =: [n/n]_f,$$

and call it the diagonal Padé approximant of f of order n.

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If the power series for f is convergent and Γ encircles infinity within the disk of convergence, then

$$0 = \oint_\Gamma z^k (q_n f - p_n)(z) \mathrm{d}z = \oint_\Gamma z^k q_n(z) f(z) \mathrm{d}z$$

for $k = \overline{0, n-1}$ and z belonging to the exterior of Γ. This can be rewritten as

$$0 = \int x^k q_n(x) \mathrm{d}\mu(x), \quad f(z) = \int \frac{\mathrm{d}\mu(x)}{z - x},$$

where μ is in general complex measure.

Markov's Theorem

Assume that μ is a positive measure on an interval $[a,b] \subset \mathbb{R}$. Since

$$0 = \int x^k q_n(x) \mathrm{d}\mu(x), \quad k = \overline{0, n-1},$$

it holds that $q_n(x) = \prod_{i=1}^n (x - x_{n,i})$ and $x_{n,i} \in [a,b]$. Therefore,

$$[n/n]_f(z) = \frac{p_n(z)}{q_n(z)} = \sum_{i=1}^n \frac{\lambda_{n,i}}{z - x_{n,i}} =: \int \frac{\mathrm{d}\mu_n(x)}{z-x}.$$

Then the asymptotics

$$\mathcal{O}\left(z^{-2n-1}\right) = \int \frac{\mathrm{d}(\mu - \mu_n)(x)}{z-x} = \frac{1}{z} \int \sum_{k=0}^{\infty} \left(\frac{x}{z}\right)^k \mathrm{d}(\mu - \mu_n)(x)$$

implies that

$$\int x^k \mathrm{d}\mu(x) = \int x^k \mathrm{d}\mu_n(x), \quad k = \overline{0, 2n}.$$

Since $(z-x)^{-1}$ is a continuous function of x on $[a,b]$, it holds that

$$[n/n]_f(z) \to f(z) = \int \frac{\mathrm{d}\mu(x)}{z-x}$$

locally uniformly in $\overline{\mathbb{C}} \setminus [a,b]$. **Can we quantify this convergence?**

Equilibrium Measures

Let ν be a compactly supported positive Borel measure. A function

$$V^{\nu}(z) := -\int \log|z-x| \mathrm{d}\nu(x)$$

is called the logarithmic potential of ν. Moreover, the number

$$I[\mu] := -\iint \log|z-x| \mathrm{d}\nu(x) \mathrm{d}\nu(z)$$

is called the logarithmic energy of ν.

Given a compact set K, either every Borel measure supported on K has infinite logarithmic energy, in which case K is called polar, or there exists the unique probability Borel measure ω_K such that

$$I[\omega_K] = \inf I[\nu],$$

where the infimum is taken over all probability Borel measures supported on K. The measure ω_K is called the equilibrium measure of K.

Green's Function

Let K be a non-polar compact set with connected complement D. There exists the unique function $g_K(z,\infty)$, which is called Green's function for D, such that

- $g_K(z,\infty)$ is positive and harmonic in $D \setminus \{\infty\}$;
- $g_K(z,\infty) - \log|z|$ is bounded near infinity;
- $g_K(z,\infty) = 0$ for quasi every (up to a polar set) $z \in \partial D$.

The quantity

$$\operatorname{cap}(K) := \exp\left\{ \lim_{z\to\infty} \left(\log|z| - g_K(z,\infty) \right) \right\}$$

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In fact, it holds that

$$g_K(z,\infty) = I[\omega_K] - V^{\omega_K}(z) \quad \Rightarrow \quad \operatorname{cap}(K) = e^{-I[\omega_K]}.$$

Moreover, if D is simply connected, we have that

$$g_K(z,\infty) = \log|\Phi(z)|,$$

where Φ is a conformal map of D onto $|z| > 1$ such that $\Phi(\infty) = \infty$.

Ullman-Stahl-Totik Regularity of Measures

Let μ be a positive Borel measure with compact support K. Let Q_n be a monic polynomial of degree n such that

$$\int \overline{z}^k Q_n(z) \mathrm{d}\mu(z) = 0, \quad k = \overline{0, n-1}.$$

The measure μ is called UST-regular if

$$\lim_{n\to\infty} \left(\int |Q_n|^2 \mathrm{d}\mu \right)^{1/2n} = \mathrm{cap}(K).$$

Equivalently, μ is UST-regular if

$$\lim_{n\to\infty} |Q_n(z)|^{1/n} = e^{-V^{\omega_K}(z)}$$

locally uniformly outside of the convex hull of K.

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In particular, if

$$\mathrm{cap}\left(\left\{ z \in K : \limsup_{\delta\to 0+} \frac{\log \mu\{w : |w-z| < \delta\}}{\log \delta} < \infty \right\} \right) = \mathrm{cap}(K),$$

the measure μ is UST-regular.

Theorem (Stahl-Totik)

Let μ be a UST-regular positive Borel measure with compact support $K \subset \mathbb{R}$ and

$$f(z) = \int \frac{\mathrm{d}\mu(x)}{z - x}.$$

Write $q_n(z) = \prod_{i=0}^{n}(z - x_{n,i})$. Then $\frac{1}{n}\sum_{i=0}^{n}\delta(x_{n,i}) \xrightarrow{*} \omega_K$. Moreover,

$$\lim_{n\to\infty} |f(z) - [n/n]_f(z)|^{1/2n} = e^{-g_K(z,\infty)}$$

locally uniformly outside of the convex hull of K.

Stahl's Class

Let f be a holomorphic germ at infinity. We say that $f \in \mathcal{S}$ if it can be meromorphically continued along any path in $\overline{\mathbb{C}} \setminus E_f$, where E_f is polar and there exists at least one point in $\overline{\mathbb{C}} \setminus E_f$ with distinct continuations.

Functions in class $\mathcal{S}$ are necessarily multi-valued, while Padé approximants are single-valued. Hence, if they converge at all, they need to select a single-valued branch. **Which one?**

A compact set K is called admissible for f if $\overline{\mathbb{C}} \setminus K$ is connected and f has a meromorphic and single-valued extension there.

Theorem (Stahl)

Let $f \in \mathcal{S}$. There exists the "unique" admissible compact Δ_f such that

$$\mathrm{cap}(\Delta_f) \le \mathrm{cap}(K)$$

for any admissible K. Moreover, for any compact set $F \subset D_f := \overline{\mathbb{C}} \setminus \Delta_f$ and $\varepsilon > 0$, it holds that

$$\lim_{n\to\infty} \mathrm{cap}\left\{ z \in F : \left| |f(z) - [n/n]_f(z)|^{1/2n} - e^{-g_{\Delta_f}(z,\infty)} \right| > \varepsilon \right\} = 0.$$

The domain D_f is optimal in the sense that the convergence does not hold in any other domain D such that $D \setminus D_f \neq \varnothing$.

Theorem (Stahl)

The minimal capacity contour Δ_f can be decomposed as

$$\Delta_f = E_0 \cup E_1 \cup \bigcup \Delta_j,$$

where $E_0 \subseteq E_f$, E_1 consists of isolated points to which f has unrestricted continuations from infinity leading to at least two distinct function elements, and Δ_j are open analytic arcs. Green's function for D_f satisfies

$$\frac{\partial g_{\Delta_f}}{\partial n_+} = \frac{\partial g_{\Delta_f}}{\partial n_-} \quad \text{on} \quad \bigcup \Delta_j,$$

where $\partial/\partial n_\pm$ are the one-sided normal derivatives on $\bigcup \Delta_j$.

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We call a collection $\mathcal{I} = \{I_n\}$, $I_n = \{v_{n,i}\}_{i=1}^{2n}$, an interpolation scheme if $v_{n,i}$ are not necessarily distinct nor finite and belong to the domain of analyticity of f.

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A rational function $p_n/q_n =: [n/n; \mathcal{I}]_f$ is called a multipoint Padé approximant of f associated with and interpolation scheme $\mathcal{I}$ if

$$\frac{q_n(z)f(z) - p_n(z)}{v_n(z)} = \mathcal{O}\left(z^{-n-1}\right)$$

has the same region of analyticity as f, where $v_n(z) := \prod_{|v_{n,i}|<\infty}(z - v_{n,i})$. Again, it holds that the rational function $[n/n; \mathcal{I}]_f$ is uniquely defined.

Green's Functions and Potentials

Let K be a non-polar compact set with connected complement D. Given $w \in D \setminus \{\infty\}$, there exists the unique function $g_K(z, w)$, which is called Green's function for D with pole at w, such that

- $g_K(z, w)$ is positive and harmonic in $D \setminus \{w\}$;
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Let ω be a positive Borel measure supported in D. Then

$$G_K(z; \omega) := \int g(z, w) \mathrm{d}\omega(w)$$

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$$G_K(z;\omega) := \int g(z,w)\mathrm{d}\omega(w)$$

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It is known that there exists the unique measure $\widehat{\omega}$ on K such that

$$G_K(z;\omega) = V^{\omega}(z) - V^{\widehat{\omega}}(z) + c_\omega$$

for some constant c_ω. The measure $\widehat{\omega}$ is called the balayage measure of ω relative to D (for measures with unbounded support one needs to spherically renormalize logarithmic potentials).

Theorem (Stahl-Totik)

Let μ be a UST-regular positive Borel measure with compact support $K \subset \mathbb{R}$ and

$$f(z) = \int \frac{\mathrm{d}\mu(x)}{z - x}.$$

Let $\mathcal{I}$ be a conjugate symmetric interpolation scheme for f asymptotic to some measure ω, i.e.,

$$\frac{1}{2n} \sum_{i=1}^{2n} \delta(v_{n,i}) \xrightarrow{*} \omega,$$

which is supported in $\overline{\mathbb{C}} \setminus K$. Write $q_n(z) = \prod_{i=0}^{n}(z - x_{n,i})$. Then

$$\frac{1}{n} \sum_{i=0}^{n} \delta(x_{n,i}) \xrightarrow{*} \widehat{\omega}.$$

Moreover,

$$\lim_{n\to\infty} \left| f(z) - [n/n; \mathcal{I}]_f(z) \right|^{1/2n} = e^{-G_K(z;\omega)}$$

locally uniformly outside of the convex hull of K.

Contours Symmetric w.r.t. a Measure

Let $f \in \mathcal{S}$ and ω be a probability measure supported in $\overline{\mathbb{C}} \setminus E_f$. An admissible compact Δ is called a symmetric contour for f with respect to ω if it consists of open analytic arcs and their endpoints and

$$\frac{\partial G_\Delta(\cdot;\omega)}{\partial n_+} = \frac{\partial G_\Delta(\cdot;\omega)}{\partial n_-}$$

at every smooth point of Δ.

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at every smooth point of Δ.

Theorem (Gonchar-Rakhmanov)

Let $f \in S$ and Δ be symmetric for f w.r.t. ω and the jump of f across Δ be non-zero almost everywhere. If $\mathcal{I}$ is an interpolation scheme asymptotic to ω, then for any compact set $F \subset \overline{\mathbb{C}} \setminus \Delta_f$ and $\varepsilon > 0$, it holds that

$$\lim_{n\to\infty} \operatorname{cap}\left\{ z \in F : \left| |f(z) - [n/n;\mathcal{I}]_f(z)|^{1/2n} - e^{-G_\Delta(z;\omega)} \right| > \varepsilon \right\} = 0.$$

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Unlike the case of classical Padé approximants, the existence of a symmetric contour is not shown but assumed.

Contours Symmetric w.r.t. an Interpolation Scheme

Let Δ be a rectifiable Jordan arc connecting ± 1. Further, let

$$w(z) = \sqrt{z^2 - 1}$$

be the branch holomorphic off Δ that behaves like z at infinity. Define

$$\Phi(z) = z + w(z)$$

which is an analytic continuation of the standard conformal map of $\overline{\mathbb{C}} \setminus [-1, 1]$ to the complement of the unit disk to $\overline{\mathbb{C}} \setminus \Delta$.

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$$\Phi(z, v) = \frac{\Phi(z) - \Phi(v)}{1 - \Phi(z)\Phi(v)}, \quad \Phi(z, \infty) = \frac{1}{\Phi(z)}.$$

Notice that $\Phi^{+}(x, v)\Phi^{-}(x, v) \equiv 1$ on Δ.

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Notice that $\Phi^+(x, v)\Phi^-(x, v) \equiv 1$ on Δ.

It is said that Δ is symmetric w.r.t. an interpolation scheme $\mathcal{I} \subset D := \overline{\mathbb{C}} \setminus \Delta$ if

$$\left|\Phi_n^\pm\right| = \mathcal{O}(1) \quad \text{and} \quad |\Phi_n| = o(1) \quad \text{as} \quad n \to \infty$$

uniformly on Δ and locally uniformly in D, where $\Phi_n(z) = \prod_{i=1}^{2n} \Phi(z, v_{n,i})$.

Theorem (Baratchart-Ya.)

Let Δ be a rectifiable Jordan arc connecting ± 1 with additional technical condition around around the endpoints. Then the following are equivalent:

(a) *there exists an interpolation scheme $\mathcal{I}$, supported in D, such that Δ is symmetric w.r.t. $\mathcal{I}$;*

(b) *there exists a positive compactly supported Borel measure ω, supported in D, such that Δ is symmetric w.r.t. ω;*

(c) *Δ is an analytic Jordan arc, i.e., there exists a univalent function $\Xi(z)$ holomorphic in some neighborhood of $[-1,1]$ such that $\Delta = \Xi([-1,1])$.*

Theorem (Baratchart-Ya.)

Let Δ be an analytic Jordan arc connecting ± 1 that is symmetric w.r.t. $\mathcal{I}$. Let

$$f_\rho(z) := \frac{1}{2\pi\mathrm{i}} \int_\Delta \frac{\rho(t)}{t-z} \frac{\mathrm{d}t}{w^+(t)},$$

where ρ is a non-vanishing Lipschitz continuous, generally complex-valued, function on Δ. Then

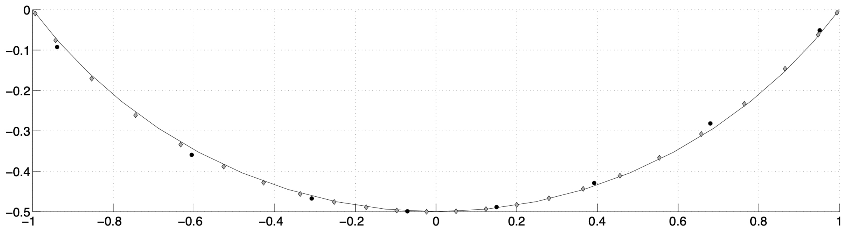
$$f_\rho(z) - [n/n;\mathcal{I}]_{f_\rho}(z) = \frac{1+o(1)}{w(z)} S_\rho^2(z)\Phi_n(z)$$

locally uniformly in $D = \overline{\mathbb{C}} \setminus \Delta$, where

$$S_\rho(z) := \exp\left\{ \frac{w(z)}{2\pi\mathrm{i}} \int_F \frac{\log\rho(t)}{t-z} \frac{\mathrm{d}t}{w^+(t)} \right\}$$

is the Szegő function of ρ.

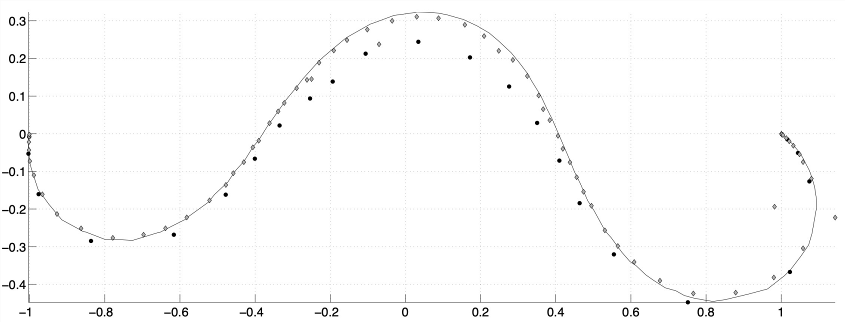
Strong Asymptotics



Zeros of q_8 and q_{24} when $\rho(t) = e^t$ and the interpolation points are equally distributed between 0 and $-4\mathrm{i}/3$. In this case

$$\Delta = \left\{ \frac{\mathrm{i} - 2x}{2 - \mathrm{i}x} : \ x \in [-1, 1] \right\}.$$

Strong Asymptotics



Zeros of q_{24} and q_{66} when $\rho(t) = t$ if $\mathrm{Im}(t) \geq 0$ and $\rho(t) = \overline{t}$ if $\mathrm{Im}(t) < 0$. The interpolation points are equally distributed between $(\mathrm{i} - 3)/4$, $(87 + 6\mathrm{i})/104$, and $-\mathrm{i}/10$.

Theorem (Stahl)

Let $f \in \mathcal{S}$ and $\Delta_f = E \cup \bigcup \Delta_j$ be its minimal capacity (symmetric) contour. Define

$$h(z) := \partial_z g_{\Delta_f}(z), \quad 2\partial_z := \partial_x - \mathrm{i}\partial_y.$$

The function h^2 is holomorphic in $\overline{\mathbb{C}} \setminus E$ with a double zero at infinity and the arcs Δ_j are orthogonal critical trajectories of the quadratic differential $h^2(z)\mathrm{d}z^2$.

Assume in addition that f is a germ of an algebraic function (E_f is necessarily finite). For each point $e \in E$ denote by $i(e)$ the bifurcation index of e, that is, the number of different arcs Δ_j incident with e. Then

$$h^2(z) = \prod_{e \in E}(z-e)^{i(e)-2} \prod_{e \in E_2}(z-e)^{2j(e)},$$

where E_2 is the set of critical points of g_{Δ_f} and $j(e)$ is the order of $e \in E_2$.

Algebraic Contours Symmetric w.r.t. an Interpolation Scheme

Let $\mathfrak{R}$ be the Riemann surface of h and $\boldsymbol{E}$ be the set of its ramification points. Let symbol $\cdot^*$ stand for the conformal involution $\boldsymbol{z}^* = (z, -h)$ if $\boldsymbol{z} = (z, h)$. If E has $2g+2$ points, then the genus of $\mathfrak{R}$ is g.

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Given $\boldsymbol{v} \in \mathfrak{R} \setminus \boldsymbol{E}$, denote by $g(\cdot, \boldsymbol{v})$ a function that is harmonic in $\mathfrak{R} \setminus \{\boldsymbol{v}, \boldsymbol{v}^*\}$, normalized so that $g(\boldsymbol{e}, \boldsymbol{v}) = 0$ for $\boldsymbol{e} \in \boldsymbol{E}$, and such that

$$g(\boldsymbol{z}, \boldsymbol{v}) \pm \begin{cases} \log|z - v|, & |v| < \infty, \\ -\log|z|, & v = \infty, \end{cases}$$

are harmonic around $\boldsymbol{v}$ $(+)$ and $\boldsymbol{v}^*$ $(-)$, respectively.

Algebraic Contours Symmetric w.r.t. an Interpolation Scheme

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Let Δ be a system of open analytic arcs and their endpoints and $\mathcal{I}$ be an interpolation scheme in $\overline{\mathbb{C}} \setminus \Delta$. We say that Δ is symmetric w.r.t. $(\mathfrak{R}, \mathcal{I})$ if

- $\mathfrak{R} \setminus \boldsymbol{\Delta}$, $\boldsymbol{\Delta} := \pi^{-1}(\Delta)$, consists of two disjoint domains, say $D^{(0)}$ and $D^{(1)}$, and no closed subset of Δ has this property;
- the sums $\sum_{i=0}^{2n} g\left(\cdot, v_{n,i}^{(0)}\right)$ are uniformly bounded above and below on $\boldsymbol{\Delta}$ and go to $-\infty$ locally uniformly in $D^{(1)}$, where $z^{(i)} = \pi^{-1}(z) \cap D^{(i)}$.

Algebraic Contours Symmetric w.r.t. an Interpolation Scheme

Fact

If Δ is symmetric w.r.t. $(\mathfrak{R}, \mathcal{I})$ and $\mathcal{I}$ is asymptotic to some measure ω, then Δ is symmetric w.r.t. ω.

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Theorem (Ya.)

Let $c > 0$ be a constant such that $L_c := \{s : g_{\Delta_f}(s) = c\}$ is a smooth Jordan curve. If $\Xi(z)$ is a conformal function in the interior of L_c such that $\Xi(e) = e$ for every $e \in E$, then there exists an interpolation scheme $\mathcal{I}$ in $\overline{\mathbb{C}} \setminus \Xi(\Delta)$ such that $\Xi(\Delta_f)$ is symmetric with respect to $(\mathfrak{R}, \mathcal{I})$.

Proposition (Ya.)

Let ρ be a Lipschitz continuous and non-vanishing function on Δ. There exists a sectionally meromorphic in $\mathfrak{R} \setminus \boldsymbol{\Delta}$ function $\Psi_n(\boldsymbol{z})$ with the zero/pole divisor

$$(n-g)\infty^{(1)} + \boldsymbol{z}_{n,1} + \cdots + \boldsymbol{z}_{n,g} - n\infty^{(0)}$$

for some set of g points $\boldsymbol{z}_{n,i}$ on $\mathfrak{R}$, and whose traces on $\boldsymbol{\Delta}$ are continuous and satisfy

$$\Psi_{n-}(\boldsymbol{s}) = \big(\rho(s)/v_n(s)\big)\Psi_{n+}(\boldsymbol{s}), \quad \boldsymbol{s} \in \boldsymbol{\Delta}.$$

If functions $\Psi(\boldsymbol{z}), \Psi_*(\boldsymbol{z})$ have these properties, then $\Psi(\boldsymbol{z})/\Psi_*(\boldsymbol{z}) = R(\pi(\boldsymbol{z}))$ for some rational function $R(z)$ with at most $g/2$ poles. In particular, if $\{\boldsymbol{z}_{n,i}\}$ does not contain involution-symmetric pairs ($\boldsymbol{z}_{n,i} = \boldsymbol{z}^*_{n,j}$ for some $i \neq j$), then $\Psi_n(\boldsymbol{z})$ is unique up to a multiplicative constant.

Theorem (Ya.)

Let Δ be symmetric w.r.t. $(\mathfrak{R},\mathcal{I})$ and set $w^2(z)=\prod_{e\in E}(z-e)$. Assume that there exists an infinite subsequence $\mathbb{N}_*$ such that the closure of $\left\{\{\boldsymbol{z}_{n,i}\}_{i=1}^{g}\right\}_{n\in\mathbb{N}_*}$ contains no divisor with an involution-symmetric pair nor with $\infty^{(0)}$. Let

$$f_\rho(z):=\frac{1}{2\pi\mathrm{i}}\int_\Delta\frac{\rho(t)}{t-z}\frac{\mathrm{d}t}{w^+(t)},$$

where ρ is a non-vanishing Lipschitz smooth function on Δ. Then

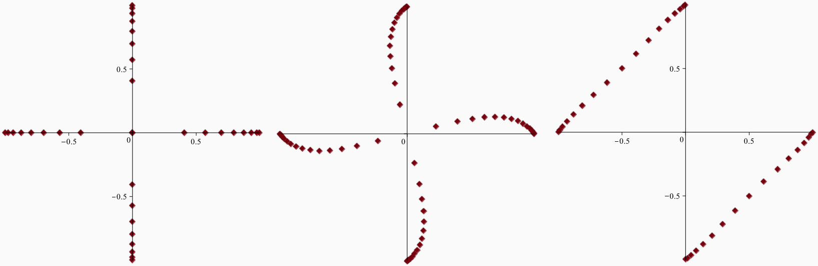
$$f_\rho(z)-[n/n;\mathcal{I}]_{f_\rho}(z)=\frac{v_n(z)}{w(z)}\frac{\Psi_n\left(z^{(1)}\right)}{\Psi_n\left(z^{(0)}\right)}\frac{1+\varepsilon_{n1}(z)+\varepsilon_{n2}(z)\Upsilon_n\left(z^{(1)}\right)}{1+\varepsilon_{n1}(z)+\varepsilon_{n2}(z)\Upsilon_n\left(z^{(0)}\right)}$$

where $\varepsilon_{ni}(z)=o(1)$ locally uniformly in D and vanish at infinity and Υ_n is a rational function on $\mathfrak{R}$ that vanishes at $\infty^{(0)}$ and whose divisor of poles is equal to $\boldsymbol{z}_{n,1}+\cdots+\boldsymbol{z}_{n,g}+\infty^{(1)}$. Moreover,

$$\left|\frac{v_n(z)}{w(z)}\frac{\Psi_n\left(z^{(1)}\right)}{\Psi_n\left(z^{(0)}\right)}\right|\leq C_K\exp\left\{\sum_{i=1}^{2n}g\left(z^{(1)},v_{n,i}^{(0)}\right)\right\}=o(1)$$

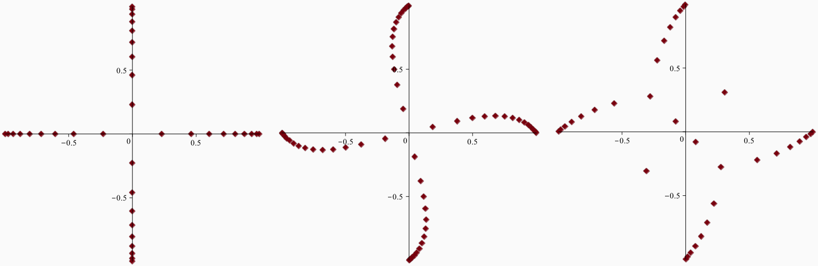
for every closed subset $K\subset D$.

Strong-type Asymptotics



Zeros of q_{36}, q_{60}, and q_{34} to $(z^4-1)^{-1/2}$ corresponding to the interpolation schemes $\{\pm1\pm\mathrm{i}\}$, $\{1/4+\mathrm{i}, -1/4-\mathrm{i}, 1-\mathrm{i}/4, -1+\mathrm{i}/4\}$, and $\{1+\mathrm{i}, -1-\mathrm{i}\}$.

Strong-type Asymptotics



Zeros of q_{36}, q_{60}, and q_{34} to $(z^4 - 1)^{-1/4}$ corresponding to the interpolation schemes $\{\pm 1 \pm \mathrm{i}\}$, $\{1/4 + \mathrm{i}, -1/4 - \mathrm{i}, 1 - \mathrm{i}/4, -1 + \mathrm{i}/4\}$, and $\{1 + \mathrm{i}, -1 - \mathrm{i}\}$.