Large Deviations, Linear Statistics, and Scaling Limits for Mahler Ensemble of Complex Random Polynomials

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The *Mahler measure* of a polynomial $p(z) = a\prod(z - \alpha_n)$ is given by

$$M(p) := \exp\left\{\int_{\mathbb{T}} \log|p(\tau)| \frac{|\mathrm{d}\tau|}{2\pi}\right\},$$

where $\mathbb{T} := \{|z| = 1\}$. It follows from Jensen's formula that

$$M(p) = |a| \prod \max\{1, |\alpha_n|\} = |a| \prod \exp\{\log^+ |\alpha_n|\}.$$

Associate to each element $\mathbf{v} \in \mathbb{C}^{N+1}$ a polynomial $p_{\mathbf{v}}$. The following quantity is of number theoretic interest:

$$\#\left\{\mathbf{v} \in \mathbb{Z}[\mathrm{i}]^{N+1} : M(p_{\mathbf{v}}) \leq T\right\}.$$

Clearly, this quantity is equal to

$$\#\mathbb{Z}[\mathrm{i}]^{N+1} \cap \left\{\mathbf{v} \in \mathbb{C}^{N+1} : M(p_{\mathbf{v}}) \leq T\right\}.$$

Chern & Vaaler[1] have shown that this quantity is bounded by

$$T^{2N+2}\mathrm{vol}\left\{\mathbf{v} \in \mathbb{C}^{N+1} : \ M(p_{\mathbf{v}}) \leq 1\right\}.$$

[1] The distribution of values of Mahler's measure, J. Reine Angew. Math., 540:1—47, 2001

They further computed that

$$\mathrm{vol}\left\{\mathbf{v} \in \mathbb{C}^{N+1} : \ M(p_{\mathbf{v}}) \leq 1\right\} = \frac{\pi}{N+1} H_N(N+1),$$

where

$$H_N(s) := \int_{\mathbb{C}^N} M^{-2s}(P_{\mathbf{u}})\,\mathrm{d}A^{\otimes N} = \frac{\pi^N}{N!} \prod_{n=1}^{N} \frac{s}{s-n}$$

and $P_{\mathbf{u}}$ is the **monic** polynomial of degree $N+1$ with the non-leading coefficients described by the vector $\mathbf{u} \in \mathbb{C}^N$.

Question

Where do the zeros of a **typical** polynomial from this volume lie? (Is $z^N - 1$ or $(z-1)^N$ more typical?)

Definition

By a *random polynomial* from a **complex Mahler ensemble** we will mean a polynomial chosen according to the density $M^{-2s}(P_{\mathbf{u}})/H_N(s)$.

Remark

True interest of a number theorists lies in polynomials with **integer** coefficients which leads to **real Mahler ensemble**. Please, stay for the talk by Chris Sinclair where this more complicated case is addressed.

As was observed by Chern & Vaaler, a change of variables from the coefficients of polynomials to their roots, gives

$$H_N(s) := \frac{1}{N!} \int_{\mathbb{C}^N} D_{N,s}(\alpha_1, \ldots, \alpha_N) \mathrm{d}A^{\otimes N}(\alpha_1, \ldots, \alpha_N),$$

where

$$\begin{aligned} D_{N,s} &:= \prod_n \exp\left\{ -2s \log^+ |\alpha_n| \right\} \prod_{n<m} |\alpha_n - \alpha_m|^2 \\ &= \prod_n \exp\left\{ -2s \int_{\mathbb{T}} \log|\tau - \alpha_n| \frac{|\mathrm{d}\tau|}{2\pi} \right\} \prod_{n<m} |\alpha_n - \alpha_m|^2. \end{aligned}$$

For any probability Borel measure on $\mathbb{C}$, say ν, set

$$I[\nu] := \int \log \frac{1}{|z-u|} \mathrm{d}\nu^{\otimes 2}(z,u)$$

to be its *logarithmic energy*. For any compact set K the *logarithmic capacity* of K is defined by

$$\mathrm{cp}(K) := \exp\left\{ - \inf_{\mathrm{supp}(\nu)\subseteq K} I[\nu] \right\}.$$

It is known that either $\mathrm{cp}(K) = 0$ (K is *polar*) or else there exists the unique measure ω_K, the *logarithmic equilibrium distribution* on K, that realizes the infimum. The measure $\frac{|\mathrm{d}\tau|}{2\pi}|_{\mathbb{T}}$ is the equilibrium distribution on both $\mathbb{T}$ and $\overline{\mathbb{D}}$.

g_K, *Green's function* with a pole at ∞ for the unbounded component of K^c, the complement of a compact set K, is the unique harmonic function which is zero q.e. on ∂K^c and behaves like $\log|z|$ at ∞. In particular,

$$g_{\overline{\mathbb{D}}}(z) = g_{\mathbb{T}}(z) = \log^+|z|.$$

Put $g_K \equiv 0$ in $\mathbb{C} \setminus \overline{K^c}$. If it is continuous in $\mathbb{C}$, K is called *regular w.r.t. Dirichlet problem*.

Let K be such that $\mathrm{cp}(K) = 1$. The Mahler measure of a polynomial p with respect to K is defined by

$$M_K(p) := \exp\left\{\int \log|p| \mathrm{d}\omega_K\right\} = |a| \exp\left\{\sum g_K(\alpha_n)\right\}.$$

Let K be a compact set. The joint density of *random configurations* (zeros of random polynomials or equivalently eigenvalues of normal random matrices) is defined by

$$\Omega_{N,s}(\mathbf{z}) := \frac{1}{Z_{N,s}} \exp\left\{-2s\sum_{n=1}^{N} g_K(z_n)\right\} \prod_{m<n} |z_n - z_m|^2,$$

where $s - N + 1 > 1 + c_0$ for some $c_0 > 0$ and

$$Z_{N,s} = \int_{\mathbb{C}^N} \exp\left\{-2s\sum_{n=1}^{N} g_K(z_n)\right\} \prod_{m<n} |z_n - z_m|^2 \, \mathrm{d}A^{\otimes N}.$$

Let $\boldsymbol{\eta} = \{\eta_1, \ldots, \eta_N\}$ be a random configuration chosen according to the law $\Omega_{N,s}$. To any such configuration we associate the *empirical measure* defined as

$$\omega_{\boldsymbol{\eta}} := \frac{1}{N} \sum_{k=1}^{N} \delta_{\eta_k},$$

where δ_z is the classical Dirac delta with the unit mass at z.

Question

Where is it most likely to find $\omega_{\boldsymbol{\eta}}$ when N is large? That is, where it is most likely for random polynomials to have their zeros?

Let ν and μ be two probability Borel measures on $\mathbb{C}$. The distance between them is defined by

$$\operatorname{dist}(\nu,\mu) = \sup_f \left| \int f \mathrm{d}\nu - \int f \mathrm{d}\mu \right|,$$

where the supremum is taken over all functions f that are bounded by 1 in modulus and satisfy the Lipschitz condition with constant 1 on $\mathrm{supp}(\nu) \cup \mathrm{sup}(\mu)$.

For measures supported on a compact set it holds that $\operatorname{dist}(\nu,\nu_n) \to 0$ as $n \to \infty$ if and only if $\nu_n \xrightarrow{*} \nu$, where $\xrightarrow{*}$ stands for the convergence in the weak* topology of measures.

The following theorem takes place.[2]

Theorem (M.Y.)

Let K be a compact set with connected complement which is regular with respect to the Dirichlet problem and such that $K = \overline{K^\circ}$. Then

$$\lim_{\epsilon \to 0} \lim_{N \to \infty} \frac{1}{N^2} \log \mathrm{Prob} \left\{ \mathrm{dist}\,(\nu, \omega_{\boldsymbol{\eta}}) < \epsilon \right\} = -\big(I_\ell[\nu] - I[\omega_K] \big),$$

for any probability Borel measure ν, $\mathrm{supp}(\nu) \subset \mathbb{C}$, where

$$I_\ell[\nu] := I[\nu] + \frac{2}{\ell} \int g_K \mathrm{d}\nu, \quad \ell := \lim_{N \to \infty} s^{-1} N,$$

and it holds that $I_\ell[\omega_K] = I[\omega_K] < I_\ell[\nu]$, $\nu \neq \omega_K$.

[2] Large Deviations and Linear Statistics for Potential Theoretic Ensembles Associated with Regular Closed Sets, Probab. Theory Relat. Fields., 2014

Let η be a random configuration chosen according to $\Omega_{N,s}$ and ω_{η} be the corresponding empirical measure. ω_{η} can be considered as a simple point process on $\mathbb{C}$.

The *correlation functions* of ω_{η} w.r.t. $\mathrm{d}A$ are functions (if they exists) $R_n : \mathbb{C}^n \to [0, \infty)$ such that for any family of mutually disjoint subsets $O_1, \dots, O_n$ it holds that

$$\mathrm{E}\left[\prod_{k=1}^{n} \omega_{\eta}(O_k)\right] = \int_{O_1 \times \cdots \times O_n} R_n(z_1, \dots, z_n) \mathrm{d}A^{\otimes n}(z_1, \dots, z_n)$$

and $R_n(z_1, \dots, z_n)$ vanishes whenever $z_i = z_k$ for $i \neq k$.

Thus, $\int_O R_1 \mathrm{d}A$ is the expected number of zeros that lie in the set O.

Exercise

$$R_n(z_1, \ldots, z_n) = \frac{N!}{(N-n)!} \int_{\mathbb{C}^{N-n}} \Omega_{N,s} \mathrm{d}A^{\otimes(N-n)}(z_{n+1}, \ldots, z_N).$$

Theorem (M.Y.)

Under the conditions of the previous theorem, it holds that

$$\lim_{N\to\infty} \frac{(N-n)!}{N!} \int_{\mathbb{C}^n} f\, R_n \mathrm{d}A^{\otimes n} = \int f \mathrm{d}\omega_K^{\otimes n}$$

for each $f \in \mathrm{C}_b(\mathbb{C}^n)$, $n \in \mathbb{N}$, where $\mathrm{C}_b(\mathbb{C}^n)$ is the Banach space of bounded continuous functions on $\mathbb{C}^n$.

Remark

In particular, $\mathsf{E}(\omega_\eta(O)) \simeq N\omega_K(O)$.

Define a sequence of orthonormal polynomials $\{p_n\}$ such that

$$\int_{\mathbb{C}} p_n(z)\overline{p_m(z)}e^{-2sg_K(z)}\mathrm{d}A = \delta_{nm}.$$

The following fact is by now standard, see Deift[3] or Mehta[4],

$$R_n(z_1, \ldots, z_n) = \det\left[K_N(z_i, z_k)\right]_{i,k=1}^{n},$$

where

$$K_N(z, w) := e^{-s(g_K(z)+g_K(w))}\sum_{n=0}^{N-1} p_n(z)\overline{p_n(w)}.$$

[3] Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach. Volume 3 of Courant Lectures in Mathematics. Amer. Math. Soc., Providence, RI, 2000.

[4] Random Matrices. Volume 142 of Pure and Applied Mathematics, Elsevier/Academic Press, Amsterdam, 2004

Then the following theorem takes place[5].

Theorem (M.Y and C. Sinclair)

Let K be a Jordan domain whose boundary ∂K is a Jordan curve of class $C^{1,\alpha}$, $\alpha > 1/2$. Then

$$p_n = (1 + o(1))\sqrt{\frac{n+1}{\pi}\left(1 - \frac{n+1}{s}\right)}\Phi^n\Phi'$$

uniformly on $\overline{K^c}$, where Φ is the conformal map from $K^c \to \{|z| > 1\}$.

Remark

Observe that $|\Phi(z)| = \exp\left\{g_K(z)\right\}$ for $z \in K^c$.

[5] Universality for ensembles of matrices with potential theoretic weights on domains with smooth boundary, J. Approx. Theory, 164(5):682—708, 2012

Denote by $K(z, w)$ the **reproducing kernel** for the Bergman space on K°. That is,

$$f(z) = \int_{K^\circ} f(w) K(z, w) \mathrm{d}A(w)$$

for every holomorphic f such that $\int_{K^\circ} |f|^2 \mathrm{d}A < \infty$.

Theorem (M.Y. and C. Sinclair)

Under the conditions of the previous theorem, $K_N(z, w)$ converges to $K(z, w)$ locally uniformly in $K^\circ \times K^\circ$.

Remark

For all N large, random polynomials are expected to have a “fixed” number of zeros in each set of positive Lebesgue measure.

$$K_N(z,w) = \left|\Phi(z)\overline{\Phi(w)}\right|^{-s} \sum_{n=0}^{N-1} p_n(z)\overline{p_n(w)}, \quad z,w \in K^c.$$

Theorem (M.Y. and C. Sinclair)

Under the conditions of the previous theorem, it holds that

$$\frac{\left|\Phi(z)\overline{\Phi(w)}\right|^s}{\left(\Phi(z)\overline{\Phi(w)}\right)^N} \frac{K_N(z,w)}{s-N} \to \frac{1}{\pi} \frac{\Phi'(z)\overline{\Phi'(w)}}{\Phi(z)\overline{\Phi(w)}-1} \left[1 + \frac{c^{-1}}{\Phi(z)\overline{\Phi(w)}-1}\right]$$

locally uniformly in $K^c \times K^c$, where $c := \lim_{N\to\infty}(s-N)$. In particular, $K_N(z,w) \to 0$ when $s-N\to\infty$.

Remark

When $c < \infty$ and N is large, random polynomials are expected to have a "fixed" number of zeros in each set of positive Lebesgue measure.

From the linear statistics we know that

$$\mathrm{E}\left[\omega_{\boldsymbol{\eta}}(\mathbb{D}_\varepsilon(\tau))\right] \sim N\omega_K(\mathbb{D}_\varepsilon(\tau)) \sim \varepsilon N\omega_K'(\tau).$$

Thus, to see a non-trivial behavior around τ we need to scale $\varepsilon \sim N^{-1}$.
We also know that

$$\begin{aligned}\mathrm{E}\left[\omega_{\boldsymbol{\eta}}(\mathbb{D}_{\frac{1}{N}}(\tau))\right] &= \int_{\mathbb{D}_{\frac{1}{N}}(\tau)} R_1(z)\mathrm{d}A(z) = \int_{\mathbb{D}_{\frac{1}{N}}(\tau)} K_N(z,z)\mathrm{d}A(z) \\ &= \int_{\mathbb{D}} \frac{1}{N^2} K_N\left(\tau + \frac{z}{N}, \tau + \frac{z}{N}\right)\mathrm{d}A(z).\end{aligned}$$

Thus, we expect integrand to converge and therefore set

$$K_\tau(z,w) := \lim_{N\to\infty} \frac{1}{N^2} K_N\left(\tau + \frac{z}{N}, \tau + \frac{w}{N}\right).$$

Theorem (M.Y. and C. Sinclair)

Under the conditions of the previous theorem, it holds that

$$K_\tau(z,w) = \frac{\omega(\tau,z)\omega(\tau,w)}{\pi}\int_0^1 x(1-\ell x)e^{\left(a(\tau,z)+\overline{a(\tau,w)}\right)x}\mathrm{d}x,$$

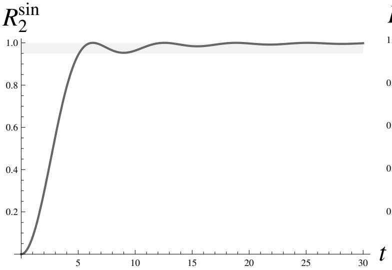
where $a(\tau,z) := z\Phi'(\tau)\overline{\Phi(\tau)}$ (the argument of $a(\tau,z)$ is equal to the angle between z and the outward normal to ∂K), $\ell = \lim_{N\to\infty} s^{-1}N$, and

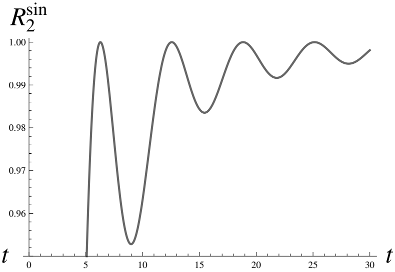
$$\omega(\tau,z) := \lim_{N\to\infty} \exp\left\{-sg_K\left(\tau+\frac{z}{N}\right)\right\}.$$

Remark

$$\int_0^1 x(1-\ell x)e^{\eta x}\mathrm{d}x = (1-\ell)\frac{e^\eta(\eta-1)+1}{\eta^2} + \ell\frac{e^\eta(\eta-2)+\eta+2}{\eta^3}.$$

$$K^{\sin}(a,b) = \frac{\sin(a-b)}{a-b} = \frac{e^{\eta}-e^{-\eta}}{2\eta}, \quad \eta = \mathrm{i}(a-b)$$

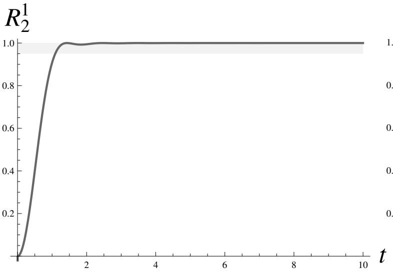


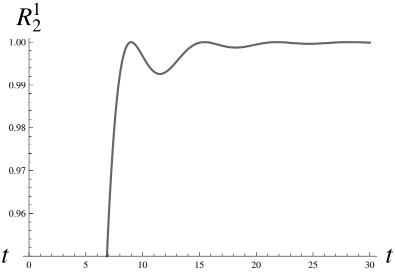


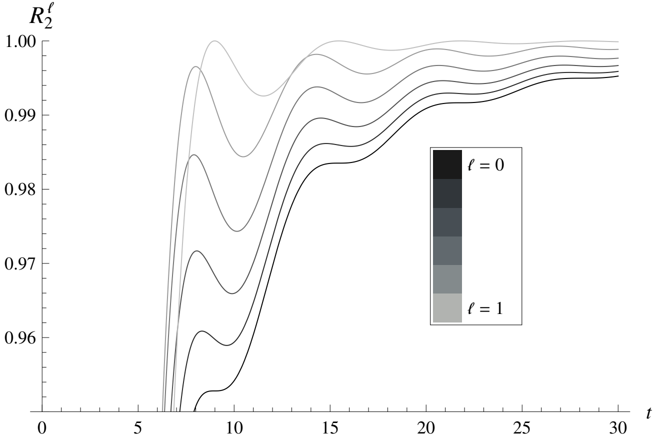
Plots of $R_2^{\sin}(a,b) = 1 - K^{\sin}(a,b)^2$ as a function of $2(a-b)$. The second plot is an enlargement of the shaded region.

$$\frac{\pi K_\tau(z,w)}{\omega(\tau,z)\omega(\tau,w)} = (1-\ell)\frac{e^\eta(\eta-1)+1}{\eta^2} + \ell\frac{e^\eta(\eta-2)+\eta+2}{\eta^3}$$

where $\eta = a(\tau,z) + \overline{a(\tau,w)}$ which can be parametrized as $\mathrm{i}(a-b)$ in the tangential direction (in which case $\omega(\tau,z)=1$).







Plot of the interpolation between R_2^0 and R_2^1 along a tangent line.