On Hermite-Padé approximants for a pair of Cauchy transforms with overlapping symmetric supports

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Hermite-Padé Approximants: Definition

Let $\vec{f} = (f_1, \ldots, f_m)$ be a vector of functions holomorphic and vanishing at infinity:

$$f_i(z) = \frac{f_{i1}}{z} + \frac{f_{i2}}{z^2} + \cdots + \frac{f_{in}}{z^n} + \cdots .$$

Let $\vec{n} \in \mathbb{N}^m$ be a multi-index, while $P_{\vec{n}}^{(1)}(z), \ldots, P_{\vec{n}}^{(m)}(z)$ and $Q_{\vec{n}}(z)$ be polynomials such that $\deg(Q_{\vec{n}}) \leq |\,\vec{n}\,| := n_1 + \cdots + n_m$ and

$$R_{\vec{n}}^{(i)}(z) := \left(Q_{\vec{n}} f_i - P_{\vec{n}}^{(i)}\right)(z) = \mathcal{O}\left(z^{-n_i-1}\right) \quad \text{as} \quad z \to \infty.$$

The vector of rational functions

$$\left(P_{\vec{n}}^{(1)}/Q_{\vec{n}}, \ldots, P_{\vec{n}}^{(m)}/Q_{\vec{n}}\right)$$

is called the *type II Hermite-Padé approximant* to $\vec{f}(z)$ corresponding to $\vec{n}$.

Hermite-Padé Approximants: Orthogonality

It follows from Cauchy integral formula that

$$f_i(z) = \int \frac{\mathrm{d}\mu_i(s)}{z - s}$$

for some compactly supported Borel generally speaking complex measure μ_i. Since $R_{\vec{n}}^{(i)}(z) = \mathcal{O}\big(z^{-n_i-1}\big)$, it holds that

$$0 = \int_\Gamma z^k R_{\vec{n}}^{(i)}(z)\mathrm{d}z = \int_\Gamma z^k Q_{\vec{n}}(z) f_i(z)\mathrm{d}z = \int s^k Q_{\vec{n}}(s)\mathrm{d}\mu_i(s)$$

for $k = \overline{0, n_i - 1}$, where Γ is any Jordan curve encircling the support of μ_i. In what follows, it assumed that $Q_{\vec{n}}(z)$ is the *monic polynomial of minimal degree*.

Padé Approximants: Markov Functions

Let μ be a positive Borel measure compactly supported on the real line. Then

$$f(z) = \int \frac{\mathrm{d}\mu(x)}{z - x}$$

is called a *Markov function*. The n-th Padé approximant is defined by the condition

$$R_n(z) = \big(Q_n f - P_n\big)(z) = \mathcal{O}\big(z^{-n-1}\big)$$

In this case it holds that

$$\int x^k Q_n(x)\mathrm{d}\mu(x) = 0, \quad k = \overline{0, n-1}.$$

That is, $Q_n(z)$ is the n-th *orthogonal polynomial* with respect to the measure μ.

Padé Approximants: Distribution of Poles

Denote by σ_n the normalized *counting measure of zeros* of $Q_n(z)$. That is,

$$\sigma_n := \frac{1}{n}\sum_{i=1}^{n}\delta(x_{n,i}), \quad Q_n(x) = \prod_{i=1}^{n}(x - x_{n,i}),$$

where $\delta(x)$ is the Dirac δ-distribution with mass at x. Recall that a sequence of measures converges weak*, $\nu_n \xrightarrow{*} \nu$, if $\int F \mathrm{d}\nu_n \to \int F \mathrm{d}\nu$ for any function F continuous on a compact set containing the supports of ν_n.

Theorem

If $\operatorname{supp}(\mu) = [-1, 1]$ and $\mu' > 0$ a.e. on $[-1, 1]$, then $\sigma_n \xrightarrow{*} \omega$, where

$$\mathrm{d}\omega(x) = \frac{\mathrm{d}x}{\pi\sqrt{1 - x^2}}.$$

Theorem (Szegő)

Let $\rho(x)$ be a non-negative function satisfying $\int_{[-1,1]} \log \rho \mathrm{d}\omega > -\infty$. Set

$$f(z) := \frac{1}{2\pi} \int_{[-1,1]} \frac{1}{z-x} \frac{\rho(x)\mathrm{d}x}{\sqrt{1-x^2}}.$$

Then it holds locally uniformly in $\overline{\mathbb{C}} \setminus [-1,1]$ that

$$\begin{cases} Q_n(z) & \approx & \gamma_n \big(\Phi^n S_\rho\big)(z), \\ R_n(z) & \approx & \gamma_n \big(h\Phi^n S_\rho\big)^{-1}(z), \end{cases}$$

where $h(z) = 1/\sqrt{z^2-1}$, γ_n is the normalizing constant, $S_\rho(z)$ is the *Szegő function* of $\rho(x)$ (non-vanishing and holomorphic with traces satisfying $S_{\rho+}(x)S_{\rho-}(x) = \rho^{-1}(x)$ on $[-1,1]$) and

$$\Phi(z) = z + \sqrt{z^2-1}.$$

- $-\log|\Phi(z)|$ is the logarithmic potential of the logarithmic equilibrium measure for $[-1, 1]$

Logarithmic potential and energy of a compactly supported Borel measure ν are defined by $V^\nu(z) = -\int \log|z - w| d\nu(w)$ and $I[\nu] = \int V^\nu(z) d\nu(z)$.

Given a compact set K, either every Borel measure supported on K has infinite logarithmic energy, in which case K is called *polar*, or there exists the unique probability Borel measure ω_K such that $I[\omega_K] = \inf I[\nu]$, where the infimum is taken over all probability Borel measures supported on K. The measure ω_K is called the *equilibrium measure* of K.

It holds that $\omega_{[-1,1]} = \omega$ and $I[\omega] = 0$.

Padé Approximants: Function $\Phi(z)$

- $-\log|\Phi(z)|$ is the logarithmic potential of the logarithmic equilibrium measure for $[-1,1]$
- $\log|\Phi(z)|$ is the Green's function for $\overline{\mathbb{C}} \setminus [-1,1]$ with poles at infinity

Let K be a compact set and D be the unbounded component of $\overline{\mathbb{C}} \setminus K$. Then $g_K(z;\infty)$, *Green's function for K with pole at ∞*, is uniquely characterized by

- $g_K(z;\infty)$ is harmonic in $D \setminus \{\infty\}$
- $g_K(z;\infty) - \log|z|$ is bounded near ∞
- $g_K(z;\infty) = 0$ for quasi every (up to a polar set) $z \in \partial D$

It holds that $g(z;\infty) = I[\omega_K] - V^{\omega_K}(z)$. The constant $\operatorname{cap}(K) = e^{-I[\omega_K]}$ is called the *logarithmic capacity* of K.

Padé Approximants: Function $\Phi(z)$

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- $\log|\Phi(z)|$ is the Green's function for $\overline{\mathbb{C}} \setminus [-1,1]$ with poles at infinity
- $\log|\Phi(z)| = \operatorname{Re}\left(\int_1^z h(s)\mathrm{d}s\right)$

Let $h(z) = 1/\sqrt{z^2-1}$ be the branch holomorphic in $\overline{\mathbb{C}} \setminus [-1,1]$ and such that $h(z) = 1/z + \mathcal{O}(z^{-2})$ as $z \to \infty$. Then

$$d\omega_{[-1,1]}(x) = d\omega(x) = -\frac{1}{\pi \mathrm{i}} h_+(x)\mathrm{d}x$$

Padé Approximants: Function $\Phi(z)$

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- $\log|\Phi(z)|$ is the Green's function for $\overline{\mathbb{C}}\setminus[-1,1]$ with poles at infinity
- $\log|\Phi(z)| = \mathrm{Re}\left(\int_1^z h(s)\mathrm{d}s\right)$
- part of a rational function on a certain Riemann surface

Let $\mathfrak{R}^{(0)}$ and $\mathfrak{R}^{(1)}$ be two copies of $\overline{\mathbb{C}}\setminus[-1,1]$ cut across $[-1,1]$ and $\mathfrak{R}$ be the surface obtained by gluing $\mathfrak{R}^{(0)}$ and $\mathfrak{R}^{(1)}$ crosswise across the cuts to each other. Denote by $\mathbf{z}$ a point on $\mathfrak{R}$ with natural projection $\pi(\mathbf{z}) = z$. Put

$$\Phi(\mathbf{z}) = \begin{cases} \Phi(z), & \mathbf{z}\in\mathfrak{R}^{(0)}, \\ \Phi^{-1}(z), & \mathbf{z}\in\mathfrak{R}^{(1)}. \end{cases}$$

Notice that $\Phi^{-1}(z) = z - \sqrt{z^2-1}$. Then $\Phi(\mathbf{z})$ is a rational function on $\mathfrak{R}$ with the zero/pole divisor $\infty^{(1)} - \infty^{(0)}$.

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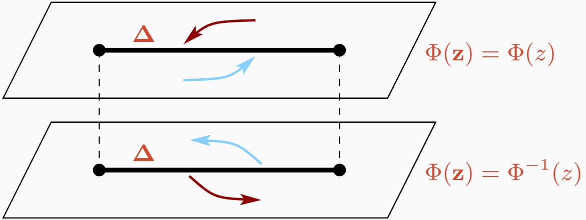

Observe also that $\log|\Phi(\mathbf{z})|$ is harmonic on $\mathfrak{R} \setminus \{\infty^{(0)}, \infty^{(1)}\}$ for which the cycle $\boldsymbol{\Delta} := \pi^{-1}([-1,1])$ is the zero level line.

Padé Approximants: Minimal Capacity Contours

It is said that $f \in \mathcal{S}$ if it can be meromorphically continued along any path in $\overline{\mathbb{C}} \setminus E_f$, where E_f is polar and there exists at least one point in $\overline{\mathbb{C}} \setminus E_f$ with distinct continuations.

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A compact set K is called *admissible* for $f(z)$ if $\overline{\mathbb{C}} \setminus K$ is connected and $f(z)$ has a meromorphic and single-valued extension there.

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A compact set K is called *admissible* for $f(z)$ if $\overline{\mathbb{C}} \setminus K$ is connected and $f(z)$ has a meromorphic and single-valued extension there.

Theorem (Stahl)

Let $f \in \mathcal{S}$. There exists a unique admissible compact Δ_f such that

$$\mathrm{cap}(\Delta_f) \leq \mathrm{cap}(K)$$

for any admissible K. The normalized counting measures of zeros of $Q_n(z)$ converge to ω_{Δ_f} in the weak* sense and it holds that

$$|f(z) - (P_n/Q_n)(z)|^{1/2n} \approx e^{-g_{\Delta_f}(z;\infty)}$$

on compact subsets of $\mathbb{C} \setminus \Delta_f$.

Padé Approximants: Symmetry Property

Theorem (Stahl)

The minimal capacity contour Δ_f can be decomposed as

$$\Delta_f = E_0 \cup E_1 \cup \bigcup \Delta_j,$$

where $E_0 \subseteq E_f$, E_1 consists of isolated points to which f has unrestricted continuations from infinity leading to at least two distinct function elements, and Δ_j are open analytic arcs. Green's function for Δ_f satisfies

$$\frac{\partial g_{\Delta_f}}{\partial n_+} = \frac{\partial g_{\Delta_f}}{\partial n_-} \quad \text{on} \quad \bigcup \Delta_j,$$

where $\partial/\partial n_\pm$ are the one-sided normal derivatives on $\bigcup \Delta_j$.

Theorem (Stahl)

Let $f \in \mathcal{S}$ and Δ_f be its minimal capacity (symmetric) contour. Define

$$h(z) := \partial_z g_{\Delta_f}(z), \quad 2\partial_z := \partial_x - \mathrm{i}\partial_y.$$

The function $h^2(z)$ is holomorphic in $\overline{\mathbb{C}} \setminus (E_0 \cup E_1)$ with a double zero at infinity and the arcs Δ_j are *orthogonal critical trajectories* of the quadratic differential $h^2(z)\mathrm{d}z^2$ (that is, $h^2(z(t))(z'(t))^2 < 0$).

Assume in addition that E_f is finite. For each point $e \in E_0 \cup E_1$ denote by $i(e)$ the bifurcation index of e, that is, the number of different arcs Δ_j incident with e. Then

$$h^2(z) = \prod_{e \in E_0 \cup E_1} (z-e)^{i(e)-2} \prod_{e \in E_2} (z-e)^{2j(e)},$$

where E_2 is the set of critical points of $g_{\Delta_f}(z;\infty)$ and $j(e)$ is the order of $e \in E_2$.

Padé Approximants: Function $\Phi(z)$

Let $f \in \mathcal{S}$ be such that E_f is finite.

Let $\mathfrak{R}^{(0)}$ and $\mathfrak{R}^{(1)}$ be two copies of $\overline{\mathbb{C}} \setminus \Delta_f$ cut across Δ_f and $\mathfrak{R}$ be the surface obtained by gluing $\mathfrak{R}^{(0)}$ and $\mathfrak{R}^{(1)}$ crosswise across the cuts to each other.

Set $h(\mathbf{z}) = (-1)^k h(z)$, $\mathbf{z} \in \mathfrak{R}^{(k)}$, which is a rational function on $\mathfrak{R}$. Put

$$\Phi(\mathbf{z}) = \exp\left\{ \int^{\mathbf{z}} h(\mathbf{s}) \mathrm{d}s \right\}.$$

Then $\Phi(\mathbf{z})$ is meromorphic on $\mathfrak{R}$ expect for the unimodular jumps on a homology basis for $\mathfrak{R}$ with the zero/pole divisor $\infty^{(1)} - \infty^{(0)}$ and such that

- $\log|\Phi(\mathbf{z})|$ is harmonic in $\mathfrak{R} \setminus \{\infty^{(0)}, \infty^{(1)}\}$
- $\pi^{-1}(\Delta_f)$ is the zero level line of $\log|\Phi(\mathbf{z})|$
- $\log|\Phi(\mathbf{z})| = g_{\Delta_f}(z; \infty)$ for $\mathbf{z} \in \mathfrak{R}^{(0)}$

Padé Approximants: Symmetric Contours through Riemann Surfaces

Take $\mathfrak{R} := \{w^2 = P(z)\}$, where $P(z)$ has degree $2g+2$. It is a hyperelliptic surface of genus g ($\pi(\mathbf{z}) = z$, $\mathbf{z} = (z, w)$).

There exists a function $g(\mathbf{z})$ on $\mathfrak{R}$ that is harmonic in $\mathfrak{R} \setminus \{\infty^{(0)}, \infty^{(1)}\}$ and behaves like $(-1)^k \log|z|$ as $\mathbf{z} \to \infty^{(k)}$. This function is involution-symmetric, i.e, $g((z,w)) = g((z,-w))$. Define

$$\Delta := \pi\left(\{\mathbf{z} \in \mathfrak{R} : g(\mathbf{z}) = 0\}\right)$$

Then Δ is a symmetric (minimal capacity) contour for some function and $g_\Delta(z) = g(\mathbf{z})$, $\mathbf{z} \in \mathfrak{R}^{(0)}$, where $\mathfrak{R}^{(0)}$ is the closure of the connected component of $\mathfrak{R} \setminus \{g(\mathbf{z}) = 0\}$ containing $\infty^{(0)}$.

Padé Approximants: Strong Asymptotics

Theorem (Aptekarev-Ya.)

Let $f \in \mathcal{S}$ be such that E_f is finite and $(P_n/Q_n)(z)$ be the n-th diagonal Padé approximant. Then

$$\begin{aligned} Q_n(z) &\approx \gamma_n \Psi_n(\mathbf{z}) \approx \mathrm{cap}^n(\Delta_f)\Phi^n(\mathbf{z}), \quad \mathbf{z} \in \mathfrak{R}^{(0)}, \\ R_n(z) &\approx \gamma_n \Psi_n(\mathbf{z}) \approx \mathrm{cap}^n(\Delta_f)\Phi^n(\mathbf{z}), \quad \mathbf{z} \in \mathfrak{R}^{(1)}, \end{aligned}$$

where $\Psi_n(\mathbf{z})$ is meromorphic in $\mathfrak{R} \setminus \pi^{-1}(\Delta_f)$ with the zero/pole divisor $(n-g)\infty^{(1)} + \sum_{i=1}^{g} \mathbf{z}_{n,i} - n\infty^{(0)}$ that solves a certain boundary value problem on $\pi^{-1}(\Delta_f)$ (g is the genus of $\mathfrak{R}$) and γ_n is a normalizing constant.

Angelesco Systems: Orthogonality

We shall say that a vector function $\vec{f} = (f_1, \ldots, f_m)$ forms an *Angelesco system* if

$$f_i(z) = \int \frac{\mathrm{d}\mu_i(x)}{z - x}, \quad \mu_i > 0, \;\; \mathrm{supp}(\mu_i) = [a_i, b_i], \;\; [a_i, b_i] \cap [a_j, b_j] = \varnothing.$$

Given a multi-index $\vec{n} = (n_1, \ldots, n_m)$, $|\vec{n}| = n_1 + \cdots + n_m$, we can write

$$\int x^k Q_{\vec{n}}(x) \mathrm{d}\mu_i(x) = 0, \quad k = \overline{0, n_i - 1}.$$

Hence, $Q_{\vec{n}}(z)$ has n_i simple zeros on $[a_i, b_i]$. Denote by $\sigma_{\vec{n},i}$ their counting measure normalized by $|\vec{n}|$. That is, $|\sigma_{\vec{n},i}| = n_i/|\vec{n}|$.

Theorem (Gonchar-Rakhmanov)

Assume that $\mu_i' > 0$ a.e. on $[a_i, b_i]$. Let $\{\vec n\}$ be a sequence of multi-indices such that $\vec n|\vec n|^{-1} \to \vec c \in (0,1)^m$, $|\,\vec c\,| = 1$. Then there exists a vector equilibrium measure $(\omega_{\vec c,1}, \dots, \omega_{\vec c,m})$ (unique minimizer of a certain energy functional) such that

$$\sigma_{\vec n,i} \xrightarrow{*} \omega_{\vec c,i}.$$

Moreover, it holds that $\operatorname{supp}(\omega_{\vec c,i}) = [a_{\vec c,i}, b_{\vec c,i}] \subseteq [a_i, b_i]$ and

$$\begin{cases} |\vec n|^{-1} \log |Q_{\vec n}(z)| & \approx -V^{\omega_{\vec c}}(z), \quad \omega_{\vec c} = \omega_{\vec c,1} + \cdots + \omega_{\vec c,m}, \\ |\vec n|^{-1} \log |R_{\vec n}^{(i)}(z)| & \approx V^{\omega_{\vec c,i}}(z) - \ell_{\vec c,i}, \quad i = \overline{1,m}, \end{cases}$$

for some constants $\ell_{\vec c,i}$.

Angelesco Systems: Divergence Domains

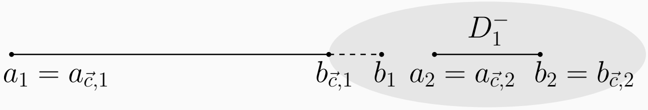
It follows from the previous theorem that

$$|\vec{n}|^{-1}\log\left|f_i(z)-(P_{\vec{n}}^{(i)}/Q_{\vec{n}})(z)\right| = V^{\omega_{\vec{c}}+\omega_{\vec{c},i}}(z)-\ell_{\vec{c},i}$$

Define the divergence domain by

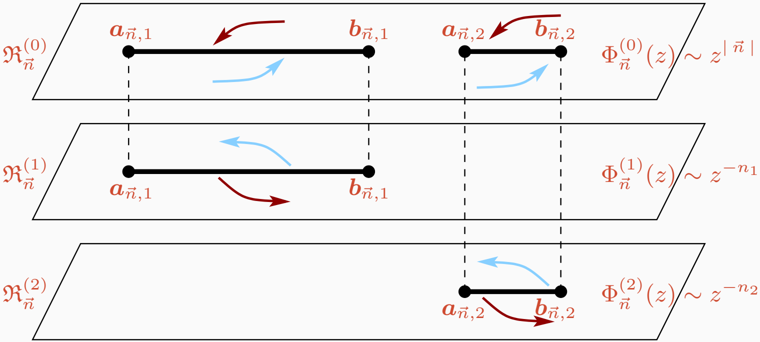
$$D_{\vec{c},i}^{-}=\left\{z:\ell_{\vec{c},i}-V^{\omega_{\vec{c}}+\omega_{\vec{c},i}}(z)<0\right\}$$

It might happen that $D_{\vec{c},i}^{-}$ is non-empty, but it is always bounded.



Angelesco Systems: Riemann Surface

Let $\vec{\omega}_{\vec{n}}$ be the vector equilibrium measure for $\vec{n}/|\vec{n}|$. Define $\mathfrak{R}_{\vec{n}}$ w.r.t. $\vec{\omega}_{\vec{n}}$ by



Angelesco Systems: Function $\Phi_{\vec{n}}(\mathbf{z})$

The surface $\mathfrak{R}_{\vec{n}}$ has genus 0. Let $\Phi_{\vec{n}}(\mathbf{z})$ be the rational function on $\mathfrak{R}_{\vec{n}}$ with the zero/pole divisor and normalization given by

$$(\Phi_{\vec{n}}) = n_1\infty^{(1)} + \cdots + n_m\infty^{(m)} - |\vec{n}|\infty^{(0)}, \quad \prod \Phi_{\vec{n}}^{(k)}(z) \equiv 1.$$

There exist points $\mathbf{z}_i$, $i = \overline{1, m-1}$, "in the gaps" on $\mathfrak{R}^{(0)}$ and a rational function $h_{\vec{n}}(\mathbf{z})$ with the zero pole/divisor and normalization

$$(h_{\vec{n}}) = \sum_{i=1}^{m-1} \mathbf{z}_i + \sum_{k=0}^{m} \infty^{(k)} - \sum_{i=1}^{m} \left(\mathbf{a}_{\vec{n},i} + \mathbf{b}_{\vec{n},i}\right), \quad h^{(0)}(z) \sim 1/z,$$

such that

$$\Phi_{\vec{n}}(\mathbf{z}) = \exp\left\{\int^{\mathbf{z}} h_{\vec{n}}(\mathbf{s})\mathrm{d}s\right\}.$$

Angelesco Systems: Function $\Phi_{\vec{n}}(\mathbf{z})$

Moreover, it holds that

$$\frac{1}{|\vec{n}|}\log|\Phi_{\vec{n}}(\mathbf{z})| = \begin{cases} -V^{\omega_{\vec{n}}}(z) + \frac{1}{m+1}\sum_{i=1}^{m}\ell_{\vec{n},i}, & \mathbf{z}\in\mathfrak{R}^{(0)}, \\ V^{\omega_{\vec{n},k}}(z) - \ell_{\vec{n},k} + \frac{1}{m+1}\sum_{i=1}^{m}\ell_{\vec{n},i}, & \mathbf{z}\in\mathfrak{R}^{(k)}, \end{cases}$$

and

$$\mathrm{d}\omega_{\vec{n}}(x) = \left(h_{\vec{n}-}^{(0)}(x) - h_{\vec{n}+}^{(0)}(x)\right)\frac{\mathrm{d}x}{2\pi\mathrm{i}}.$$

In particular, the boundary between convergence and divergence domains can be described as

$$\partial D_{\vec{n},i}^{-} = \left\{s : |\Phi_{\vec{n}}^{(0)}(s)| = |\Phi_{\vec{n}}^{(i)}(s)|\right\}$$

That is, it is an orthogonal trajectory of $\left(h_{\vec{n}}^{(0)}(s) - h_{\vec{n}}^{(i)}(s)\right)^2\mathrm{d}s^2$.

Angelesco Systems: Strong Asymptotics

Theorem (Ya.)

Let $\rho_i(x)$ be a Fisher-Hartwig perturbation of a non-vanishing holomorphic function on $[a_i, b_i]$ and

$$f_i(z) := \frac{1}{2\pi \mathrm{i}} \int_{[a_i,b_i]} \frac{\rho_i(x)\mathrm{d}x}{x-z}.$$

Further, let $\{\vec{n}\}$ be a sequence of multi-indices such that $\vec{n}/|\vec{n}| \to \vec{c} \in (0,1)^m$. Then

$$\begin{cases} Q_{\vec{n}}(z) & \approx & \left(\Phi_{\vec{n}} S\right)^{(0)}(\mathbf{z}), \\ R_{\vec{n}}^{(i)}(z) & \approx & \left(\Phi_{\vec{n}} S\right)^{(i)}(\mathbf{z}), \end{cases}$$

where $S(\mathbf{z})$ is a Szegő-type function on $\mathfrak{R}_{\vec{c}}$.

Previous works by Kalyagin, Aptekarev, Aptekarev–Lysov, and subsequent work by Aptekarev-Denisov-Ya. ($\vec{c} \in [0,1]^2$ for $m = 2$).

We say that a vector function $\vec{f} = (f_1, f_2)$ forms a *symmetric Stahl system* if

$$f_i \leftrightarrow \mu_i, \quad \mathrm{supp}(\mu_1) = [-1, a], \quad \mathrm{supp}(\mu_2) = [-a, 1], \quad a \in (0, 1).$$

Let h be an algebraic function given by

$$A(z)h^3 - 3B_2(z)h - 2B_1(z) = 0$$

where, for some parameter $p > 0$, we set

$$\begin{cases} A(z) & := (z^2 - 1)(z^2 - a^2), \\ B_2(z) & := z^2 - p^2, \\ B_1(z) & := z, \end{cases}$$

Symmetric Stahl Systems: Riemann Surface

Denote by $\mathfrak{R}$ the Riemann surface of h. We are looking $\mathfrak{R}$ such that

$$\operatorname{Re}\left(\int^{\mathbf{z}} h(\mathbf{s})\mathrm{d}s\right) \text{ is } \textit{single-valued} \text{ and harmonic on } \mathfrak{R} \quad (*)$$

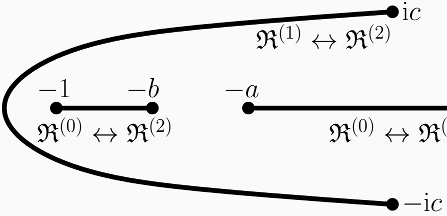
Theorem (Aptekarev-Van Assche-Ya.)

(I) If $a \in (0, 1/\sqrt{2})$, then there exists $p \in \left(a, \sqrt{(1+a^2)/3}\right)$ such that condition $(*)$ is fulfilled. In this case $\mathfrak{R}$ has 8 ramification points whose projections are $\{\pm 1, \pm a\}$ and $\{\pm b, \pm ic\}$ for some uniquely determined $b \in (a, p)$ and $c > 0$.

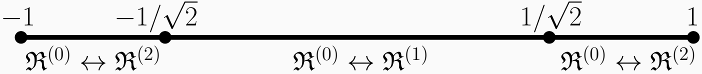
(II) If $a = 1/\sqrt{2}$, then $(*)$ is fulfilled with $p = 1/\sqrt{2}$. In this case $\mathfrak{R}$ has 4 ramification points whose projections are $\{\pm 1, \pm 1/\sqrt{2}\}$.

(III) If $a \in (1/\sqrt{2}, 1)$, then $(*)$ is fulfilled for $p = \sqrt{(1+a^2)/3}$. In this case $\mathfrak{R}$ has 6 ramification points whose projections are $\{\pm 1, \pm a\}$ and $\{\pm b\}$, $b \in (p, a)$.

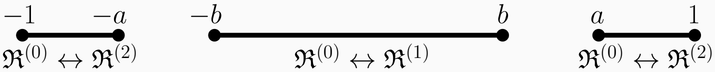
Symmetric Stahl Systems: Riemann Surface



(a) Case I



(b) Case II



(c) Case III

Symmetric Stahl Systems: Nuttall-Szegő Functions

Put

$$\Phi(\mathbf{z}) = \exp\left\{\int^{\mathbf{z}} h(\mathbf{s})\mathrm{d}s\right\}$$

Then $\Phi(\mathbf{z})$ is meromorphic on $\mathfrak{R}$ expect for the unimodular jumps on a homology basis for $\mathfrak{R}$ with the zero/pole divisor $\infty^{(1)} + \infty^{(2)} - 2\infty^{(0)}$. Moreover, $\log|\Phi(\mathbf{z})|$ is harmonic on $\mathfrak{R} \setminus \{\infty^{(0)}, \infty^{(1)}, \infty^{(2)}\}$.

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To $\Phi^n(\mathbf{z})$ and $\rho_1(z), \rho_2(z)$ there corresponds a function $\Psi_n(\mathbf{z})$ that is meromorphic away from the cycles that separate sheets $\mathfrak{R}^{(0)}, \mathfrak{R}^{(1)}, \mathfrak{R}^{(2)}$ and on those cycles it solve a certain boundary value problem (in Case I the jump on $\mathfrak{R}^{(1)} \cap \mathfrak{R}^{(2)}$ depends on $(\rho_1/\rho_2)(z)$).

Each of the functions $\Psi_n(\mathbf{z})$ has a wandering zero (*two* in Case I) and there exists a subsequence $\mathbb{N}_*$ such that

- $|\Psi_n| \leq C(\mathbb{N}_*)\,|\Phi^n|$ uniformly away from the branch points of $\mathfrak{R}$
- $|\Psi_n| \geq C(\mathbb{N}_*)^{-1}\,|\Phi^n|$ uniformly in a neighborhood of $\infty^{(0)}$

Symmetric Stahl Systems: Strong-type Asymptotics

Theorem (Aptekarev-Van Assche-Ya.)

Let $f_i(z) \leftrightarrow \mu_i$, $\mathrm{d}\mu_i(x) = \rho_i(x)\mathrm{d}x/(2\pi\mathrm{i})$, where $\rho_i(z)$ are as before and we assume in addition that the ratio $(\rho_2/\rho_1)(z)$ extends from $(-a, a)$ to a holomorphic and non-vanishing function

- in a domain that contains in its interior the closure of all the bounded components of the regions Ω_{ijk} in Case I;
- in a domain whose complement is compact and belongs to the right-hand component of Ω_{021} in Cases II and IIIa;
- in the extended complex plane, i.e., the ratio is a non-zero constant, in Case IIIb,

where $\Omega_{ijk} := \left\{ z : |\Phi^{(i)}(z)| > |\Phi^{(j)}(z)| > |\Phi^{(k)}(z)| \right\}$. Then it holds that

$$\begin{cases} Q_{(n,n)}(z) & \approx & \gamma_n \Psi_n^{(0)}(z), \\ R_{(n,n)}^{(i)}(z) & \approx & \gamma_n \widehat{\Psi}_n^{(i)}(z), \end{cases} \qquad n \in \mathbb{N}_*.$$

Symmetric Stahl Systems: Strong-type Asymptotics

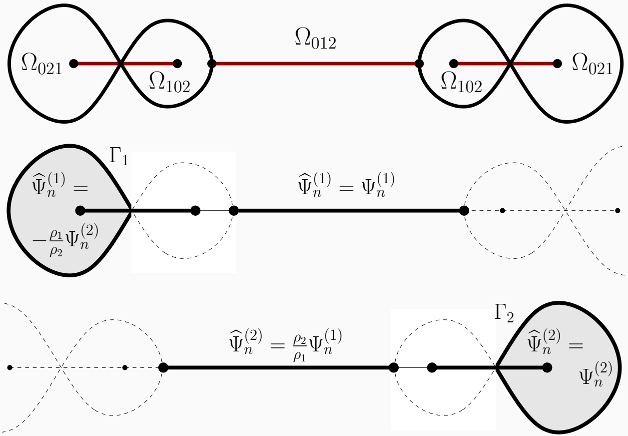
Case IIIb: $\gamma := (\rho_2/\rho_1)(z)$ is a constant.

$$\widehat{\Psi}_n^{(1)}(z) = \Psi_n^{(1)}(z) \quad \text{and} \quad \widehat{\Psi}_n^{(2)}(z) = \gamma\Psi_n^{(1)}(z)$$

The functions $f_i(z) - \left(P_{(n,n)}^{(i)}/Q_{(n,n)}\right)(z)$ diverge in both components of Ω_{102}.

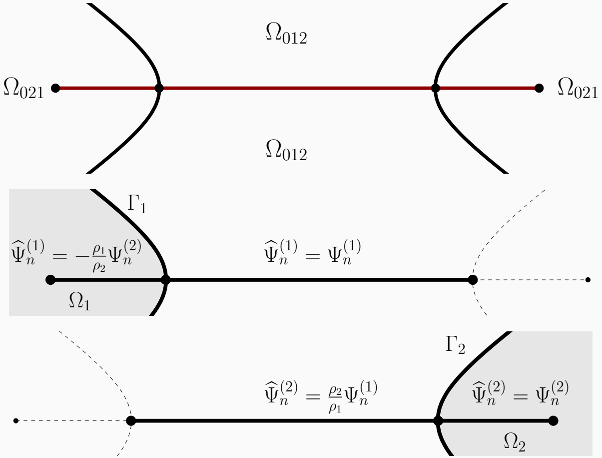
Symmetric Stahl Systems: Strong-type Asymptotics

Case IIIa: extension to a domain whose complement belongs to the right-hand component of Ω_{021}.



Again, divergence in both components of Ω_{102}.

Case II: extension to a domain whose complement compactly belongs to the right-hand component of Ω_{021}.



Case I: extension to a domain that contains in its interior the closure of the bounded components of Ω_{ijk}.