Spurious Poles in Padé Approximation of Algebraic Functions

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In 1844 Liouville[1] constructed the first example of a transcendental number by using continued fractions.

Carefully studying similarities between simultaneous diophantine approximation of real numbers and rational approximation of holomorphic functions, Hermite[2] proved in 1873 that e is transcendental.

[1] Sur des class trè étendues de quantités dont la valeur n'est ni algébrique, ni même réductible à des irrationelles algébriques. C.R. Acad. Sci. Paris, 18:883--885, 910--911, 1844

[2] Sur la fonction exponentielle. C.R. Acad. Sci. Paris, 77:18--24, 74--79, 226--233, 285--293, 1873

Hermite's proof is based on the following criterion.

Lemma

a is transcendental if for any $m \in \mathbb{N}$ and any $\varepsilon > 0$ there exist $m+1$ linearly independent vectors of integers $(q_j, p_{j1}, \ldots, p_{jm})$, $j = \overline{0, m}$, such that $\left|q_j a^k - p_{jk}\right| \leq \varepsilon$, $k = \overline{1, m}$.

If a is algebraic, then for some $m \in \mathbb{N}$ there exist $a_k \in \mathbb{Z}$, $k = \overline{0, m}$, such that $\sum_{k=0}^{m} a_k a^k = 0$. Hence,

$$\sum_{k=1}^{m} a_k (q_j a^k - p_{jk}) + a_0 q_j + \sum_{k=1}^{m} a_k p_{jk} = 0.$$

Then for some $0 \leq j_0 \leq m$, it holds that

$$1 \leq \left|\sum_{k=1}^{m} a_k (q_{j_0} a^k - p_{j_0 k})\right| \leq \varepsilon \sum_{k=1}^{m} |a_k|.$$

Let $n_0, n_1, \ldots, n_m$ be non-negative integers. Set $N := n_0 + \cdots + n_m$ and consider the following system:

$$Q(z)e^{kz} - P_k(z) = \mathcal{O}\left(z^{N+1}\right),$$

where $\deg(Q) \leq N - n_0$ and $\deg(P_k) \leq N - n_k$.

Hermite proceeded to explicitly construct these polynomials, which as it turned out have integer coefficients. By evaluating these polynomials at 1 he succeeded in applying the above criterion.

Let $F(z) = \sum_{k=0}^{\infty} f_k z^k$ be a function holomorphic at the origin. Consider the following system:

$$Q(z)F(z) - P(z) = \mathcal{O}\left(z^{m+n+1}\right),$$

where $\deg(Q) \leq n$ and $\deg(P) \leq m$. This system always has a solution. Indeed,

$$Q(z)F(z) = \sum_{k=0}^{\infty} \left(\sum_{j+i=k, i \leq n} f_j q_i \right) z^k.$$

Set $f_{-k} := 0$ for $k > 0$. Then

$$\begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_m \end{pmatrix} = \begin{pmatrix} f_0 & f_{-1} & \cdots & f_{-n} \\ f_1 & f_0 & \cdots & f_{1-n} \\ \vdots & \vdots & \ddots & \vdots \\ f_m & f_{m-1} & \cdots & f_{m-n} \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ \vdots \\ q_n \end{pmatrix}$$

and

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} f_{m+1} & f_m & \cdots & f_{m+1-n} \\ f_{m+2} & f_{m+1} & \cdots & f_{m+2-n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m+n+1} & f_{m+n} & \cdots & f_{m+1} \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ \vdots \\ q_n \end{pmatrix}$$

The latter is a linear system of n equations with $n+1$ unknowns. Such a system always has a solution. A solution may not be unique, but the ratio $[m/n]_F := P/Q$ always is.

Indeed, let $Q_1(z), P_1(z)$ and $Q_2(z), P_2(z)$ be solutions. Then

$$Q_2(z)\Big(Q_1(z)F(z) - P_1(z)\Big) = \mathcal{O}\Big(z^{m+n+1}\Big)$$

and

$$Q_1(z)\Big(Q_2(z)F(z) - P_2(z)\Big) = \mathcal{O}\Big(z^{m+n+1}\Big).$$

Therefore,

$$Q_2(z)P_1(z) - Q_1(z)P_2(z) = \mathcal{O}\Big(z^{m+n+1}\Big).$$

However,

$$\deg\Big(Q_2P_1 - Q_1P_2\Big) \leq m + n.$$

$$\begin{array}{lcccccc}
\text{0-th row} \longrightarrow & [0/0]_F & [1/0]_F & \cdots & [m/0]_F & \cdots \\
\text{1-st row} \longrightarrow & [0/1]_F & [1/1]_F & \cdots & [m/1]_F & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
n\text{-th row} \longrightarrow & [0/n]_F & [1/n]_F & \cdots & [m/n]_F & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots
\end{array}$$

Theorem

Let $F(z)$ be an analytic function in $|z| \leq R$. Then $[m/0]_F(z)$ converge to $F(z)$ uniformly in $|z| \leq R$ as $m \to \infty$.

The following theorem is due to de Montessus de Ballore[3].

Theorem

Let $F(z)$ be a meromorphic function in $|z| \leq R$ with N poles contained in $0 < |z| < R$. Then $[m/N]_F(z)$ converge to $F(z)$ in $|z| \leq R$ in the spherical metric as $m \to \infty$.

[3] Sur les fractions continues algébriques. Bull. Soc. Math. de France, 30:28--36, 1902.

The following theorem is due to Gonchar[4] and Suetin[5][6].

Theorem

Let $F(z)$ be a holomorphic function at the origin. If the poles of Padé approximants $[m/N]_F(z)$ converge to the points $z_1, \dots, z_N$ as $m \to \infty$, then $F(z)$ can be meromorphically continued to $|z| < R_N := \max |z_k|$ and all the points z_k are singularities of $F(z)$ (polar if $|z_k| < R_N$).

[4] Poles of rows of the Padé table and meromorphic continuation of functions, Math. USSR-Sb., 43(4):527--546, 1982

[5] On poles of the m-th row of a Padé table, Math. USSR-Sb., 48(2):493--497, 1984

[6] On an inverse problem for the m-th row of a Padé table, Math. USSR-Sb., 52(1):231--244, 1985

The following theorem is due to Beardon[7].

Theorem

Let $F(z)$ be an analytic function in $|z| \leq R$. Then an infinite subsequence of $[m/1]_F(z)$ converges to $F(z)$ uniformly in $|z| \leq R$ as $m \to \infty$.

[7] On the location of poles of Padé approximants. J. Math. Anal. Appl., 21:469--474, 1968.

The following example is due to Lubinsky and Saff[8].

Theorem

Set $F_q(z) := 1 + z + qz^2 + q^3z^3 + q^6z^4 + \cdots$, where $q = e^{2\pi i\partial}$ with ∂ irrational. Then, for each fixed $N \geq 1$, Padé approximants $[m/N]_{F_q}(z)$ converge to $F_q(z)$ locally uniformly in $|z| < R_{q,N}$ as $m \to \infty$ for some $R_{N,q} < 1$. Moreover, the circle $|z| = R_{q,N}$ necessarily contains limit points of the poles of $[m/N]_{F_q}(z)$ and no subsequence of approximants converges to $F_q(z)$ locally uniformly in $|z| < 1$.

Observe that $F_q(z)$ is holomorphic in $|z| < 1$ with the unit circle being the natural boundary of analyticity.

[8] Convergence of Padé approximants of partial theta function and Rogers-Szegő polynomials. Constr. Approx., 3:331–361, 1987.

The following theorem is due to Buslaev, Gonchar, and Suetin[9].

Theorem

Let $F(z)$ be a holomorphic function in $|z| < R$. Then for each N there exists $R_N < R$ such that some subsequence of $[m/N]_F(z)$ converges to $F(z)$ uniformly in $|z| \leq R_N$ as $m \to \infty$.

[9] On convergence of subsequences of the m-th row of a Padé table, Math. USSR-Sb., 48(2):535--540, 1984

The following theorem is due to Zinn-Justin[10].

Theorem

Let $F(z)$ be a meromorphic function in $|z| \leq R$ with N poles contained in $0 < |z| \leq R$. Then $[m_k/n_k]_F(z)$ converge to $F(z)$ in measure in $|z| < R$ for $n_k \geq N$ as $k \to \infty$ and $m_k/n_k \to \infty$.

[10] Convergence of Padé approximants in the general case. In Colloquium on Advanced Computing Methods in Theoretical Physics, A. Visconti (ed.), pp. 88--102, C.R.N.S., Marseille, 1971.

The following theorem is due to Arms and Edrei[11].

Theorem

Let

$$F(z) = e^{cz} \prod_{k=1}^{\infty} \left(1 + a_k z\right)\left(1 - b_k z\right)^{-1},$$

where $c, a_k, b_k \geq 0$, and $\sum_{k=1}^{\infty}(a_k + b_k) < \infty$. If $m_k/n_k \to \mathfrak{I} \in (0, \infty)$ as $k \to \infty$, then

$$P_{m_k}(z) \quad \to \quad \exp\left\{\frac{cz}{1+\mathfrak{I}}\right\} \prod_{k=1}^{\infty}\left(1 + a_k z\right)$$

$$Q_{n_k}(z) \quad \to \quad \exp\left\{\frac{-c\mathfrak{I}z}{1+\mathfrak{I}}\right\} \prod_{k=1}^{\infty}\left(1 - b_k z\right)$$

locally uniformly in the complex plane, where $[m_k/n_k]_F = P_{m_k}/Q_{n_k}$.

[11] The Padé tables and continued fractions generated by totally positive sequences. In Mathematical Essays, H. Shankar (ed.), pp. 1–21, Ohio University Press, Athens, Ohio, 1970.

The following theorem is due to Lubinsky[12].

Theorem

Let $F(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ be such that $a_{k-1}a_{k+1}/a_k^2 \to a$, $|a| < 1$, as $k \to \infty$. Then $[m_k/n_k]_F(z)$ converge to $F(z)$ locally uniformly in the complex plane as $k \to \infty$ and $m_k \to \infty$.

[12] Padé tables of entire functions of very slow and smooth growth, II. Constr. Approx., 4(1):321--339, 1988.

Let $F(z) = \sum_{k=1}^{\infty} f_k z^{-k}$ be a function holomorphic at infinity. Consider the following system:

$$Q_n(z)F(z) - P_n(z) = \mathcal{O}\left(z^{-(n+1)}\right),$$

where $\deg(Q_n), \deg(P_n) \leq n$. This system always has a solution and for any solution the rational function $[n/n]_F = P_n/Q_n$ is unique.

From the equality $Q_n(z)F(z) - P_n(z) = \mathcal{O}\left(z^{-(n+1)}\right)$, it follows that

$$0 = \oint_\Gamma z^k \left(Q_n(z)F(z) - P_n(z)\right) \mathrm{d}z$$

for $k \in \{0, \ldots, n-1\}$, where Γ is any Jordan curve in the domain of holomorphy of $F(z)$ encircling the point at infinity. However, since $z^k P_n(z)$ is holomorphic in the interior domain of Γ, it holds that

$$0 = \oint_\Gamma z^k Q_n(z)F(z) \mathrm{d}z.$$

In particular, if $F(z) = \int \frac{\mathrm{d}\mu(x)}{z-x}$, where μ is a positive measure compactly supported on the real line ($F(z)$ is a Markov function), then

$$0 = \int x^k Q_n(x)\mathrm{d}\mu(x), \quad k \in \{0, \dots, n-1\}.$$

Using the above orthogonality Markov[13] showed the following.

Theorem

Let F be as above. Padé approximants $[n/n]_F(z)$ converge to $F(z)$ locally uniformly (including at infinity) outside of the convex hull of $\mathrm{supp}(\mu)$.

[13] Deux demonstrations de la convergence de certaines fractions continues. Act. Math., 19:93--104, 1895.

Based on the analytical and numerical evidence, Baker, Gammel, and Wills[14] put forward the following conjecture.

Padé Conjecture

Let $F(z)$ be a holomorphic function in $|z| < R$ except for N poles contained in $0 < |z| < R$ and one point on the boundary $|z| = R$ where it is continuous. Then at least a subsequence of $[n/n]_F(z)$ converges locally uniformly to $F(z)$ in $\{ |z| < R \} \setminus \{ \text{poles of } F \}$.

[14] An investigation of the applicability of the Padé approximant method. J. Math. Anal. Appl., 2:405–418, 1961.

For q which is not a root of unity and $|q| = 1$, define

$$H_q(z) = 1 + \frac{qz|}{|1} + \frac{q^2z|}{|1} + \frac{q^3z|}{|1} + \cdots .$$

The following result is due to Lubinsky[15].

Theorem

Let $q = e^{2\pi i\partial}$, where $\partial = 2/(99 + \sqrt{5})$. Then $H_q(z)$ is meromorphic in $|z| < 1$ and holomorphic at the origin. Moreover, there does not exist any subsequence of $[n/n]_{H_q}(z)$ that converges to $H_q(z)$ uniformly on compact subsets of $\{\ |z| < 0.46\ \} \setminus \{\ \text{poles of } H_q\ \}$.

[15] Rogers-Ramanujan and the Baker-Gammel-Wills (Padé) conjecture. Ann. of Math., 157:847–889, 2003.

Let ω be a compactly supported probability Borel measure. The logarithmic energy of ω is defined by

$$I[\omega] := \iint \log \frac{1}{|z-u|} \mathrm{d}\omega(u) \mathrm{d}\omega(z).$$

Let K be a compact set. The logarithmic capacity of K is defined as

$$\mathrm{cp}(K) := \exp\left\{ - \inf I[\omega] \right\},$$

where infimum is taken over all probability Borel measures on K.

In particular, if D, the unbounded component of the complement of K, is simply connected and Φ is the conformal map of D onto $|z| > 1$ such that $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$, then

$$\Phi(z) = \frac{z}{\mathrm{cp}(K)} + \text{terms analytic at infinity.}$$

A polar set is a set that cannot support a single positive Borel measure with finite logarithmic energy. Polar sets are totally disconnected.

A property is said to hold quasi everywhere (q.e.) if it holds everywhere except on a polar set.

From now on, all the Padé approximants interpolate at infinity.

The following result is due to Nuttall[16] and Pommerenke[17].

Theorem

Let $F(z)$ be a meromorphic and single-valued function in the extended complex plane except for a compact polar set. Then, as $n \to \infty$, the diagonal Padé approximants $[n/n]_F(z)$ converge in capacity to $F(z)$ in the domain of meromorphy of $F(z)$ and the convergence is faster than geometric.

[16] The convergence of Padé approximants of meromorphic functions, J. Math. Anal. Appl. 31, 129--140, 1970

[17] Padé approximants and convergence in capacity, J. Math. Anal. Appl. 41, 775--780, 1973

A tremendous step forward in the investigation of the behavior of Padé approximants was done by Herbert Stahl[18].

Theorem

Let $F(z)$ be holomorphic at infinity, multi-valued, and with all its singularities contained in a compact polar set E. Then

(i) there exists the unique maximal domain D, such that $[n/n]_F(z)$ converge in capacity to $F(z)$ in D as $n \to \infty$;

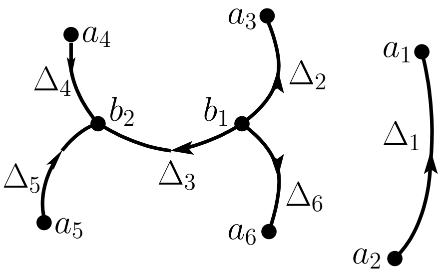
(ii) $\Delta := \overline{\mathbb{C}} \setminus D$ is characterized as the set of the smallest logarithmic capacity among all compact sets that make $F(z)$ single-valued in their complement.

18 The convergence of Padé approximants to functions with branch points, J. Approx. Theory, 91, 139--204, 1997

Moreover, it holds that[19]

$$\Delta = E \cup E_0 \cup \bigcup \Delta_j,$$

where E_0 is finite and Δ_j are open analytic arcs connecting the points in $E \cup E_0$.



[19] H. Stahl, The structure of extremal domains associated with an analytic function, Complex Variables Theory Appl. 4, 339--354, 1985

Let F be holomorphic in the extended complex plane except at finitely many finite points where it has algebro-logarithmic branching. Then

$$\Delta = \{a_1, \ldots, a_p\} \cup \{b_1, \ldots, b_{p-2}\} \cup \bigcup \Delta_j,$$

where $\{a_1, \ldots, a_p\}$ are some of the branch points of F (the ones that belong to the considered sheet of the Riemann surface), $\{b_1, \ldots, b_{p-2}\}$ are not necessarily distinct, and the arcs Δ_j are the negative critical trajectories of the quadratic differential

$$\frac{(z-b_1)\cdot\ldots\cdot(z-b_{p-2})}{(z-a_1)\cdot\ldots\cdot(z-a_p)}(\mathrm{d}z)^2.$$

That is, for any smooth parametrization $z(t)$, $t \in [0,1]$, of Δ_j, it holds

$$\frac{(z(t)-b_1)\cdot\ldots\cdot(z(t)-b_{p-2})}{(z(t)-a_1)\cdot\ldots\cdot(z(t)-a_p)}(z'(t))^2 < 0, \qquad t \in (0,1).$$

A function is called hyperelliptic if it is of the form $r_1 + r_2\sqrt{p}$, where p is a polynomial and r_1, r_2 are rational functions.

Herbert Stahl raised the following question[20]: is the Padé conjecture true for hyperelliptic functions?

[20] Orthogonal polynomials with respect to complex-valued measures. Ann. Comput. Appl. Math., pages 139--154, 1991. IMACS 1990.

This question was settled in negative by Buslaev[21].

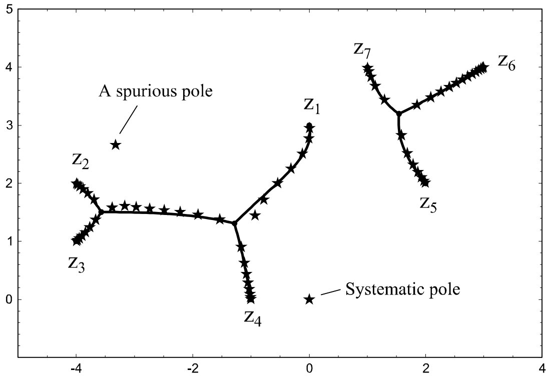
Theorem

Let $j := (-1 + \sqrt{3}\mathrm{i})/2$ and set

$$F(z) = \frac{-27 + 6z^2 + 3(9 + j)z^3 + \sqrt{81(3 - (3 + j)z^3)^2 + 4z^6}}{2z(9 + 9z + (9 + j)z^2)}.$$

There does not exist a subsequence of Padé approximants at the origin $[n/n]_F(z)$ that converges to $F(z)$ simultaneously at z, jz, and j^2z, $|z| < 1$.

[21] On the Baker-Gammel-Wills conjecture in the theory of Padé approximants. Mat Sb., 193(6):25--38, 2002.



The poles[22] of Padé approximant $[63/63]_F$ to function

$$F(z) = \sqrt[4]{\prod_{k=1}^{4}(1 - z_k/z)} + \sqrt[3]{\prod_{k=5}^{7}(1 - z_k/z)}.$$

22 The picture is taken from H. Stahl, Sets of Minimal Capacity and Extremal Domains, manuscript, 2006

Young man, in mathematics you don't understand things. You just get used to them.

Jon von Neumann

The following is an ``explanation'' of what is going wrong with the uniform convergence of Padé approximants to generic algebraic, in particular, hyperelliptic functions.

Let $F(z)$ be a holomorphic function in the extended complex plane except at finitely many finite points where it has algebro-logarithmic branching of integrable order. Then

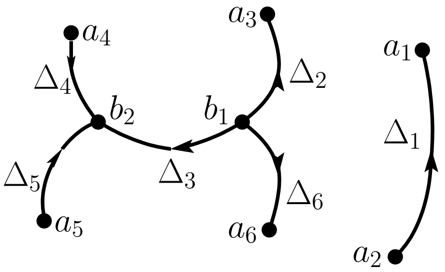
$$\Delta = \{a_1, \ldots, a_p\} \cup \{b_1, \ldots, b_{p-2}\} \cup \bigcup \Delta_j,$$

The arcs Δ_j are the negative critical trajectories of the quadratic differential $h^2(z)\mathrm{d}z^2$, where

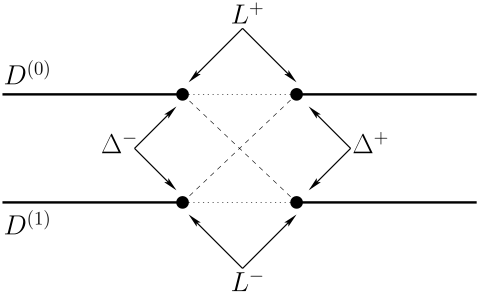
$$h^2(z) := \frac{(z - b_1) \cdot \ldots \cdot (z - b_{p-2})}{(z - a_1) \cdot \ldots \cdot (z - a_p)},$$

where $h(z)z \to 1$ as $z \to \infty$.

Assume that each $b_k \in \Delta$ is incident with exactly three arcs Δ_j.



Let $\mathfrak{R}$ be the Riemann surface of $h(z)$ and g be the genus $\mathfrak{R}$.



Further, let L be the chain on $\mathfrak{R}$ that lies above Δ.

Behind the question of convergence of Padé approximants to algebraic functions lies a certain boundary value problem on L.

Boundary Value Problem

For each $n \in \mathbb{N}$, find S_n holomorphic in $\mathfrak{R} \setminus \left(L \cup \{\infty^{(0)}\}\right)$ and such that it has a pole of order n at $\infty^{(0)}$, a zero of order n at $\infty^{(1)}$ and satisfies

$$S_n^- = JS_n^+ \quad \text{on} \quad L$$

with prescribed behavior at the branching points of $\mathfrak{R}$, where J is the jump of F across Δ lifted to L.

(i) Generically, given $\{P_1, \ldots, P_k\}$ and $\{Z_1, \ldots, Z_{k-g}\}$ on $\mathfrak{R}$ there exist a unique (up to normalization) rational function on $\mathfrak{R}$ with poles P_j and zeros Z_j as the ratio of two such functions will have at most g poles.

(ii) A collection of points $\{P_1, \ldots, P_l\}$, $l \leq g$, is called special if there exists a rational function on $\mathfrak{R}$ with poles only among the points P_j counting multiplicities.

(iii) Generically, the function S_n is unique and has g additional zeros on $\mathfrak{R}$ (the ratio S_n/S_{n-1} is a rational function on $\mathfrak{R}$ and therefore generically should have at least $g+1$ poles and $g+1$ zeros).

"Theorem"

Denote by $\mathbb{N}_{\mathrm{ni}}$ the subsequence of indices for which the function S_n uniquely exists in a proper sense. The gaps in $\mathbb{N}_{\mathrm{ni}}$ are at most of size $g+1$. Let $\{Z_{n1},\dots,Z_{ng}\}$ be the additional zeros of S_n, $n \in \mathbb{N}_{\mathrm{ni}}$. Then

(i) if Z_{nj} belongs to $D^{(0)}$, then $[n/n]_F$ has a pole next to the projection of Z_{nj};

(ii) if Z_{nj} belongs to $D^{(1)}$, then $[n/n]_F$ overinterpolates F at a point next to the projection of Z_{nj};

(iii) the rest of the poles of $[n/n]_F$ converge to Δ and uniform type formulae can be provided.

Akhiezer[23] and Widom[24]: Szegő densities on disjoint subintervals of $\mathbb{R}$

Nuttall[25]: $F(z) = \prod_{j=1}^{3}(z-a_j)^{\alpha_j}$, $\sum_{j=1}^{3}\alpha_j = 0$

Suetin[26]: Hölder-continuous/Chebyshëv weight for disjoint arcs

Baratchart-Ya.[27]: Dini-continuous/Chebyshëv weight for $\{a_1, a_2, a_3\}$

Martínez Finkelstein-Rakhmanov-Suetin[28]: $F(z) = \prod_{j=1}^{p}(z-a_j)^{\alpha_j}$, $\sum_{j=1}^{p}\alpha_j = 0$

Aptekarev-Ya.[29]: the above described setting + Cauchy-type integrals

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[23] Orthogonal polynomials on several intervals. Soviet Math. Dokl., 1:989--992, 1960.

[24] Extremal polynomials associated with a system of curves in the complex plane. Adv. Math., 3:127--232, 1969.

[25] Asymptotics of generalized Jacobi polynomials. Constr. Approx., 2:59--77, 1986

[26] Uniform convergence of Padé diagonal approximants for hyperelliptic functions. Mat. Sb., 191(9):81--114, 2000

[27] Asymptotics of Padé approximants to a certain class of elliptic-type functions, arXiv

[28] Heine, Hilbert, Padé, Riemann, and Stieltjes: a John Nuttall's work 25 years later, arXiv

[29] Padé approximants for functions with branch points -- strong asymptotics of Nuttall-Stahl polynomials, arXiv