Nuttall's Theorem for Padé Approximants

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In 1844 Liouville[1] constructed the first example of a transcendental number by using continued fractions.

Studying similarities between simultaneous diophantine approximation of real numbers and rational approximation of holomorphic functions, Hermite[2] proved in 1873 that e is transcendental.

[1] Sur des class trè étendues de quantités dont la valeur n'est ni algébrique, ni même réductible à des irrationelles algébriques, 1844

[2] Sur la fonction exponentielle. C.R. Acad. Sci. Paris, 1873

Hermite's proof is based on the following criterion.

Lemma

α is transcendental if for any $m \in \mathbb{N}$ and any $\varepsilon > 0$ there exist $m+1$ linearly independent vectors of integers $(q_j, p_{j1}, \ldots, p_{jm})$, $j = \overline{0, m}$, such that $\left|q_j \alpha^k - p_{jk}\right| \leq \varepsilon$, $k = \overline{1, m}$.

If α is algebraic, then for some $m \in \mathbb{N}$ there exist $a_k \in \mathbb{Z}$, $k = \overline{0, m}$, such that $\sum_{k=0}^m a_k \alpha^k = 0$. Hence,

$$\sum_{k=1}^{m} a_k (q_j \alpha^k - p_{jk}) + a_0 q_j + \sum_{k=1}^{m} a_k p_{jk} = 0.$$

Then for some $0 \leq j_0 \leq m$, it holds that

$$1 \leq \left| \sum_{k=1}^{m} a_k (q_{j_0} \alpha^k - p_{j_0 k}) \right| \leq \varepsilon \sum_{k=1}^{m} |a_k|.$$

Let $n_0, n_1, \ldots, n_m$ be non-negative integers. Set $N := n_0 + \cdots + n_m$ and consider the following system:

$$Q(z)e^{kz} - P_k(z) = \mathcal{O}\left(z^{N+1}\right),$$

where $\deg(Q) \leq N - n_0$ and $\deg(P_k) \leq N - n_k$.

Hermite proceeded to explicitly construct these polynomials, which as it turned out have integer coefficients. By evaluating these polynomials at 1 he succeeded in applying the above criterion.

Let $F(z) = \sum_{k=0}^{\infty} f_k z^k$ be a function holomorphic at the origin. Consider the following system:

$$Q(z)F(z) - P(z) = \mathcal{O}\left(z^{m+n+1}\right),$$

where $\deg(Q) \leq n$ and $\deg(P) \leq m$. This system always has a solution. Indeed,

$$Q(z)F(z) = \sum_{k=0}^{\infty} \left(\sum_{j+i=k, i\leq n} f_j q_i \right) z^k.$$

Set $f_{-k} := 0$ for $k > 0$. Then

$$\begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_m \end{pmatrix} = \begin{pmatrix} f_0 & f_{-1} & \cdots & f_{-n} \\ f_1 & f_0 & \cdots & f_{1-n} \\ \vdots & \vdots & \ddots & \vdots \\ f_m & f_{m-1} & \cdots & f_{m-n} \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ \vdots \\ q_n \end{pmatrix}$$

and

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} f_{m+1} & f_m & \cdots & f_{m+1-n} \\ f_{m+2} & f_{m+1} & \cdots & f_{m+2-n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m+n+1} & f_{m+n} & \cdots & f_{m+1} \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ \vdots \\ q_n \end{pmatrix}$$

The latter is a linear system of n equations with $n+1$ unknowns. Such a system always has a solution. A solution may not be unique, but the ratio $[m/n]_F := P/Q$ always is.

Indeed, let $Q_1(z), P_1(z)$ and $Q_2(z), P_2(z)$ be solutions. Then

$$Q_2(z)\big(Q_1(z)F(z) - P_1(z)\big) = \mathcal{O}\left(z^{m+n+1}\right)$$

and

$$Q_1(z)\big(Q_2(z)F(z) - P_2(z)\big) = \mathcal{O}\left(z^{m+n+1}\right).$$

Therefore,

$$Q_2(z)P_1(z) - Q_1(z)P_2(z) = \mathcal{O}\left(z^{m+n+1}\right).$$

However,

$$\deg\left(Q_2P_1 - Q_1P_2\right) \leq m+n.$$

Theorem (de Montessus de Ballore[3])

Let $F(z)$ be a meromorphic function in $|z| \leq R$ with N poles contained in $0 < |z| < R$. Then $[m/N]_F(z)$ converge to $F(z)$ in $|z| \leq R$ in the spherical metric as $m \to \infty$.

[3] Sur les fractions continues algébriques, 1902.

[4] Poles of rows of the Padé table and meromorphic continuation of functions, 1982

[5] On poles of the m-th row of a Padé table, 1984

[6] On an inverse problem for the m-th row of a Padé table, 1985

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Theorem (Gonchar[4] and Suetin[5,6])

Let $F(z)$ be a holomorphic function at the origin. If the poles of Padé approximants $[m/N]_F(z)$ converge to the points $z_1, \ldots, z_N$ as $m \to \infty$, then $F(z)$ can be meromorphically continued to $|z| < R_N := \max|z_k|$ and all the points z_k are singularities of $F(z)$ (polar if $|z_k| < R_N$).

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[3] Sur les fractions continues algébriques, 1902.

[4] Poles of rows of the Padé table and meromorphic continuation of functions, 1982

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Theorem (Lubinsky and Saff[7])

They constructed a one-parameter family of functions F_q, holomorphic in $\{|z| < 1\}$ with the unit circle being the boundary of analyticity, such that the Padé approximants $[m/N]_{F_q}(z)$, $N \geq 1$, had poles clustering on $\{|z| = R_q < 1\}$ as $m \to \infty$.

[7] Convergence of Padé approximants of partial theta function and Rogers-Szegő polynomials, 1987.

[8] Convergence of Padé approximants in the general case, 1971.

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Theorem (Zinn-Justin[8])

Let $F(z)$ be a meromorphic function in $|z| \leq R$ with n poles contained in $0 < |z| \leq R$. Then $[m/N]_F(z)$ converge to $F(z)$ in measure in $|z| < R$ for any $N \geq n$ as $m \to \infty$.

[7] Convergence of Padé approximants of partial theta function and Rogers-Szegő polynomials, 1987.

[8] Convergence of Padé approximants in the general case, 1971.

Let f be a function holomorphic and vanishing at infinity:

$$f(z) = \frac{f_1}{z} + \frac{f_2}{z^2} + \cdots + \frac{f_n}{z^n} + \cdots .$$

Further, let p_n, q_n be a pair of polynomials of degree at most n solving the linear system

$$(q_n f - p_n)(z) = \mathcal{O}\left(z^{-n-1}\right) \quad \text{as} \quad z \to \infty.$$

Such a pair always exists but might not be unique. However, the rational function $[n/n]_f := p_n/q_n$ is unique and is called the *diagonal Padé approximant* to f of order n.

For any probability Borel measure on $\mathbb{C}$, say ν, set

$$I[\nu] := \int \log \frac{1}{|z-u|} \mathrm{d}\nu(z)\mathrm{d}\nu(u)$$

to be *logarithmic energy*. For any compact set K the *logarithmic capacity* of K is defined by

$$\mathrm{cp}(K) := \exp\left\{ -\inf_{\mathrm{supp}(\nu)\subseteq K} I[\nu] \right\}.$$

It is known that either $\mathrm{cp}(K) = 0$, i.e., K is *polar*, or there exists the unique measure ω_K, the *logarithmic equilibrium distribution* on K, that realizes the infimum. That is,

$$\mathrm{cp}(K) = \exp\left\{ -I[\omega_K] \right\}.$$

In particular, if D, the unbounded component of the complement of K, is simply connected and Φ is the conformal map of D onto $\{|z|>1\}$ such that $\Phi(\infty)=\infty$ and $\Phi'(\infty)>0$, then

$$\Phi(z)=\frac{z}{\mathrm{cp}(K)}+\text{terms analytic at infinity.}$$

For $K_r:=\{|z|=r\}$, it holds that $\Phi(z)=z/r$ and therefore $\mathrm{cp}(K_r)=r$.

It is said that a property holds *quasi everywhere (q.e.)* if it holds everywhere except on a polar set.

Theorem (Nuttall[9] and Pommerenke[10])

Let f be meromorphic function in the complement of a compact polar set F. Then for any $E \subset \mathbb{C} \setminus F$ and $\varepsilon > 0$, it holds that

$$\lim_{n\to\infty} \operatorname{cp}\left\{z \in E : \ |(f - [n/n]_f)(z)|^{1/2n} > \varepsilon\right\} = 0.$$

In other words, Padé approximants $[n/n]_f$ converge to f in capacity and the convergence is faster than geometric.

In the case of Pólya frequency series[11] and entire functions of very slow and smooth growth[12] the convergence is, in fact, uniform.

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[9] The convergence of Padé approximants of meromorphic functions, 1970

[10] Padé approximants and convergence in capacity, 1973

[11] Arms and Edrei. The Padé tables and continued fractions generated by totally positive sequences, 1970.

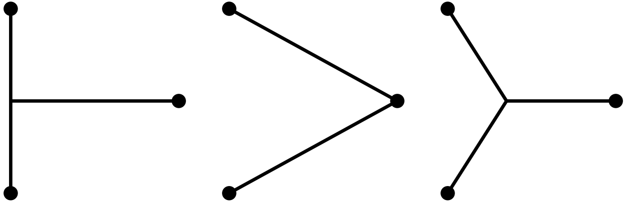
[12] Lubinsky. Padé tables of entire functions of very slow and smooth growth, II, 1988.

Theorem (Rakhmanov[13])

Let D be an unbounded domain such that $\mathrm{cp}(\partial D) > 0$. Then there exists a function f holomorphic in D such that any $z \in D \setminus \{\infty\}$ has a neighborhood in which $[n/n]_f \rightrightarrows \infty$ for $n \in \mathbb{N}_z \subset \mathbb{N}$.

[13] On the convergence of Padé approximants in classes of holomorphic functions, 1980

$$1/\sqrt[3]{z^3-1}$$



Each of the three contours is a valid branch cut for this function.

Theorem (Stahl[14,15])

Let $F(z)$ be holomorphic at infinity, multi-valued, and with all its singularities contained in a compact polar set E. Then

(i) there exists the unique maximal domain D, such that $[n/n]_F(z)$ converge in capacity to $F(z)$ in D as $n \to \infty$;

[14] Extremal domains associated with an analytic function. I, II, 1985.

[15] Structure of extremal domains associated with an analytic function, 1985.

Theorem (Stahl[14,15])

Let $F(z)$ be holomorphic at infinity, multi-valued, and with all its singularities contained in a compact polar set E. Then

(i) there exists the unique maximal domain D, such that $[n/n]_F(z)$ converge in capacity to $F(z)$ in D as $n \to \infty$;

(ii) $\Delta := \overline{\mathbb{C}} \setminus D$ is characterized as the set of the smallest logarithmic capacity among all compact sets that make $F(z)$ single-valued in their complement;

[14] Extremal domains associated with an analytic function. I, II, 1985.

[15] Structure of extremal domains associated with an analytic function, 1985.

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(iii) it holds that $\Delta = E_0 \cup E_1 \cup \bigcup \Delta_j$, where $E_0 \subseteq E$, E_1 is finite, and Δ_j are open analytic arcs connecting the points in $E_0 \cup E_1$.

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[14] Extremal domains associated with an analytic function. I, II, 1985.

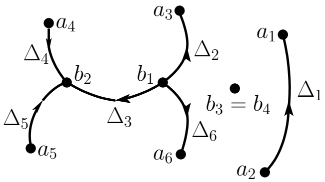
[15] Structure of extremal domains associated with an analytic function, 1985.

In particular, if $F(z)$ is an algebraic function, then

$$\Delta = \{a_1, \ldots, a_p\} \cup \{b_1, \ldots, b_{p-2}\} \cup \bigcup \Delta_j,$$

where a_j are some of the branch points, b_j are not necessarily distinct, and the arcs Δ_j are the negative critical trajectories of the rational quadratic differential

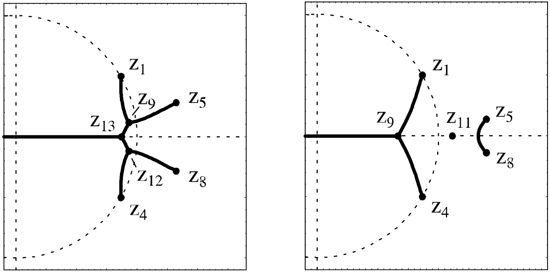
$$\frac{(z-b_1)\cdots(z-b_{p-2})}{(z-a_1)\cdots(z-a_p)}(\mathrm{d}z)^2.$$



The following examples are due to Herbert Stahl[16]. Take

$$f(z)=\sqrt{\sqrt{\prod_{j=1}^{4}\left(1-\frac{z_j}{z}\right)}-c}, \quad f(z)\sim\frac{1}{z} \quad \text{as} \quad z\to\infty,$$

$z_j=e^{i\phi_j}$, $\phi_j\in\left\{\frac{\pi}{6},\frac{5\pi}{6},\frac{7\pi}{6},\frac{11\pi}{6}\right\}$. Then for $c=\sqrt{.70}$ and $c=\sqrt{.74}$



[16] Sets of minimal capacity and extremal domains, manuscript, 2006

It follows from the Cauchy theorem that

$$0 = \int_\Gamma z^k(q_n f - p_n)(z)\mathrm{d}z = \int_\Gamma z^k(q_n f)(z)\mathrm{d}z, \quad k \in \{0,\dots,n-1\},$$

if f is holomorphic in the exterior domain of a Jordan curve Γ.

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if f is holomorphic in the exterior domain of a Jordan curve Γ. Hence, if

$$f(z) = \int \frac{\mathrm{d}\mu(x)}{x - z}$$

is a Markov function (μ is a positive measure compactly supported on $\mathbb{R}$), then

$$\int x^k q_n(x) \mathrm{d}\mu(x) = 0, \quad k \in \{0, \ldots, n-1\}.$$

Theorem (Bernstein[17] and Szegő[18])

If $p(x)$ is a positive polynomial on $[-1,1]$ and $\mathrm{d}\mu(x) = \frac{\mathrm{d}x}{\pi p(x)\sqrt{1-x^2}}$, then

$$\left(f_p - [n/n]_{f_p}\right)(z) = \frac{2}{\sqrt{z^2-1}} \frac{\Psi_n^{(1)}(z)}{\left(\Psi_n^{(0)} + p\Psi_n^{(1)}\right)(z)},$$

where S_p is the unique holomorphic and non-vanishing function in $\overline{\mathbb{C}} \setminus [-1,1]$ such that $|S_p^{\pm}|^2 = p$ on $[-1,1]$ and

$$\begin{cases} \Psi_n^{(0)}(z) & := & \left(z + \sqrt{z^2-1}\right)^n S_p(z) \\ \Psi_n^{(1)}(z) & := & \left(z - \sqrt{z^2-1}\right)^n / S_p(z) \end{cases}, \quad z \in \overline{\mathbb{C}} \setminus [-1,1].$$

[17] Selected papers, volume 1, 1952

[18] Orthogonal Polynomials, volume 23 of Colloquium Publications, 1999

Notice that

(i) f_p is an algebraic function with two branch points ± 1 and the segment $[-1,1]$ is the minimal capacity contour for f_p;

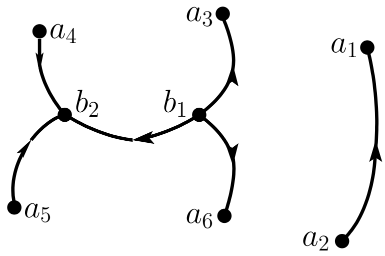
(ii) the function $\Psi_n^{(0)}$ has a pole of order n at infinity, the function $\Psi_n^{(1)}$ has a zero of order n there, and $(\Psi_n^{(0)})^{\pm} = p(\Psi_n^{(1)})^{\mp}$ on $[-1,1]$;

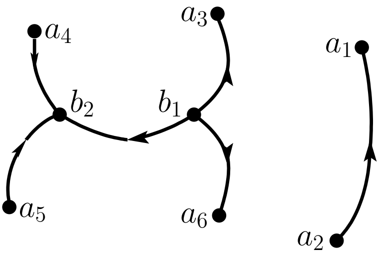
(iii) the Padé approximants $[n/n]_{f_p}$ converge to f_p locally uniformly in $\overline{\mathbb{C}} \setminus [-1,1]$.

Strategy

Take an arbitrary algebraic function together with its minimal capacity contour. Find analogs of $\Psi_n^{(k)}$.

Chebyshëv-type Weight





$$w_\Delta^2(z) := \prod_{e\in E_\Delta}(z-e),$$

where $E_\Delta \subseteq \{a_1,\ldots,a_p\}\cup\{b_1,\ldots,b_{p-2}\}$ is the subset of points with odd bifurcation index, and the function is normalized so

$$z^{-g-1}w_\Delta(z)\to 1 \quad \text{as} \quad z\to\infty.$$

Theorem (Nuttall-Singh[19] and unknowingly Y)

Let Δ be the minimal capacity contour for some algebraic function F. Further, let p be a non-vanishing polynomial on Δ and

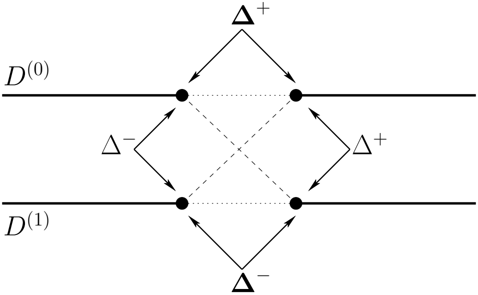
$$f_p(z) := \frac{1}{\pi \mathrm{i}} \int_\Delta \frac{1}{x - z} \frac{\mathrm{d}x}{p(x) w_\Delta^+(x)}.$$

Then

$$f_p - [n/n]_{f_p} = \frac{2}{w_\Delta} \frac{\Psi_n^{(1)}}{\Psi_n^{(0)} + p \Psi_n^{(1)}}.$$

[19] Orthogonal polynomials and Padé approximants associated with a system of arcs, 1977

Denote by $\mathfrak{R}$ be the Riemann surface of w_Δ. The genus of $\mathfrak{R}$ is g.



Further, let $\boldsymbol{\Delta}$ be the chain on $\mathfrak{R}$ that lies above Δ.

(i) Given $\{P_1, \dots, P_k\}$ and $\{Z_1, \dots, Z_{k-g}\}$ for some $k > g$, there exist $\{Z_{k-g+1}, \dots, Z_g\}$ such that the divisor $\mathscr{D} = \sum_{j=1}^{k} Z_j - \sum_{j=1}^{k} P_j$ is principal. The collection $\{Z_{k-g+1}, \dots, Z_g\}$ is either unique or special.

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(ii) A collection of points $\{P_1,\ldots,P_l\}$, $l\leq g$, from $\mathfrak{R}$ is called *special* if there exists a rational function on $\mathfrak{R}$ with poles only among the points P_j counting multiplicities. On $\mathfrak{R}$ as described, it happens iff it contains at least one pair of involution-symmetric points.

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(iii) The problem of finding $\{Z_{k-g+1}, \ldots, Z_g\}$, given $\{P_1, \ldots, P_k\}$ and $\{Z_1, \ldots, Z_{k-g}\}$, is a particular case of the more general Jacobi Inversion Problem. Solution of JIP is either special or unique.

Proposition

Denote by $\mathscr{D}_n$ the unique solutions of a special JIP that depends on the periods of Green and holomorphic differentials on $\mathfrak{R}$, the weight p, and the index n whenever the solution is unique. Denote further by $\mathbb{N}_{JIP}$ the subsequence of indices for which JIP is uniquely solvable and does not contain $\infty^{(0)}$. It holds that $\mathbb{N}_{JIP}$ has gaps of size at most g.

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Let $n \in \mathbb{N}_{JIP}$. Then there exists unique (up to normalization) function Ψ_n, sectionally meromorphic in $\mathfrak{R} \setminus \boldsymbol{\Delta}$, whose zeros and poles are described by the divisor $(n-g)\infty^{(1)} + \mathscr{D}_n - n\infty^{(0)}$, and which has continuous traces on $\boldsymbol{\Delta} \setminus \boldsymbol{E}_\Delta$ that satisfy $\Psi_n^+ = p\Psi_n^-$. For $n \notin \mathbb{N}_{JIP}$, set $\Psi_n := \Psi_{\tilde{n}}$, where $\tilde{n}$ is the largest integer in $\mathbb{N}_{JIP}$ smaller than n.

Recall that

$$f_p - [n/n]_{f_p} = \frac{2}{w_\Delta} \frac{\Psi_n^{(1)}}{\Psi_n^{(0)} + p\Psi_n^{(1)}},$$

where $\Psi_n^{(k)} := \Psi_{n|D^{(k)}}$.

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where $\Psi_n^{(k)} := \Psi_{n|D^{(k)}}$. Write $\mathscr{D}_n = \sum_{j=1}^g Z_{nj}$. Therefore,

(i) if $Z_{nj} \in D^{(1)}$, then $[n/n]_{f_p}$ overinterpolates f_p at the projection of Z_{nj};

(ii) if $Z_{nj} \in D^{(0)}$, then $[n/n]_{f_p}$ has a pole next to the projection of Z_{nj}.

Generically, the collection $\left\{\{Z_{nj}\}_{j=1}^g\right\}_n$ is dense in $\mathfrak{R}$.

Almost a Theorem

Let Δ be the minimal capacity contour for some algebraic function F. Further, let ρ be a non-vanishing Hölder continuous function on Δ and

$$f_\rho(z) := \frac{1}{\pi \mathrm{i}} \int_\Delta \frac{1}{x-z} \frac{\mathrm{d}x}{\rho(x) w_\Delta^+(x)}.$$

Then for $n \in \mathbb{N}_{JIP}$ it holds that

$$f_\rho - [n/n]_{f_\rho} = \frac{2}{w_\Delta} \frac{\Psi_n^{(1)} \left[1 + E_n^{(1)}\right]}{\Psi_n^{(0)} \left[1 + E_n^{(0)}\right] + p_n \Psi_n^{(1)} \left[1 + E_n^{(1)}\right]},$$

where p_n, $\deg(p_n) \leq n$, is the polynomial of best uniform approximation to ρ on Δ, E_n is sectionally meromorphic on $\mathfrak{R} \setminus \boldsymbol{\Delta}$ with at most g poles only among the elements of $\mathscr{D}_n$, and $\|L_n E_n^{\pm}\|_{2,\boldsymbol{\Delta}} \ll \|\rho - p_n\|_\Delta$.

The previous theorem has been verified when

(i) $\Delta = [-1, 1]$ by Nuttall[20];

(ii) Δ consists of disjoint arcs by Suetin[21];

(iii) Δ consists of 3 arc with a common endpoint by Baratchart-Y[22];

(iv) Δ is any algebraic S-contour tentatively by Y.

[20] Padé polynomial asymptotic from a singular integral equation, 1990

[21] Uniform convergence of Padé diagonal approximants for hyperelliptic functions, 2000

[22] Asymptotics of Padé approximants to a certain class of elliptic-type functions, 2013

Theorem (Aptekarev-Y[24])

Let

(i) Δ be a minimal capacity contour such that no more than three arcs Δ_j have a common endpoint;

(ii) the weight ρ be such that $\rho_{|\Delta_j}$ is a Jacobi weight modified by a non-vanishing holomorphic function;

(iii) $\mathbb{N}^*_{JIP} \subset \mathbb{N}_{JIP}$ be such that the elements of $\mathscr{D}_{n-1}$ and $\mathscr{D}_n$ are uniformly bounded away from $\infty^{(1)}$ and$\infty^{(0)}$, respectively.

Then for $n \in \mathbb{N}^*_{JIP}$ it holds that

$$f_\rho - [n/n]_{f_\rho} = \left[1 + \mathscr{O}(1/n)\right] \frac{2}{w_\Delta} \frac{\Psi_n^{(1)}}{\Psi_n^{(0)}}$$

in $D \setminus \bigcup U_\epsilon(Z_{nj})$, where $U_\epsilon(Z)$ is the ϵ-neighborhood of the projection of Z in and $\mathscr{O}(1/n)$ is uniform for each fixed $\epsilon > 0$.

[24] Padé approximants for functions with branch points – strong asymptotics of Nuttall-Stahl polynomials.