On multiple orthogonal polynomials

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Multiple Orthogonal Polynomials

Let $\vec{\mu} = (\mu_1, \dots, \mu_d)$ be a vector of measures supported on the real line, each having infinitely many points in its support and finite moments of all orders.

Let $\vec{n} = (n_1, \dots, n_d)$ be a multi-index of non-negative integers.

Multiple orthogonal polynomial $P_{\vec{n}}(x)$ (type II) is a polynomial of degree at most $|\vec{n}| = n_1 + \cdots + n_d$ satisfying

$$\int x^k P_{\vec{n}}(x) d\mu_i(x) = 0, \quad k = \overline{0, n_i - 1}.$$

The multi-index $\vec{n}$ is called *normal* if $\deg P_{\vec{n}} = |\vec{n}|$. In this case we normalize $P_{\vec{n}}(x)$ to be monic. The vector $\vec{\mu}$ is called *perfect* if all the multi-indices are normal.

Theorem (Angelesco, 1919)

Let $\Delta_1 < \Delta_2 < \cdots < \Delta_d$, where Δ_i is the convex hull of the support of μ_i. Then $\vec{\mu}$, called an Angelesco system, is perfect.

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Theorem (López Lagomasino–Fidalgo Prieto, 2011)

Let σ_i be d auxiliary measures and F_i be the convex hulls of their supports. Assume that $F_i \cap F_{i+1} = \varnothing$. Write

$$d\langle \sigma, \nu \rangle(x) := \widehat{\nu}(x) d\sigma(x), \quad \widehat{\nu}(x) := \int (z-y)^{-1} d\nu(y).$$

Let $\langle \sigma_j, \ldots, \sigma_k \rangle := \langle \sigma_j, \langle \sigma_{j+1}, \ldots, \sigma_k \rangle \rangle$. Put

$$\begin{aligned} \mu_1 &:= \sigma_1 \\ \mu_2 &:= \langle \sigma_1, \sigma_2 \rangle \\ &\cdots \\ \mu_d &:= \langle \sigma_1, \ldots, \sigma_d \rangle. \end{aligned}$$

Then $\vec{\mu}$, called a Nikishin system, is perfect.

Lattice Recurrence Relations

Let $\{\vec{e}_i\}_{i=1}^d$ be the standard basis in $\mathbb{R}^d$. If $\vec{n}$ and $\vec{n}+\vec{e}_j$ are normal, then

$$xP_{\vec{n}}(x) = P_{\vec{n}+\vec{e}_j}(x) + b_{\vec{n},j}P_{\vec{n}}(x) + \sum_{i=1}^{d} a_{\vec{n},i}P_{\vec{n}-\vec{e}_i}(x)$$

for some coefficients $b_{\vec{n},i}, a_{\vec{n},i}$. These coefficients satisfy consistency conditions

$$\begin{aligned} b_{\vec{n}+\vec{e}_i,j} - b_{\vec{n}+\vec{e}_j,i} &= b_{\vec{n},j} - b_{\vec{n},i}, \\ \sum_{k=1}^{d} a_{\vec{n}+\vec{e}_j,k} - \sum_{k=1}^{d} a_{\vec{n}+\vec{e}_i,k} &= b_{\vec{n}+\vec{e}_j,i}b_{\vec{n},j} - b_{\vec{n}+\vec{e}_i,j}b_{\vec{n},i}, \\ a_{\vec{n},i}(b_{\vec{n},j} - b_{\vec{n},i}) &= a_{\vec{n}+\vec{e}_j,i}(b_{\vec{n}-\vec{e}_i,j} - b_{\vec{n}-\vec{e}_i,i}). \end{aligned}$$

When $d = 1$, these relations simply become classical recurrence relations for monic orthogonal polynomials

$$xP_n(x;\mu) = P_{n+1}(x;\mu) + b_n(\mu)P_n(x;\mu) + a_n^2(\mu)P_{n-1}(x;\mu).$$

Theorem (Filipuk–Haneczok–Van Assche, 2015)

If the recurrence coefficients $\{a_{\vec{n},i} b_{\vec{n},i}\}$ *satisfy*

- *consistency conditions,*
- $a_{n\vec{e}_i,i} > 0$ *and* $a_{n\vec{e}_i,j} = 0$, $i \neq j$,
- $b_{\vec{n},i} \neq b_{\vec{n},j}$ *for* $i \neq j$,

then there exists $\vec{\mu}$ *for which* $\{a_{\vec{n},i} b_{\vec{n},i}\}$ *are the recurrence coefficients.*

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Conversely, $\{a_{\vec{n},i} b_{\vec{n},i}\}$ *can be constructively recovered from* $\{b_n(\mu_i), a_n^2(\mu_i)\}$ *and the initial conditions*

$$b_{n\vec{e}_i,i} = b_n(\mu_i), \quad a_{n\vec{e}_i,i} = a_n^2(\mu_i), \quad a_{n\vec{e}_i,j} = 0, \ \ j \neq i,$$

provided $b_{\vec{n},i} \neq b_{\vec{n},j}$ *for* $i \neq j$.

The condition $b_{\vec{n},i} \neq b_{\vec{n},j}$ holds for multiple Hermite ($e^{-x^2-c_i x}$), Laguerre ($x^{\alpha_j} e^{-x}$, $x^{\alpha} e^{-c_j x}$), and Charlier ($a_i^k/k!$) polynomials as well as for Angelesco systems where $b_{\vec{n},i} < b_{\vec{n},j}$, $i < j$ (Aptekarev–Denisov-Ya., 2020).

Ratio Asymptotics

Theorem (Van Assche, 2016)

Let $\vec{n} = (\lfloor c_1 n \rfloor, \dots \lfloor c_d n \rfloor)$ *for some* $\vec{c} \in (0,1)^d$ *with* $|\vec{c}| = 1$*. Assume that*

$$\lim_{n\to\infty} n^{-2\gamma} a_{\vec{n},i} = A_{\vec{c},i} \quad \textit{and} \quad \lim_{n\to\infty} n^{-\gamma} b_{\vec{n},i} = B_{\vec{c},i}$$

for some $\gamma \geq 0$ *with* $B_{\vec{c},i} \neq B_{\vec{c},j}$*. Then*

$$\lim_{n\to\infty} \frac{P_{\vec{n}+\vec{e}_i}(n^\gamma z)}{n^\gamma P_{\vec{n}}(n^\gamma z)} = \chi_{\vec{c}}(z) - B_{\vec{c},i},$$

where $z = \chi_{\vec{c}} + \sum_i \frac{A_{\vec{c},i}}{\chi_{\vec{c}} - B_{\vec{c},i}}$ *such that* $\chi_{\vec{c}}(z) - z \to 0$ *as* $z \to \infty$*.*

When $a_n^2(\mu) \to A^2$, $b_n(\mu) \to B$, the theorem recovers

$$2\chi(z) = (z + B) + \sqrt{(z - B - 2A)(z - B + 2A)}.$$

Theorem (Gonchar–Rakhmanov, 1985)

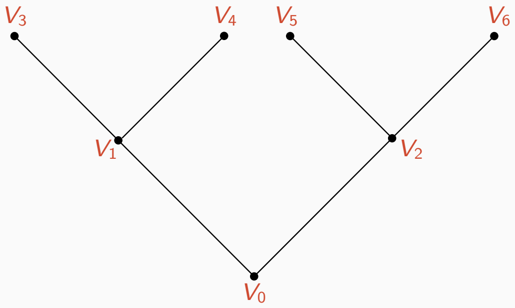
Let $F = \{F_i\}$ *be a collection of intervals,* $\theta = \{\theta_i\}$, $\theta_i > 0$, *and* $A = [a_{ij}]$ *be a positive definite symmetric matrix with* $a_{ii} > 0$ *and* $a_{ij} = 0$ *if* $F_i \cap F_j \neq \varnothing$, $i \neq j$.

Let $M_\theta(F)$ *be the set of vector measures* $\vec{\nu} = (\nu_1, \ldots, \nu_d)$ *such that* ν_i *is supported on* F_i *and* $|\nu_i| = \theta_i$. *Define*

$$I(\vec{\nu}) := -\sum a_{ij} \iint \log|x - y| d\nu_i(x) d\nu_j(y).$$

Then there exists a unique $\vec{\omega} \in M_\theta(F)$, *the vector equilibrium measure, such that* $I(\vec{\omega}) = \min_{M_\theta(F)} I(\vec{\nu})$.

Let $\mathcal{G}$ be a rooted tree with $d+1$ vertices $V_0, V_1, \ldots, V_d$, where V_0 is the root. To each V_i, $i > 0$, associate an interval F_i such that $F_i \cap F_j = \varnothing$ if V_i and V_j are either siblings or one is a child of the other.



V_1, V_2 are siblings and children of V_0; $F_1 \cap F_2 = \varnothing$;
V_3, V_4 are siblings and children of V_1; $F_i \cap F_j = \varnothing$, $i, j \in \{1, 3, 4\}$;
V_5, V_6 are siblings and children of V_2; $F_i \cap F_j = \varnothing$, $i, j \in \{2, 5, 6\}$.

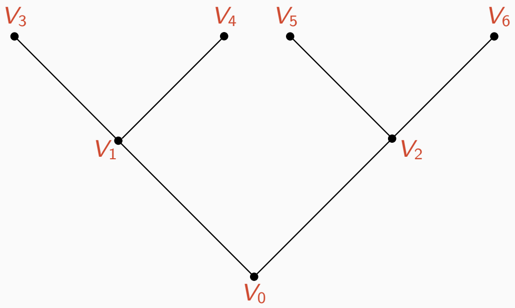
GN Systems

On each interval F_i, choose an auxiliary measure σ_i. Given V_m, let

$$V_0 \to V_{i_1} \to V_{i_2} \to \cdots \to V_{i_k} = V_m$$

be the path connecting V_0 and V_m. A *GN system* is a vector $\vec{\mu}$ with

$$\mu_m := \langle \sigma_{i_1}, \ldots, \sigma_{i_k} \rangle.$$

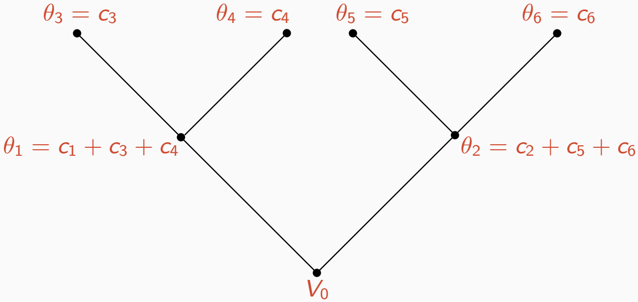


$$\mu_1 = \sigma_1, \;\; \mu_2 = \sigma_2, \;\; \mu_3 = \langle \sigma_1, \sigma_3 \rangle, \;\; \mu_4 = \langle \sigma_1, \sigma_4 \rangle, \;\; \mu_5 = \langle \sigma_2, \sigma_5 \rangle, \;\; \mu_6 = \langle \sigma_2, \sigma_6 \rangle.$$

Aptekarev-Lysov generalized this construction to graphs where multiple edges between vertices are allowed.

GN Systems

Given $\vec{c} \in (0,1)^d$, set $\theta_m = c_m + \sum c_i$, where the sum is over all descendants V_i of V_m.



Further, let $a_{ii} = 2$, $a_{ij} = -1$ if V_i, V_j is a child/parent pair, $a_{ij} = 1$ if V_i, V_j are siblings, and otherwise $a_{ij} = 0$.

Theorem (Gonchar–Rakhmanov–Sorokin, 1997)

Assume that $d\sigma_i/dx > 0$ *a.e. on* F_i*. Suppose further that* $\vec{n}$ *is such that* $n_i \leq n_j + 1$ *if* V_i *is a child of* V_j *and that* $\vec{n}/|\vec{n}| \to \vec{c}$ *as* $|\vec{n}| \to \infty$*. Let* $\vec{\omega}_{\vec{c}}$ *be the vector equilibrium measure. Then the normalized counting measures of zeros of* $P_{\vec{n}}(z)$ *converge weak** *to* $\sum \omega_{\vec{c},i}$ *where the sum is taken over the children of* V_0*.*

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Theorem (Gonchar–Rakhmanov, 1981)

For Angelesco systems it holds that the support of $\omega_{\vec{c},i}$ is an interval.

Aptekarev-Lysov claim that this is true for all GN systems.

Theorem (Geronimo–Kuijlaars–Van Assche, 2001)

Let $d\mu_i(x) = \rho_i(x)dx$. Consider the following Riemann-Hilbert problem for $(d+1)\times(d+1)$ matrices:

(a) $\boldsymbol{Y}(z)$ *is analytic in* $\mathbb{C}\setminus\mathbb{R}$ *and* $\lim_{z\to\infty} \boldsymbol{Y}(z)z^{-\sigma(\vec{n})} = \boldsymbol{I}$, *where* $\boldsymbol{I}$ *is the identity matrix and* $\sigma(\vec{n}) := \mathrm{diag}\left(|\vec{n}|, -n_1, \ldots, -n_d\right)$;

(b) *on the real line it holds that* $\boldsymbol{Y}_+(x) = \boldsymbol{Y}_-(x)(\boldsymbol{I} + \sum \rho_i(x)\boldsymbol{E}_{1,i+1})$, *where* $\boldsymbol{E}_{1,i+1}$ *has all zero entries except for* $(1, i+1)$, *which is* 1.

This problem has a unique solution whose $(1,1)$*-entry is* $P_{\vec{n}}(z)$.

The proof is the modification of the one by Fokas–Its–Kitaev in the case $d = 1$.

Let $\vec{\mu}$ be an Angelesco system corresponding to intervals $\Delta_1 < \Delta_2 < \cdots < \Delta_d$.

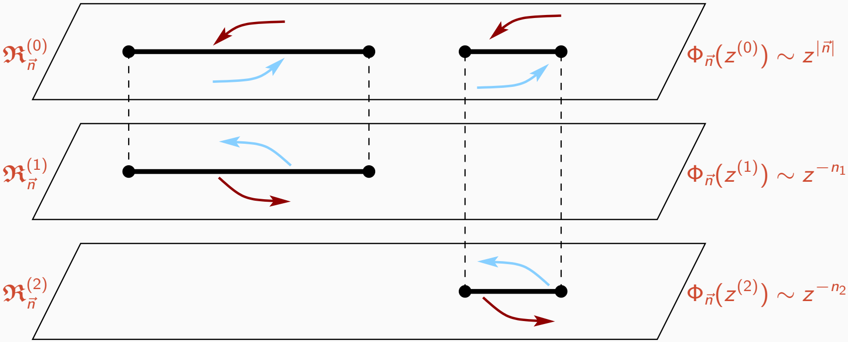
Given $\vec{c} \in (0,1)^d$, let $\vec{\omega}_{\vec{c}}$ be the vector equilibrium measure constructed before.

Denote by $\Delta_{\vec{c},i} \subseteq \Delta_i$ the support of $\omega_{\vec{c},i}$, which is an interval.

We shall assume that $d\mu_i(x) = \rho_i(x)dx$, where $\rho_i(x)$ extends to a holomorphic and non-vanishing function in a neighborhood of Δ_i (we can also consider Fisher-Hartwig perturbations).

When $\vec{c} = \vec{n}/|\vec{n}|$, we shall simply write write $\vec{\omega}_{\vec{n}}$ and $\Delta_{\vec{n},i}$.

Strong Asymptotics in Angelesco Systems



The surface $\mathfrak{R}_{\vec n}$ constructed w.r.t to cuts $\Delta_{\vec n,i}$ and has genus 0. Let $\Phi_{\vec n}(z)$ be the rational function on $\mathfrak{R}_{\vec n}$ such that

$$(\Phi_{\vec n}) = n_1\infty^{(1)} + \cdots + n_d\infty^{(d)} - |\vec n|\infty^{(0)}, \quad \prod \Phi_{\vec n}(z^{(k)}) \equiv 1.$$

Strong Asymptotics in Angelesco Systems

Theorem (Ya., 16)

If $\vec{n}/|\vec{n}| \to \vec{c} \in (0,1)^d$ *as* $|\vec{n}| \to \infty$, *then*

$$P_{\vec{n}}(z) \sim \left(\Phi_{\vec{n}} S\right)\left(z^{(0)}\right),$$

where $S(z)$ *is a Szegő-type function on* $\mathfrak{R}_{\vec{c}}$.

Similar result for Nikishin systems with $d = 2$ and $\vec{n} = n\vec{c}$ for $\vec{c} \in \mathbb{Q}^2 \cap (0,1)^2$ was proven by López Lagomasino–Van Assche, 2018.

Theorem (Aptekarev–Denisov–Ya., in prep.)

When $d = 2$, *the condition* $\vec{n}/|\vec{n}| \to \vec{c} \in (0,1)^d$ *can be replaced by* $n_i \to \infty$.

There are many other results along the diagonal sequences $(n, n, \ldots, n)$.

Theorem (Aptekarev–Denisov–Ya., 2020)

Let $\chi_{\vec{c}} : \mathfrak{R}_{\vec{c}} \to \overline{\mathbb{C}}$ *be a conformal map such that*

$$\chi_{\vec{c}}\big(z^{(0)}\big) = z + \mathcal{O}\big(z^{-1}\big) \quad \text{as} \quad z \to \infty.$$

Define constants $A_{\vec{c},i}, B_{\vec{c},i}$ *by*

$$\chi_c\big(z^{(i)}\big) = B_{\vec{c},i} + A_{\vec{c},i} z^{-1} + \mathcal{O}\big(z^{-2}\big) \quad \text{as} \quad z \to \infty.$$

Then, as $|\vec{n}| \to \infty$, $\vec{n}/|\vec{n}| \to \vec{c} \in (0,1)^d$, *it holds that*

$$\lim a_{\vec{n},i} = A_{\vec{c},i} \quad \text{and} \quad \lim b_{\vec{n},i} = B_{\vec{c},i}.$$

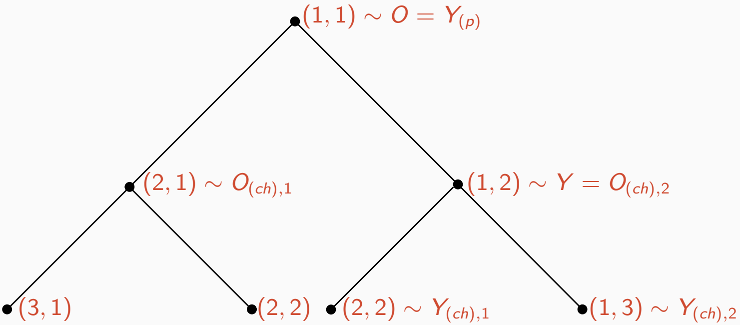
Theorem (Aptekarev–Denisov–Ya., in prep.)

When $d = 2$, *the limits of* $A_{(c,1-c),i}, B_{(c,1-c),i}$ *as* $c \to 0$ *or* $c \to 1$ *exist and*

$$\lim a_{\vec{n},i} = A_{\vec{c},i} \quad \text{and} \quad \lim b_{\vec{n},i} = B_{\vec{c},i}.$$

holds as $|\vec{n}| \to \infty$, $\vec{n}/|\vec{n}| \to \vec{c} \in [0,1]^2$.

Let $\mathcal{T}$ be the rooted tree of all possible increasing paths on $\mathbb{N}^d$ starting at $\vec{1}$.



We denote the set of all vertices of $\mathcal{T}$ by $\mathcal{V}$. We let

$$\ell : \mathcal{V} \to \{1, \ldots, d\}, \quad Y \mapsto \ell_Y \text{ such that } \Pi(Y) = \Pi(Y_{(p)}) + \vec{e}_{\ell_Y},$$

where Π is the natural projection of $\mathcal{V}$ onto $\mathbb{N}^d$.

Jacobi Operators

Let $\vec{\kappa} \in \mathbb{R}^d$, $|\vec{\kappa}| = 1$. Define two interaction functions $A, B : \mathcal{V} \to \mathbb{R}$ by

$$A_O := 1, \quad B_O := \sum \kappa_i b_{\vec{1}-\vec{e}_i,i}, \quad Y = O,$$
$$A_Y := a_{\Pi(Y_{(p)}),\ell_Y}, \quad B_Y := b_{\Pi(Y_{(p)}),\ell_Y}, \quad Y \neq O.$$

Assume now that

$$0 < a_{\vec{n},j} \text{ for all } \vec{n} \in \mathbb{Z}_+^d \text{ such that } n_j > 0,$$
$$\sup a_{\vec{n},j} < \infty\,, \ \sup |b_{\vec{n},j}| < \infty.$$

This condition is satisfied by Angelesco systems (Aptekarev–Denisov–Ya., 20).

Then, for any function $f \in \ell^2(\mathcal{V})$, the action of the operator $\mathcal{J}_{\vec{\kappa}}$ can be written in the following form

$$(\mathcal{J}_{\vec{\kappa}} f)_O := (Bf)_O + \sum_i (A^{1/2} f)_{O_{(ch),i}}, \quad Y = O,$$
$$(\mathcal{J}_{\vec{\kappa}} f)_Y := A_Y^{1/2} f_{Y_{(p)}} + (Bf)_Y + \sum_i (A^{1/2} f)_{Y_{(ch),i}}, \quad Y \neq O.$$

$\mathcal{J}_{\vec{\kappa}}$ is a bounded and self-adjoint operator on $\ell^2(\mathcal{V})$.

Proposition (Aptekarev-Denisov-Ya., 2020)

Let $\upsilon_{\vec\kappa}$ *be the spectral measure of* $\mathcal{J}_{\vec\kappa}$ *associated to an Angelesco system* (μ_1, μ_2). *Then*

$$\Theta_{\upsilon_{\vec\kappa}}(z) = \Xi(\mu_1, \mu_2)\frac{\Theta_{\mu_1}(z) - \Theta_{\mu_2}(z)}{\kappa_2\Theta_{\mu_1}(z) + \kappa_1\Theta_{\mu_2}(z)},$$

where $\Theta_\mu(z) := \int (x-z)^{-1} d\mu(x)$ *and*

$$\Xi(\mu_1, \mu_2) := \left(\int t\big(d\mu_2(t) - d\mu_1(t)\big)\right)^{-1}.$$

If the measures μ_i *are absolutely continuous w.r.t. the Lebesgue measure, then*

$$\upsilon'_{\vec\kappa}(x) = \frac{\Theta_{\mu_2}(x)\mu'_1(x) - \Theta_{\mu_1}(x)\mu'_2(x)}{\left|\kappa_1\Theta_{\mu_1}(x) + \kappa_2\Theta_{\mu_2}(x)\right|^2}.$$

Proposition (Aptekarev-Denisov-Ya., 2020)

If $\upsilon_{\vec\kappa}$ *and* $\Xi(\mu_1, \mu_2)$ *are known, then* μ_1, μ_2, *and* $\mathcal{J}_{\vec\kappa}$ *can be found uniquely.*

Theorem (Aptekarev–Denisov–Ya., in prep.)

Let $\Delta_1 < \Delta_2$ be two intervals. Write $\Delta_{c,i}$ for the support of the i-th component of the vector equilibrium measure $\vec{\omega}_{c,1-c}$.

Let $\chi_c(z)$ be the above constructed conformal map on $\mathfrak{R}_c$ that defines constants $A_{c,1}, A_{c,2}, B_{c,1}, B_{c,2}$ together with their limits as $c \to 0$ and $c \to 1$.

Let $\mathcal{J}_{\vec{\kappa}}$ be a Jacobi operator corresponding to some constants $\{a_{\vec{n},i}, b_{\vec{n},i}\}$. If for any $c \in [0,1]$ it holds that

$$\lim a_{\vec{n},i} = A_{c,i} \quad \text{and} \quad \lim b_{\vec{n},i} = B_{c,i}$$

where the limit is taken along any sequence $\vec{n}/|\vec{n}| \to (c, 1-c)$ as $|\vec{n}| \to \infty$, then $\sigma_{\mathrm{ess}}(\mathcal{J}_{\vec{\kappa}}) = \Delta_1 \cup \Delta_2$.

Symmetric Stahl Systems

We shall say that (μ_1, μ_2) forms a symmetric Stahl system if

$$\mathrm{supp}(\mu_1) = [-1, a], \quad \mathrm{supp}(\mu_2) = [-a, 1], \quad a \in (0,1).$$

Let h be an algebraic function given by

$$A(z)h^3 - 3B_2(z)h - 2B_1(z) = 0,$$

where $A(z) := (z^2-1)(z^2-a^2)$, $B_2(z) := z^2 - p^2$, and $B_1(z) := z$, for some parameter $p > 0$.

Let $\mathfrak{R}$ be the Riemann surface of h. We are looking for the surface such that

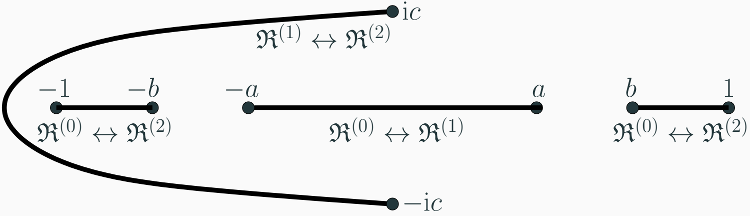
$$\mathrm{Re}\left(\int^{z} h(t)\mathrm{d}t\right) \quad \text{is a single-valued and harmonic function on } \mathfrak{R}.$$

Theorem (Aptekarev–Van Assche–Ya., 2017)

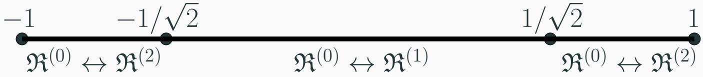
(I) *If* $a \in (0, 1/\sqrt{2})$, *then there exists* $p \in \left(a, \sqrt{(1+a^2)/3}\right)$ *such that the condition is fulfilled. In this case* $\mathfrak{R}$ *has* 8 *ramification points whose projections are* $\{\pm 1, \pm a\}$ *and* $\{\pm b, \pm \mathrm{i}c\}$ *for some uniquely determined* $b \in (a, p)$ *and* $c > 0$.

(II) *If* $a = 1/\sqrt{2}$, *then the condition is fulfilled for* $p = 1/\sqrt{2}$. *In this case* $\mathfrak{R}$ *has* 4 *ramification points whose projections are* $\{\pm 1, \pm 1/\sqrt{2}\}$.

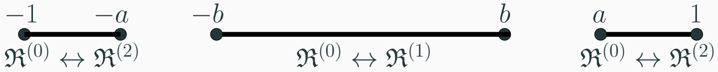
(III) *If* $a \in (1/\sqrt{2}, 1)$, *then the condition is fulfilled for* $p = \sqrt{(1+a^2)/3}$. *In this case* $\mathfrak{R}$ *has* 6 *ramification points whose projections are* $\{\pm 1, \pm a\}$ *and* $\{\pm b\}$, $b \in (p, a)$.



(a) Case I



(b) Case II



(c) Case III

Symmetric Stahl Systems

Let $\Phi(z) := \exp\left\{\int^z h(t)\mathrm{d}t\right\}$. It is a multiplicatively multi-valued function on $\mathfrak{R}$ with the divisor $\infty^{(1)} + \infty^{(2)} - 2\infty^{(0)}$.

Let ρ_1 and ρ_2 be functions holomorphic and non-vanishing in a neighborhood of $[-1,1]$. In Case I, assume also that the ratio ρ_1/ρ_2 extends holomorphically to a non-vanishing function in a neighborhood of $\mathfrak{R}^{(1)} \cap \mathfrak{R}^{(2)}$. Then $\Psi_n \leftrightarrow \Phi^n$, where

$$\begin{cases} \left(\Psi_n^{(1)}\right)^{\pm} & = & \pm\left(\Psi_n^{(0)}\right)^{\mp}\rho_1, \\ \left(\Psi_n^{(2)}\right)^{\pm} & = & \mp\left(\Psi_n^{(0)}\right)^{\mp}\rho_2, \\ \left(\Psi_n^{(2)}\right)^{\pm} & = & \pm\left(\Psi_n^{(0)}\right)^{\mp}\rho_2, \\ \left(\Psi_n^{(2)}\right)^{\pm} & = & \pm\left(\Psi_n^{(1)}\right)^{\mp}(\rho_2/\rho_1). \end{cases}$$

$\Psi_n(z)$ has a wandering zero (2 in Case I) and there exists a subsequence $\mathbb{N}_*$ such that

- $|\Psi_n| \leq C(\mathbb{N}_*)\,|\Phi^n|$ uniformly away from the branch points of $\mathfrak{R}$;
- $|\Psi_n| \geq C(\mathbb{N}_*)^{-1}\,|\Phi^n|$ uniformly in a neighborhood of $\infty^{(0)}$.

Theorem (Aptekarev-Van Assche-Ya., 2017)

Let $d\mu_i(x) = \rho_i(x)dx$ be a symmetric Stahl system, where $\rho_i(x)$ are as before and we assume in addition that the ratio $(\rho_2/\rho_1)(x)$ extends from $(-a, a)$ to a holomorphic and non-vanishing function

- *in a domain that contains in its interior the closure of all the bounded components of the regions Ω_{ijk} in Case I;*
- *in a domain whose complement is compact and belongs to the right-hand component of Ω_{021} in Cases II and IIIa;*
- *in the extended complex plane, i.e., the ratio is a non-zero constant, in Case IIIb,*

where

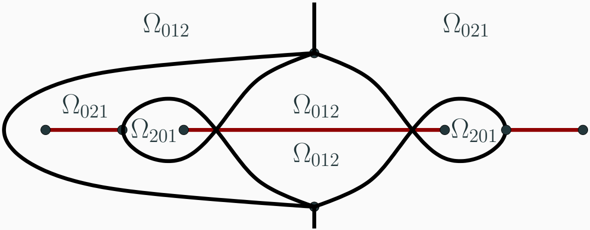
$$\Omega_{ijk} := \left\{ z : \left|\Phi^{(i)}(z)\right| > \left|\Phi^{(j)}(z)\right| > \left|\Phi^{(k)}(z)\right| \right\}.$$

Then for multi-indices $\vec{n} = (n, n)$ it holds that

$$P_{\vec{n}}(z) \sim \Psi_n^{-1}\left(\infty^{(0)}\right)\Psi_n\left(z^{(0)}\right), \quad n \in \mathbb{N}_*.$$

Symmetric Stahl Systems

Case I:



Case II:

Case IIIa:

Case IIIb: