Spectral Theory Behind Multiple Orthogonal Polynomials

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Fourier Series

Consider the equation of heat distribution in a metal rod of length π:

$$u_t(\theta, t) = k u_{\theta\theta}(\theta, t)$$
$$u_\theta(0, t) = u_\theta(\pi, t) = 0$$
$$u(\theta, 0) = f(\theta).$$

When $f(\theta) = \cos(n\theta)$, $n \in \mathbb{N}$, it can be easily checked that

$$u(\theta, t) = \cos(n\theta) e^{-n^2 kt}.$$

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In 1807 Fourier realized that if

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\theta), \quad a_n := \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos(n\theta) d\theta,$$

where $f(-\theta) = f(\theta)$, then the solution of the heat equation is given by

$$u(\theta, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\theta) e^{-n^2 kt}.$$

If the Neumann condition $u_\theta(0, t) = u_\theta(\pi, t) = 0$ is replaced with the Dirichlet condition $u(0, t) = u(\pi, t) = 0$, then we must take odd extension of $f(\theta)$ and consider the series in sines. *Fourier considered only series that were convergent*.

Fourier Series

Given an integrable function $f(\theta)$ on $[-\pi, \pi]$, one can identify it with the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(n\theta) + b_n \sin(n\theta) \right),$$

where $a_n := \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos(n\theta) d\theta$ and $b_n := \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin(n\theta) d\theta$. A natural question arises: when does this series converge to $f(\theta)$ and in which sense?

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Riesz-Fischer Theorem: If $f \in L^2$, then $\|f - S_N\|_2 \to 0$ as $N \to \infty$.

Jackson's Theorem: If f is α-Hölder continuous, $|f(\theta) - S_N(\theta)| \leq C\frac{\log N}{N^\alpha}$.

Carleson's Theorem: If $f \in L^2$, then $S_N(\theta)$ converges to $f(\theta)$ a.e.

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Observe that $\{sin(n\theta), \cos(n\theta)\}$ is an orthogonal system on $[-\pi, \pi]$.

Chebyshëv Polynomials

First kind degree n Chebyshëv polynomial $T_n(x)$ is a solution of

$$(1 - x^2)y'' - xy' + n^2 y = 0.$$

It has a famous explicit expression

$$T_n(x) = \cos(n \arccos x), \quad x \in [-1, 1].$$

Moreover, it turns out to be an orthogonal polynomial:

$$\int_{-1}^{1} x^k T_n(x) \frac{dx}{\sqrt{1 - x^2}} = 0, \quad k = \overline{0, n-1}.$$

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From the explicit expression it follows that

$$T_n(\cos\theta) = \cos(n\theta), \quad \theta \in [-\pi, \pi].$$

Hence, to a function $f(x)$ on $[-1, 1]$ we can associate

$$f(x) = f(\cos\theta) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n T_n(x),$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\cos\theta) \cos(n\theta) d\theta = \frac{2}{\pi} \int_{-1}^{1} f(x) T_n(x) \frac{dx}{\sqrt{1-x^2}}.$$

Jacobi Polynomials

Orthonormal Jacobi polynomials $\left\{p_n^{(\alpha,\beta)}(x)\right\}$, $\alpha, \beta > -1$, are defined by

$$\int_{-1}^{1} p_n^{(\alpha,\beta)}(x) p_m^{(\alpha,\beta)}(x) w^{(\alpha,\beta)}(x) dx = \delta_{mn}, \quad w^{(\alpha,\beta)}(x) := (1+x)^{\alpha}(1-x)^{\beta}.$$

To every function $f(x)$ on $[-1,1]$, one can associate a series

$$f(x) \sim \sum_{n=0}^{\infty} c_n p_n^{(\alpha,\beta)}(x), \quad c_n := \int_{-1}^{1} f(x) p_n^{(\alpha,\beta)}(x) w^{(\alpha,\beta)}(x) dx.$$

Theorem (Szegő)

If $|f(x)|$ is integrable w.r.t. $w^{(\alpha,\beta)}(x)$ and $w^{(\alpha/2-1/4,\beta/2-1/4)}(x)$, then

$$\lim_{N\to\infty} \left(S_N^{(\alpha,\beta)}(x) - w^{(-\alpha/2-1/4,-\beta/2-1/4)}(x) S_N(x) \right) = 0$$

uniformly on compact subsets of $(-1,1)$, where $S_N^{(\alpha,\beta)}(x)$ is the N-th Jacobi partial sum and $S_N(\cos\theta)$ is the N-th Fourier partial sum of $w^{(\alpha/2+1/4,\beta/2+1/4)}(\cos\theta) f(\cos\theta)$.

Orthogonal Polynomials

Let μ be a Borel measure with bounded infinite support on the real line. Orthonormal polynomials $\{p_n(x)\}$ are defined by

$$\int p_n(x)p_m(x)d\mu(x) = \delta_{nm}.$$

Theorem (Freud 1953 + Mastroianni & Totik 2000)

If the measure μ is absolutely continuous and doubling on some interval $[a, b]$ ($\mu(2I) \leq c\mu(I)$, where $2I \subseteq [a, b]$ is the interval with the same center and twice the length of an interval I) and $f(x)$ is Hölder continuous with index greater than $1/2$, then

$$\sum_{n=0}^{N-1} c_n(f)p_n(x) \rightrightarrows f \quad \text{on} \quad [a, b], \quad c_n(f) := \int f(x)p_n(x)d\mu(x).$$

Euclidean Algorithm

Let $p/q \in \mathbb{Q}$. The Euclidean Algorithm is used to find the gcd of p and q:

$$
\begin{aligned}
p &= a_0 q + r_0 \\
q &= a_1 r_0 + r_1 \\
r_0 &= a_2 r_1 + r_2 \\
&\cdots \\
r_{n-2} &= a_n r_{n-1}.
\end{aligned}
$$

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However, it also has the following consequence:

$$\begin{aligned} \frac{p}{q} &= a_0 + \frac{r_0}{q} = a_0 + \cfrac{1}{a_1 + \cfrac{r_1}{r_0}} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{r_2}{r_1}}} \\ &= a_0 + \cfrac{1}{a_1 + \cfrac{1}{\cdots + \cfrac{1}{a_n}}} = a_0 + \Phi_{k=1}^{n} \frac{1}{a_k}. \end{aligned}$$

Let now $x \in \mathbb{R}$. Then

$$\begin{aligned} x &= [x] + \frac{1}{1/\{x\}} = [x] + \cfrac{1}{[1/\{x\}] + \cfrac{1}{1/\{1/\{x\}\}}} = \cdots \\ &=: a_0(x) + \Phi_{k=1}^{\infty} \frac{1}{a_k(x)}, \end{aligned}$$

where $a_k(x) \in \mathbb{Z} \cup \{\infty\}$, which is called a *continued fraction* representation of x. Set

$$x_n := a_0(x) + \Phi_{k=1}^{n} \frac{1}{a_k(x)} = \frac{q_n}{p_n} \in \mathbb{Q}$$

to be the *n-th convergent* of the continued fraction.

Fact

Continued fraction

$$a_0(x) + \Phi_{k=1}^{\infty} \frac{1}{a_k(x)}$$

is finite if and only if $x \in \mathbb{Q}$. Moreover, if $x \in \mathbb{R} \setminus \mathbb{Q}$, then

$$\frac{1}{p_n(p_n + p_{n+1})} \leq \left| x - \frac{q_n}{p_n} \right| \leq \frac{1}{p_n p_{n+1}},$$

where q_n/p_n is the n-th convergent. Furthermore,

$$\left| x - \frac{q}{p} \right| < \frac{1}{2p^2} \quad \Rightarrow \quad \frac{q}{p} = \frac{q_m}{p_m}.$$

Continued Fraction of a Series

Start with a formal power series at infinity

$$f(z) = \sum_{k=1}^{\infty} f_k z^{-k}$$

such that the Hankel determinants of the coefficients $\{f_k\}$ are non-zero. Then

$$f(z) = \Phi_{k=1}^{\infty} \frac{b_k}{z - a_k}$$

for some well-defined constants $\{a_k, b_k\}$. Denote $[n/n]_f$ the n-th convergent:

$$[n/n]_f(z) := \Phi_{k=1}^{n} \frac{b_k}{z - a_k}.$$

Then it is known that

$$(f - [n/n]_f)(z) = \mathcal{O}\left(z^{-2n-1}\right)$$

and the above relation *uniquely* determines $[n/n]_f$. Moreover,

$$(P_n f - Q_n)(z) = \mathcal{O}\left(z^{-n-1}\right), \quad [n/n]_f =: Q_n/P_n.$$

Padé Approximants

Let f be a formal power series at infinity and polynomials Q_n, P_n be defined by

$$(P_n f - Q_n)(z) = \mathcal{O}\left(z^{-n-1}\right),$$

$\deg(Q_n), \deg(P_n) \leq n$. Such a pair of polynomials may not be unique, but their ratio *always is*. Thus, we normalize P_n to be *monic* and set

$$Q_n/P_n =: [n/n]_f$$

and call it the *diagonal Padé approximant* for f of order n.

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Moreover, if the power series for f is convergent and Γ encircles infinity within the disk of convergence, then

$$0 = \oint_\Gamma z^k (P_n f - Q_n)(z) dz = \oint_\Gamma z^k P_n(z) f(z) dz$$

for $k = \overline{0, n-1}$ and z belonging to the exterior of Γ. Thus,

$$f(z) = \int \frac{d\mu(x)}{z - x} \quad \Rightarrow \quad 0 = \int x^k P_n(x) d\mu(x).$$

Three-term Recurrence Relations

Let μ be a probability measure with bounded infinite support on the real line and $P_n(x)$ be the monic orthogonal polynomial of degree n, i.e.,

$$\int P_n(x)x^k d\mu(x) = 0, \quad k = \overline{0, n-1}.$$

These polynomials satisfy the three-term recurrence relations:

$$xP_n(x) = P_{n+1}(x) + b_n P_n(x) + a_{n-1} P_{n-1}(x)$$

with $P_{-1} := 0$, $P_0 = 1$, and $a_n > 0$. These relations can be symmetrized:

$$xp_n(x) = c_n p_{n+1}(x) + b_n p_n(x) + c_{n-1} p_{n-1}(x), \quad c_n := \sqrt{a_n},$$

where $p_n(x)$ is the n-th orthonormal polynomial. It holds that

$$c_n \leq |\Delta|/2 \quad \text{and} \quad |b_n| \leq \sup_{x \in \Delta} |x|,$$

where Δ is the convex hull of the support of μ.

Jacobi Matrices

The Jacobi matrix $\mathcal{J}$, defined by

$$\mathcal{J} := \begin{bmatrix} b_0 & c_0 & 0 & \dots \\ c_0 & b_1 & c_1 & \dots \\ 0 & c_1 & b_2 & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix},$$

is symmetric in $\ell^2(\mathbb{Z}_+)$. Since the sequences $\{a_n\}$ and $\{b_n\}$ are both bounded, the operator $\mathcal{J}$ is bounded and self-adjoint. If $p := (p_0, p_1, \dots)$, then

$$\mathcal{J} p = xp \quad \text{and} \quad (\mathcal{J} - z)r = e_0,$$

where $r := (r_0, r_1, \dots)$ and

$$r_n(z) = \int \frac{p_n(x)}{x - z} d\mu(x) = \int \left(\frac{x}{z}\right)^n \frac{p_n(x)}{x - z} d\mu(x).$$

Since $r \in \ell^2(\mathbb{Z}_+)$ for all z large,

$$r = (\mathcal{J} - z)^{-1} e_0, \quad z \notin \sigma(\mathcal{J}).$$

Therefore, μ is the spectral measure for $\mathcal{J}$ as

$$\left\langle (\mathcal{J} - z)^{-1} e_0, e_0 \right\rangle = \int \frac{d\mu(x)}{x - z}.$$

In this cycle we could have started with a bounded Jacobi operator.

Criterion for Transcendence

In 1873 Hermite proved that e is transcendental.

Criterion

α is transcendental if for any $m \in \mathbb{N}$ and any $\varepsilon > 0$ there exist $m+1$ linearly independent vectors of integers $(p_j, q_{j1}, \ldots, q_{jm})$, $j = \overline{0, m}$, such that

$$\left| p_j \alpha^k - q_{jk} \right| \le \varepsilon, \quad k = \overline{1, m}.$$

If α is algebraic, then for some $m \in \mathbb{N}$ there exist $a_k \in \mathbb{Z}$, $k = \overline{0, m}$, such that

$$\sum_{k=0}^{m} a_k \alpha^k = 0.$$

Hence,

$$\sum_{k=1}^{m} a_k (p_j \alpha^k - q_{jk}) + a_0 p_j + \sum_{k=1}^{m} a_k q_{jk} = 0.$$

Then for some $0 \le j_0 \le m$, it holds that

$$1 \le \left| \sum_{k=1}^{m} a_k (p_{j_0} \alpha^k - q_{j_0 k}) \right| \le \varepsilon \sum_{k=1}^{m} |a_k|.$$

Hermite-Padé Approximants

Set $N := n_0 + \cdots + n_m$, where $n_0, n_1, \ldots, n_m$ are non-negative integers. Let $F_1(z), \ldots, F_m(z)$ be functions holomorphic at the origin. Consider

$$P(z)F_k(z) - Q_k(z) = \mathcal{O}\left(z^{N+1}\right),$$

where $\deg(P) \leq N - n_0$ and $\deg(Q_k) \leq N - n_k$.

Such polynomials exist (their coefficients are obtained from a linear system which is always solvable), but are not necessarily unique. The m-tuple $Q_1/P, \ldots, Q_m/P$ is called an *Hermite-Padé approximant of type II*.

Theorem

The m-tuple of Hermite-Padé approximants to the system $e^z, \dots, e^{mz}$ is unique and is given up to the normalization by the formulae

$$\begin{aligned} P(z) &= \mathcal{D}^N[f](0) + \mathcal{D}^{N-1}[f](0)z + \cdots + \mathcal{D}^{n_0}[f](0)z^{N-n_0}, \\ Q_k(z) &= \mathcal{D}^N[f](k) + \mathcal{D}^{N-1}[f](k)z + \cdots + \mathcal{D}^{n_k}[f](k)z^{N-n_k}, \end{aligned}$$

where $f(s) = s^{n_0}(s-1)^{n_1} \cdots (s-m)^{n_m}$ and $\mathcal{D}$ is the diff. operator.

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where $f(s) = s^{n_0}(s-1)^{n_1} \cdots (s-m)^{n_m}$ and $\mathcal{D}$ is the diff. operator.

Theorem

For any $m, n \in \mathbb{N}$, let $P_j, Q_{j1}, \dots, Q_{jm}$, $j = \overline{0, m}$, be the m-tuples of the HP approximants to $e^z, \dots, e^{mz}$ associated with the indices $(n, n, \dots, n) - \vec{e}_j$. Set

$$p_j := P_j(1)/(n-1)! \quad \text{and} \quad q_{jk} := Q_{jk}(1)/(n-1)!.$$

Then these numbers are integers, form $m+1$ linearly independent vectors, and satisfy $|p_j e^k - q_{jk}| \leq c^n/(n-1)!$ for some constant c.

Multiple Orthogonal Polynomials

Let $f_1(z), f_2(z)$ be functions holomorphic at infinity and $\vec{n} = (n_1, n_2) \in \mathbb{Z}_+^2$. Type II Hermite-Padé approximant for f_1, f_2 at infinity corresponding to $\vec{n}$ is defined as a pair of rational functions $Q_{\vec{n},1}(z)/P_{\vec{n}}(z)$ and $Q_{\vec{n},2}(z)/P_{\vec{n}}(z)$, where

$$\left(P_{\vec{n}} f_i - Q_{\vec{n},i}\right)(z) = \mathcal{O}\left(z^{-n_i-1}\right), \quad i = 1, 2,$$

and $\deg P_{\vec{n}} \leq |\vec{n}| := n_1 + n_2$. If functions $f_i(z)$ are Markov functions

$$f_i(z) = \int \frac{d\mu_i(x)}{z - x},$$

where each μ_i is a probability measure with bounded infinite support on the real line, then

$$\int x^k P_{\vec{n}}(x) d\mu_i(x) = 0, \quad k = \overline{0, n_i - 1}.$$

The multi-index $\vec{n}$ is called *normal* if $\deg P_{\vec{n}} = |\vec{n}|$. In this case we normalize $P_{\vec{n}}(x)$ to be monic. The pair (μ_1, μ_2) is called *perfect* if all the multi-indices are normal.

Lattice Recurrence Relations

Let $\vec{e}_1 = (1,0)$ and $\vec{e}_2 = (0,1)$. If (μ_1, μ_2) is perfect, then

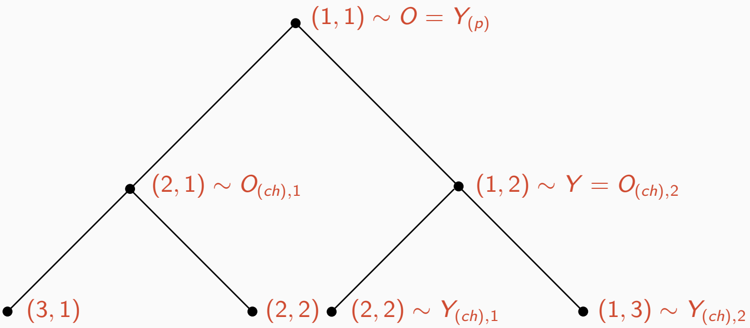
$$\begin{aligned} xP_{\vec{n}}(x) &= P_{\vec{n}+\vec{e}_1}(x) + b_{\vec{n},1}P_{\vec{n}}(x) + a_{\vec{n},1}P_{\vec{n}-\vec{e}_1}(x) + a_{\vec{n},2}P_{\vec{n}-\vec{e}_2}(x) \\ xP_{\vec{n}}(x) &= P_{\vec{n}+\vec{e}_2}(x) + b_{\vec{n},2}P_{\vec{n}}(x) + a_{\vec{n},1}P_{\vec{n}-\vec{e}_1}(x) + a_{\vec{n},2}P_{\vec{n}-\vec{e}_2}(x) \end{aligned}$$

for some coefficients $b_{\vec{n},1}, b_{\vec{n},2}, a_{\vec{n},1}, a_{\vec{n},2}$. These coefficients satisfy consistency conditions

$$\begin{aligned} b_{\vec{n}+\vec{e}_1,2} - b_{\vec{n}+\vec{e}_2,1} &= b_{\vec{n},2} - b_{\vec{n},1}, \\ \sum_{k=1}^{2} a_{\vec{n}+\vec{e}_j,k} - \sum_{k=1}^{2} a_{\vec{n}+\vec{e}_i,k} &= b_{\vec{n}+\vec{e}_j,i}b_{\vec{n},j} - b_{\vec{n}+\vec{e}_i,j}b_{\vec{n},i}, \\ a_{\vec{n},i}(b_{\vec{n},j} - b_{\vec{n},i}) &= a_{\vec{n}+\vec{e}_j,i}(b_{\vec{n}-\vec{e}_i,j} - b_{\vec{n}-\vec{e}_i,i}). \end{aligned}$$

Homogeneous Rooted Tree

Let $\mathcal{T}$ be the rooted tree of all possible increasing paths on $\mathbb{N}^2$ starting at $(1,1)$.



We denote the set of all vertices of $\mathcal{T}$ by $\mathcal{V}$. We let

$$\ell : \mathcal{V} \to \{1,2\}, \quad Y \mapsto \ell_Y \text{ such that } \Pi(Y) = \Pi(Y_{(p)}) + \vec{e}_{\ell_Y},$$

where Π is the natural projection of $\mathcal{V}$ onto $\mathbb{N}^2$.

Jacobi Operator on $\mathcal{T}$

Let $\vec{\kappa} \in \mathbb{R}^2$, $\kappa_1 + \kappa_2 = 1$. Define two interaction functions $A, B : \mathcal{V} \to \mathbb{R}$ by

$$\begin{aligned} A_O &:= 1, \quad B_O := \kappa_1 b_{(0,1),1} + \kappa_2 b_{(1,0),2}, \quad Y = O, \\ A_Y &:= a_{\Pi(Y_{(p)}),\ell_Y}, \quad B_Y := b_{\Pi(Y_{(p)}),\ell_Y}, \quad Y \neq O. \end{aligned}$$

Assume now that

$$0 < a_{\vec{n},j} \text{ for all } \vec{n} \in \mathbb{Z}_+^2 \text{ such that } n_j > 0,$$
$$\sup a_{\vec{n},j} < \infty \,, \, \sup |b_{\vec{n},j}| < \infty.$$

Then, for any function $f \in \ell^2(\mathcal{V})$, the action of the operator $\mathcal{J}_{\vec{\kappa}}$ can be written in the following form

$$\begin{aligned} (\mathcal{J}_{\vec{\kappa}} f)_O &:= (Bf)_O + (A^{1/2} f)_{O_{(ch),1}} + (A^{1/2} f)_{O_{(ch),2}}, \quad Y = O, \\ (\mathcal{J}_{\vec{\kappa}} f)_Y &:= A_Y^{1/2} f_{Y_{(p)}} + (Bf)_Y + (A^{1/2} f)_{Y_{(ch),1}} + (A^{1/2} f)_{Y_{(ch),2}}, \quad Y \neq O. \end{aligned}$$

$\mathcal{J}_{\vec{\kappa}}$ is a bounded and self-adjoint operator on $\ell^2(\mathcal{V})$.

Angelesco Systems

The measures (μ_1, μ_2) form an *Angelesco system* if the convex hulls of their supports, Δ_1 and Δ_2, are disjoint. We assume that $\Delta_1 < \Delta_2$.

Theorem (Aptekarev & Denisov & Ya.)

If (μ_1, μ_2) is an Angelesco system, then it is perfect and $0 < a_{\vec{n},j}$ for all $\vec{n} \in \mathbb{Z}_+^2$ with $n_j > 0$ while $\sup a_{\vec{n},j} < \infty$, $\sup |b_{\vec{n},j}| < \infty$. Moreover, $b_{\vec{n},1} < b_{\vec{n},2}$, $\vec{n} \in \mathbb{Z}_+^2$.

AS: Asymptotics of the Recurrence Coefficients

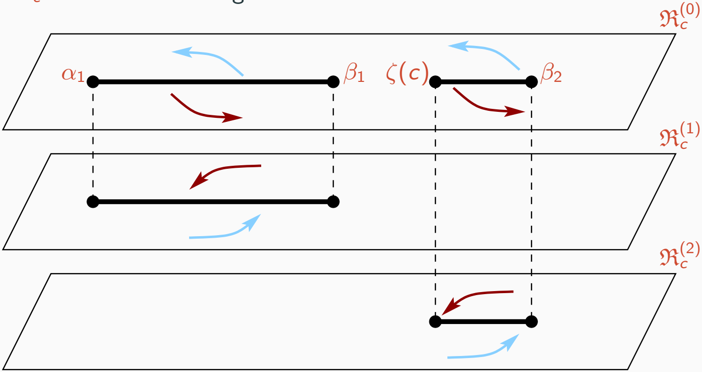
Assume now that $\mathrm{supp}(\mu_i) = \Delta_i = [\alpha_i, \beta_i]$ and let $\mathcal{N}_c \subset \mathbb{Z}_+^2$ be a such that

$$\frac{n_1}{n_1+n_2} \to c \in [0,1] \text{ and therefore } \frac{n_2}{n_1+n_2} \to 1-c.$$

There is a function $\zeta : [0,1] \to [\alpha_1, \beta_2]$, which comes from a certain energy minimization problem, that continuously increases from α_1 to β_2. Put

$$\Delta_{c,1} := \Delta_1 \cap [\alpha_1, \zeta(c)] \quad \text{and} \quad \Delta_{c,2} := \Delta_2 \cap [\zeta(c) \cap \beta_2].$$

Define $\mathfrak{R}_c$ to be the following Riemann surface:



AS: Asymptotics of the Recurrence Coefficients

Theorem (Aptekarev & Denisov & Ya.)

For each $c \in (0,1)$, let $\mathfrak{R}_c$ be as before and $\chi_c : \mathfrak{R}_c \to \overline{\mathbb{C}}$ be a conformal map such that

$$\chi_c\left(z^{(0)}\right) = z + \mathcal{O}\left(z^{-1}\right) \quad \text{as} \quad z \to \infty.$$

Define constants $A_{c,1}, A_{c,2}, B_{c,1}, B_{c,2}$ by

$$\chi_c\left(z^{(i)}\right) = B_{c,i} + A_{c,i} z^{-1} + \mathcal{O}\left(z^{-2}\right) \quad \text{as} \quad z \to \infty.$$

Assume that $\mu_i'(x)$ is analytic and non-vanishing on Δ_i. Then it holds that

$$\lim_{\mathcal{N}_c} a_{\vec{n},i} = A_{c,i} \quad \text{and} \quad \lim_{\mathcal{N}_c} b_{\vec{n},i} = B_{c,i}.$$

The constants $A_{c,i}$ and $B_{c,i}$ are continuous functions of the parameter c and have well defined limits as $c \to 0$ and $c \to 1$.

Quasi AS: Essential Spectrum

Theorem (Aptekarev & Denisov & Ya.)

Let constants $A_{c,1}, A_{c,2}, B_{c,1}, B_{c,2}$ be as above (coming from some intervals $\Delta_1 < \Delta_2$). Further, let $\mathcal{J}_{\vec{\kappa}}$ be a Jacobi operator constructed as before for some constants $\{a_{\vec{n},1}, a_{\vec{n},2}, b_{\vec{n},1}, b_{\vec{n},2}\}_{\vec{n}\in\mathbb{Z}_+^2}$. If

$$\lim_{\mathcal{N}_c} a_{\vec{n},i} = A_{c,i} \quad \text{and} \quad \lim_{\mathcal{N}_c} b_{\vec{n},i} = B_{c,i}$$

for any $\mathcal{N}_c$ and $c \in [0,1]$, then $\sigma_{\text{ess}}(\mathcal{J}_{\vec{\kappa}}) = \Delta_1 \cup \Delta_2$.

AS: Spectral Theorem

Theorem (Denisov & Ya.)

Let $\mathcal{J}_{\vec{\kappa}}$ be a Jacobi operator constructed as before for the recurrence coefficients $\{a_{\vec{n},1}, a_{\vec{n},2}, b_{\vec{n},1}, b_{\vec{n},2}\}_{\vec{n}\in\mathbb{Z}_+^2}$ coming from an Angelesco system. Then $\ell^2(\mathcal{V})$ can be decomposed as an infinite orthogonal sum of cyclic subspaces of $\mathcal{J}_{\vec{\kappa}}$ whose spectral measures admit a semi-explicit expressions. In particular, it holds that

$$\sigma(\mathcal{J}_{\vec{\kappa}}) \subseteq \Delta_1 \cup \Delta_2 \cup E_{\vec{\kappa}},$$

where $E_{\vec{\kappa}}$ is either a single real point or is empty. If $\operatorname{supp}\mu_i = \Delta_i$, $i \in \{1,2\}$, then inclusion becomes equality. If $d\mu_i(x) = \mu_i'(x)dx$ and $(\mu_i')^{-1} \in L^\infty(\Delta_i)$, $i \in \{1,2\}$, then the spectrum of $\mathcal{J}_{\vec{e}_k}$ is purely absolutely continuous.