Root statistics of random polynomials with bounded Mahler measure

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Mahler Measure

The **Mahler measure** of a polynomial $P(z) = a\prod_{n=1}^{N}(z-\alpha_n)$ is defined as

$$\begin{aligned} M(P) &:= |a|\prod_{n=1}^{N}\max\left\{1,|\alpha_n|\right\} \\ &= \exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\log\left|P\left(e^{\mathrm{i}\theta}\right)\right|\mathrm{d}\theta\right\}. \end{aligned}$$

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Theorem (Kronecker, 1857)

$M(P)=1$ for a polynomial P with integer coefficients iff P is a product of monomials and cyclotomic polynomials (divisors of z^n-1). Necessarily, such a polynomial has all its roots in $\mathbb{T}\cup\{0\}$.

The **Mahler measure** of a polynomial $P(z) = a\prod_{n=1}^{N}(z-\alpha_n)$ is defined as

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Theorem (Kronecker, 1857)

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Conjecture (Lehmer, 1933)

Is 1 an isolated point of the range of $M(\cdot)$ on integer polynomials?

Lehmer himself constructed the smallest known example:

$$M(z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1) \approx 1.18.$$

Theorem (Chern-Vaaler, 2001)

The number of integer polynomials of height at most T behaves as

$$\mathrm{vol}\left(B_N\right) T^{N+1} + \mathcal{O}(T^N), \quad T \to \infty,$$

where B_N is the Mahler measure unit star body. Moreover,

$$\mathrm{vol}\left(B_N\right) = \frac{2}{N+1} F_N\left(N+1\right),$$

where

$$F_N(s) = C_N \prod_{m=0}^{\lfloor (N-1)/2 \rfloor} \frac{s}{s-(N-2m)}$$

and C_N is an explicit constant.

Notice that Lehmer's conjecture asks what happens when $T \to 1$.

Observe also that both $(z-1)^N$ and $z^N - 1$ belong to B_N but have drastically different coefficient vectors.

More generally, the **λ-homogeneous Mahler measure** is given by

$$M^{\lambda}(P) = |a|^{\lambda} \prod_{n=1}^{N} \max\left\{1, |\alpha_n|\right\}.$$

The corresponding unit star body is defined as

$$B_N^{\lambda} := \left\{(a_1, \ldots, a_{N+1}) \in \mathbb{R}^{N+1} : M^{\lambda}\left(\sum_{n=0}^{N} a_{n+1} z^n\right) \leq 1\right\}.$$

Theorem (Chern-Vaaler, 2001)

$$\mathrm{vol}\left(B_N^{\lambda}\right) = \frac{2}{N+1} F_N\left(\frac{N+1}{\lambda}\right).$$

Volumes of Star Bodies

$$\begin{aligned}\mathrm{vol}\left(B_N^\lambda\right) &= \int_{-\infty}^{\infty} \mathrm{vol}\left\{\boldsymbol{b} : M^\lambda\left(cz^N + \sum_{n=0}^{N-1} b_{n+1}z^n\right) \leq 1\right\} \mathrm{d}c \\ &= \int_{-\infty}^{\infty} \mathrm{vol}\left\{c\boldsymbol{b} : M^\lambda\left(cz^N + \sum_{n=0}^{N-1} cb_{n+1}z^n\right) \leq 1\right\} \mathrm{d}c.\end{aligned}$$

Using the λ-homogeneity of M^λ one then gets

$$\begin{aligned}\mathrm{vol}\left(B_N^\lambda\right) &= 2\int_0^\infty c^N \mathrm{vol}\left\{\boldsymbol{b} : M(\boldsymbol{b}) \leq c^{-\lambda}\right\} \mathrm{d}c \\ &= \frac{2}{\lambda}\int_0^\infty \xi^{-(N+1)/\lambda} \mathrm{vol}\{\boldsymbol{b} : M(\boldsymbol{b}) \leq \xi\} \frac{\mathrm{d}\xi}{\xi},\end{aligned}$$

where $M(\boldsymbol{b})$ is the Mahler measure of $z^N + \sum_{n=0}^{N-1} b_{n+1}z^n$. Integration by parts then gives

$$\begin{aligned}\mathrm{vol}\left(B_N^\lambda\right) &= \frac{2}{N+1}\int_0^\infty \xi^{-(N+1)/\lambda} \mathrm{d}\mathrm{vol}\{\boldsymbol{b} : M(\boldsymbol{b}) \leq \xi\} \\ &= \frac{2}{N+1}\int_{\mathbb{R}^N} M(\boldsymbol{b})^{-(N+1)/\lambda} \mathrm{d}\mu_{\mathbb{R}}^N(\boldsymbol{b}).\end{aligned}$$

Volumes of Star Bodies

Making a change of variables from coefficients of polynomials to their roots gives

$$F_N(s) := \int_{\mathbb{R}^N} M(\boldsymbol{b})^{-s} \mathrm{d}\mu_{\mathbb{R}}^N(\boldsymbol{b}) = \sum_{L+2M=N} \frac{Z_{L,M}(s)}{L!M!},$$

where L and M stand for the number of real and complex roots, and

$$Z_{L,M}(s) = \int_{\mathbb{R}^L} \int_{\mathbb{C}^M} \prod_{l=1}^{L} \Phi(\alpha_l)^{-s} \prod_{m=1}^{M} \Phi(\beta_m)^{-2s} |\Delta(\boldsymbol{\alpha}, \boldsymbol{\beta})| \mathrm{d}\mu_{\mathbb{R}}^L(\boldsymbol{\alpha}) \mathrm{d}\mu_{\mathbb{C}}^M(\boldsymbol{\beta})$$

with $\Delta(\boldsymbol{\alpha}, \boldsymbol{\beta})$ being the Vandermonde of $\alpha_1, \ldots, \alpha_L, \beta_1, \overline{\beta}_1, \ldots, \beta_M, \overline{\beta}_M$.

The summands $Z_{L,M}(s)$ are not simple and Chern-Vaaler went through a dozen pages of rational function identities to show that

$$F_N(s) = C_N \prod_{m=0}^{\lfloor (N-1)/2 \rfloor} \frac{s}{s - (N - 2m)}.$$

Complex Star Bodies

In fact, one could consider polynomials with complex coefficients. Set

$$B_N^\lambda(\mathbb{C}) := \left\{ (a_1, \ldots, a_{N+1}) \in \mathbb{C}^{N+1} : M^\lambda \left(\sum_{n=0}^{N} a_{n+1} z^n \right) \le 1 \right\}.$$

Then

$$\begin{aligned} \mathrm{vol}\big(B_N^\lambda(\mathbb{C})\big) &= \int_{\mathbb{C}} \mathrm{vol}\left\{ \boldsymbol{b} : M^\lambda \left(cz^N + \sum_{n=0}^{N-1} b_{n+1} z^n \right) \le 1 \right\} \mathrm{d}c \\ &= \int_{\mathbb{C}} |c|^{2N} \mathrm{vol}\left\{ \boldsymbol{b} : M(\boldsymbol{b}) \le |c|^{-\lambda} \right\} \mathrm{d}c \\ &= \frac{\pi}{N+1} \int_0^\infty \xi^{-2(N+1)/\lambda} \mathrm{d}\mathrm{vol}\{ \boldsymbol{b} : M(\boldsymbol{b}) \le 1 \} \\ &= \frac{\pi}{N+1} \int_{\mathbb{C}^N} M(\boldsymbol{b})^{-2(N+1)/\lambda} \mu_{\mathbb{C}}^N(\boldsymbol{b}). \end{aligned}$$

As before, making a change of variables from the coefficients to the roots gives

$$G_N(s) := \int_{\mathbb{C}^N} M(\boldsymbol{b})^{-2s} \mathrm{d}\mu_{\mathbb{C}}^N(\boldsymbol{b}) = \frac{Z_N(s)}{N!},$$

where

$$\begin{aligned} Z_N(s) &= \int_{\mathbb{C}^N} \prod_{n=1}^{N} \Phi(\lambda_n)^{-2s} |\Delta(\boldsymbol{\lambda})|^2 \mathrm{d}\mu_{\mathbb{C}}^N(\boldsymbol{\lambda}) \\ &= (2\pi)^N \sum_{\sigma} \left(\prod_{n=1}^{N} \int_0^\infty \Phi(\rho_n)^{-2s} \rho_n^{2\sigma(n)-1} \mathrm{d}\rho_n \right) \\ &= (2\pi)^N \sum_{\sigma} \left(\prod_{n=1}^{N} \frac{s}{2\sigma(n)(s-\sigma(n))} \right) = \pi^N \prod_{n=1}^{N} \frac{s}{s-n}. \end{aligned}$$

Theorem (Chern-Vaaler, 2001)

$$\operatorname{vol}\big(B_N^\lambda(\mathbb{C})\big) = \frac{\pi}{N+1} G_N\left(\frac{N+1}{\lambda}\right).$$

Theorem (Sinclair, 2008)

Let $\Pi_0, \ldots, \Pi_{N-1}$ *be polynomials such that*

$$\langle \Pi_n | \Pi_m \rangle = \delta_{n,m},$$

where inner product $\langle \cdot | \cdot \rangle$ *is defined by*

$$\langle f | g \rangle_{\mathbb{C}} = \int_{\mathbb{C}} f(z) \overline{g(z)} \Phi(z)^{-2s} \mathrm{d}\mu_{\mathbb{C}}(z),$$

with $\Phi(z) := \max\{1, |z|\}$*. Then*

$$G_N(s) = \prod_{n=0}^{N-1} \gamma_n^{-2},$$

where $\Pi_k(z) = \gamma_k z^k + \cdots$*.*

Theorem (Sinclair, 2008)

Let $\pi_0, \ldots, \pi_{N-1}$ *be polynomials such that*

$$\langle \pi_{2n} | \pi_{2m} \rangle = \langle \pi_{2n+1} | \pi_{2m+1} \rangle = 0 \quad \text{and} \quad \langle \pi_{2n} | \pi_{2m+1} \rangle = \delta_{n,m},$$

where skew-symmetric inner product $\langle \cdot | \cdot \rangle = \langle \cdot | \cdot \rangle_{\mathbb{R}} + \langle \cdot | \cdot \rangle_{\mathbb{C}}$ *is defined by*

$$\begin{aligned} \langle f | g \rangle_{\mathbb{R}} &= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x) g(y) \operatorname{sgn}(y - x) \Phi(x)^{-s} \Phi(y)^{-s} \mathrm{d}\mu_{\mathbb{R}}(x) \mathrm{d}\mu_{\mathbb{R}}(y) \\ \langle f | g \rangle_{\mathbb{C}} &= -2\mathrm{i} \int_{\mathbb{C}} \overline{f(z)} g(z) \operatorname{sgn}(\operatorname{Im}(z)) \Phi(z)^{-2s} \mathrm{d}\mu_{\mathbb{C}}(z), \end{aligned}$$

with $\Phi(z) = \max\{1, |z|\}$. *Then*

$$F_N(s) = \prod_{n=0}^{\lfloor (N-1)/2 \rfloor} \left(\gamma_{2n} \gamma_{2n+1} \right)^{-1},$$

where $\pi_k(z) = \gamma_k z^k + \cdots$.

Recall that

$$G_N(s) = \int_{\mathbb{C}^N} M(\boldsymbol{b})^{-2s} \mathrm{d}\mu_{\mathbb{C}}^N(\boldsymbol{b}) \quad \text{and} \quad F_N(s) = \int_{\mathbb{R}^N} M(\boldsymbol{b})^{-s} \mathrm{d}\mu_{\mathbb{R}}^N(\boldsymbol{b}).$$

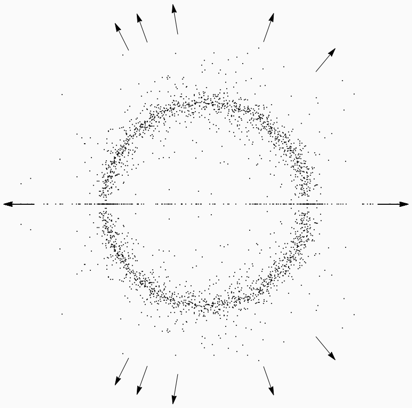
Under a **random polynomial** we mean a polynomial chosen with respect to

$$M(\boldsymbol{b})^{-2s}/G_N(s), \quad \boldsymbol{b} \in \mathbb{C}^N, \quad \text{or} \quad M(\boldsymbol{b})^{-s}/F_N(s), \quad \boldsymbol{b} \in \mathbb{R}^N.$$

This is equivalent to choosing polynomials uniformly from $B_N^{(N+1)s^{-1}}$.

We would like to study fine statistics of zeros of such random polynomials.

Numerical Simulation



A simultaneous plot of the roots of 100 random polynomials of degree 28. A ball-walk of $10,000$ steps of length .01 starting from x^{28} was performed for each polynomial. The arrows indicate directions of outlying roots.

Correlation Functions: Complex Case

Let P be a random polynomial. For $C \subset \mathbb{C}$ define $N_C := \sharp C \cap \{\text{zeros of } P\}$.

In the case of complex coefficients, a function $R_n : \mathbb{C}^n \to [0, \infty)$ is called ***n*-th correlation function** if

$$E[N_{C_1} \cdots N_{C_n}] = \int_{C_1} \cdots \int_{C_n} R_n(z) \mathrm{d}\mu_{\mathbb{C}}^n(z)$$

for pairwise disjoint sets $C_1, \ldots, C_n$. Since the joint density of the zeros is given by

$$\frac{1}{Z_N(s)} \prod_{m<n} |\lambda_n - \lambda_m|^2 \prod_{n=1}^{N} \Phi(\lambda_n)^{-2s} \mathrm{d}\mu_{\mathbb{C}}^N(\boldsymbol{\lambda}),$$

$\Phi(z) = \max\{1, |z|\}$, it is well known in random matrix theory that

$$R_n(\boldsymbol{\lambda}) = \det \left[K_N(\lambda_i, \lambda_j)\right]_{i,j=1}^{n},$$

where

$$K_N(z, w) := \Phi(z)^{-s} \Phi(w)^{-s} \sum_{n=0}^{N-1} \Pi_n(z) \overline{\Pi_n(w)}$$

and Π_n are orthonormal polynomials w.r.t. $\Phi^{-2s}(z) \mathrm{d}\mu_{\mathbb{C}}(z)$.

Correlation Functions: Real Case

In the case of real coefficients, if there is a function $R_{l,m} : \mathbb{R}^l \times \mathbb{C}_+^m \to [0, \infty)$ such that

$$E\left[N_{A_1} \cdots N_{A_l} N_{B_1} \cdots N_{B_m}\right] := \int_{A_1} \cdots \int_{A_l} \int_{B_1} \cdots \int_{B_m} R_{l,m}(\boldsymbol{x}, \boldsymbol{z}) \mathrm{d}\mu_{\mathbb{R}}^l(\boldsymbol{x}) \mathrm{d}\mu_{\mathbb{C}}^m(\boldsymbol{z})$$

for pairwise disjoint sets $A_1, \ldots, A_l \subset \mathbb{R}$ and $B_1, \ldots, B_m \subset \mathbb{C}_+$, then it is called the **(l, m)-th correlation function**.

When such functions exist, it holds in particular that

$$\deg(P) = \int_{\mathbb{R}} R_{1,0}(x, -) \mathrm{d}\mu_{\mathbb{R}}(x) + \int_{\mathbb{C}} R_{0,1}(-, z) \mathrm{d}\mu_{\mathbb{C}}(z)$$

and the first integral represents the expected number of real zeros, where we set $R_{l,m}(\cdot, \overline{z}) := R_{l,m}(\cdot, z)$.

Theorem (Borodin-Sinclair, 2009)

There exists a 2×2 *matrix kernel* $\boldsymbol{K}_N : \mathbb{C} \times \mathbb{C} \to \mathbb{C}^{2\times 2}$ *such that*

$$R_{l,m}(\boldsymbol{x}, \boldsymbol{z}) = \mathrm{Pf}\begin{bmatrix} \left[\boldsymbol{K}_N(x_i, x_j)\right]_{i,j=1}^{l} & \left[\boldsymbol{K}_N(x_i, z_n)\right]_{i,n=1}^{l,m} \\ -\left[\boldsymbol{K}_N^{\mathrm{T}}(z_k, x_j)\right]_{k,j=1}^{m,l} & \left[\boldsymbol{K}_N(z_k, z_n)\right]_{k,n=1}^{m} \end{bmatrix}.$$

In particular, it holds that

$$R_{1,0}(x, -) = \mathrm{Pf}\,\boldsymbol{K}_N(x, x) \quad \textit{and} \quad R_{0,1}(-, z) = \mathrm{Pf}\,\boldsymbol{K}_N(z, z).$$

Recall that we set $\langle \cdot | \cdot \rangle = \langle \cdot | \cdot \rangle_{\mathbb{R}} + \langle \cdot | \cdot \rangle_{\mathbb{C}}$, where

$$\begin{aligned} \langle f | g \rangle_{\mathbb{R}} &= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x) g(y) \mathrm{sgn}(y - x) \Phi(x)^{-s} \Phi(y)^{-s} \mathrm{d}\mu_{\mathbb{R}}(x) \mathrm{d}\mu_{\mathbb{R}}(y) \\ \langle f | g \rangle_{\mathbb{C}} &= -2\mathrm{i} \int_{\mathbb{C}} \overline{f(z)} g(z) \mathrm{sgn}(\mathrm{Im}(z)) \Phi(z)^{-2s} \mathrm{d}\mu_{\mathbb{C}}(z). \end{aligned}$$

Correlation Functions: Real Case

Theorem (Borodin-Sinclair, 2009)

Let $N = 2J$ *and* $\pi_0, \ldots, \pi_{N-1}$ *be skew-orthogonal polynomials w.r.t.* $\langle\cdot|\cdot\rangle$*. Set*

$$\kappa_N(u, v) := 2\Phi(u)^{-s}\Phi(v)^{-s}\sum_{n=0}^{J}\left(\pi_{2n}(u)\pi_{2n+1}(v) - \pi_{2n}(v)\pi_{2n+1}(u)\right).$$

Then

$$\boldsymbol{K}_N(u, v) = \begin{bmatrix} \kappa_N(u, v) & \kappa_N\epsilon(u, v) \\ \epsilon\kappa_N(u, v) & \epsilon\kappa_N\epsilon(u, v) + \frac{1}{2}\mathrm{sgn}(u - v) \end{bmatrix},$$

where $\mathrm{sgn}(\cdot) = 0$ *for non-real arguments and* ϵ *is the operator*

$$\epsilon f(u) := \begin{cases} \frac{1}{2}\int_{\mathbb{R}} f(t)\mathrm{sgn}(t - u)\mathrm{d}\mu_{\mathbb{R}}(t), & u \in \mathbb{R}, \\ \mathrm{i}\cdot\mathrm{sgn}(\mathrm{Im}(u))f(\overline{u}), & u \in \mathbb{C}\setminus\mathbb{R}, \end{cases}$$

which acts on u *when written on the left and on* v *when written on the right.*

The following results are from Sinclair-Ya. 2012 (complex case) and 2015 (real case).

Theorem

It holds that

$$\pi_{2n}(z) = \frac{2}{\pi} \sum_{k=0}^{n} \frac{\Gamma(k+3/2)\Gamma(n-k+1/2)}{\Gamma(k+1)\Gamma(n-k+1)} z^{2k}$$

and

$$\pi_{2n+1}(z) = -\frac{1}{2\pi} \sum_{k=0}^{n} \frac{s-(2k+2)}{2s} \frac{\Gamma(k+3/2)\Gamma(n-k-1/2)}{\Gamma(k+1)\Gamma(n-k+1)} z^{2k+1}.$$

It is also true that

$$\Pi_n(z) = \sqrt{\frac{n+1}{\pi}\left(1-\frac{n+1}{s}\right)} z^n.$$

Expected Number of Real Zeros

Write $\widetilde{\pi}_k(z) := \pi_k(z)\Phi(z)^{-s}$. Given $A \subseteq \mathbb{R}$ and N even, it holds that

$$
\begin{aligned}
E[N_A] &= \int_A \mathrm{Pf}\,\boldsymbol{K}_N(x,x)\mathrm{d}\mu_{\mathbb{R}}(x) \\
&= \int_A \mathrm{Pf}\begin{bmatrix} 0 & \kappa_N\epsilon(x,x) \\ \epsilon\kappa_N(x,x) & 0 \end{bmatrix} \mathrm{d}\mu_{\mathbb{R}}(x) \\
&= 2\sum_{n=0}^{N/2}\int_A \left(\widetilde{\pi}_{2n}(x)\epsilon\widetilde{\pi}_{2n+1}(x) - \widetilde{\pi}_{2n+1}(x)\epsilon\widetilde{\pi}_{2n}(x)\right) d\mu_{\mathbb{R}}(x).
\end{aligned}
$$

Theorem

Let N_{in} and N_{out} be the number of real roots on $[-1,1]$ and $\mathbb{R} \setminus (-1,1)$. Then

$$\begin{cases} E[N_{\text{in}}] &= \dfrac{1}{\pi} \log N + O_N(1) \\ E[N_{\text{out}}] &= -\dfrac{1}{\pi} \dfrac{\sqrt{N(2s-N)}}{s} \log\left(1 - Ns^{-1}\right) + \sqrt{Ns^{-1}}\, O_N(1), \end{cases}$$

where the implicit constants are uniform with respect to s.

Observe that

$$E[N_{\text{out}}] = \begin{cases} \sqrt{Ns^{-1}}\, O_N(1), & \limsup_{N\to\infty} Ns^{-1} < 1, \\ \frac{\alpha}{\pi} \log N + O_N(1), & s = N + N^{1-\alpha}, \alpha \in [0,1], \\ \frac{1}{\pi} \log N + O_N(1), & \limsup_{N\to\infty} (s - N) < \infty. \end{cases}$$

Expected Number of Zeros Around Points of the Unit Circle

Let $\zeta \in \mathbb{T}$ and δ be small. In the complex case we have that

$$\begin{aligned} E\left[N_{\zeta+\delta\mathbb{D}}\right] &= \int_{\zeta+\delta\mathbb{D}} K_N(z,z)\mathrm{d}\mu_{\mathbb{C}}(z) \\ &= \int_{\mathbb{D}} \delta^2 K_N\big(\zeta+\delta z, \zeta+\delta z\big)\mathrm{d}\mu_{\mathbb{C}}(z). \end{aligned}$$

Similarly, in the real case we have for $\zeta \in \mathbb{T} \setminus \{\pm 1\}$ that

$$E\left[N_{\zeta+\delta\mathbb{D}}\right] = \int_{\zeta+\delta\mathbb{D}} \mathrm{Pf}\,\boldsymbol{K}_N(z,z)\mathrm{d}\mu_{\mathbb{C}}(z).$$

As we have N total zeros, the scale should be $\delta = 1/N$.

Theorem

Let $\zeta \in \mathbb{T}$. Assume that $\lambda := \lim_{N\to\infty} Ns^{-1} \in [0,1]$ exists. Then

$$\lim_{N\to\infty} \frac{1}{N^2} K_N\left(\zeta + \frac{z}{N}, \zeta + \frac{w}{N}\right) = K_\zeta(z,w),$$

where $\omega(\tau) := \min\left\{1, e^{-\mathrm{Re}(\tau)/\lambda}\right\}$ and

$$K_\zeta(z,w) = \omega\left(z\overline{\zeta}\right)\omega\left(\overline{w}\zeta\right)\frac{1}{\pi}\int_0^1 x(1-\lambda x)e^{\left(z\overline{\zeta}+\overline{w}\zeta\right)x}\mathrm{d}x.$$

It holds that $\mathrm{Re}(z\overline{\zeta}) > 0$ iff z points outside $\mathbb{D}$ at ζ.

Expected Number of Zeros Around Points of the Unit Circle

Theorem

Let $\zeta \in \mathbb{T} \setminus \{\pm 1\}$*. Assume that* $\lambda := \lim_{N\to\infty} Ns^{-1} \in [0,1]$ *exists. Then*

$$\lim_{N\to\infty} \frac{1}{N^2} \boldsymbol{K}_N\left(\zeta + \frac{z}{N}, \zeta + \frac{w}{N}\right) = \begin{bmatrix} 0 & K_\zeta(z,w) \\ -K_\zeta(w,z) & 0 \end{bmatrix},$$

That is, Pfaffian point process becomes essentially determinantal around ζ.

Expected Number of Zeros Around Points of the Unit Circle

Theorem

Let $\zeta \in \mathbb{T} \setminus \{\pm 1\}$. Assume that $\lambda := \lim_{N\to\infty} Ns^{-1} \in [0,1]$ exists. Then

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Theorem

Let $\xi \in \{\pm 1\}$. Assuming that $\lambda := \lim_{N\to\infty} Ns^{-1} \in [0,1]$ exists, it holds that

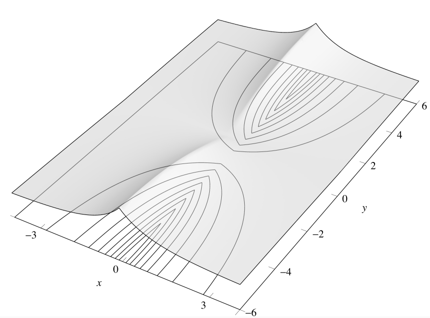
$$\lim_{N\to\infty} \frac{1}{N^2} \kappa_N\left(x + \frac{u}{N}, \xi + \frac{v}{N}\right) = \kappa_\xi(u,v)$$

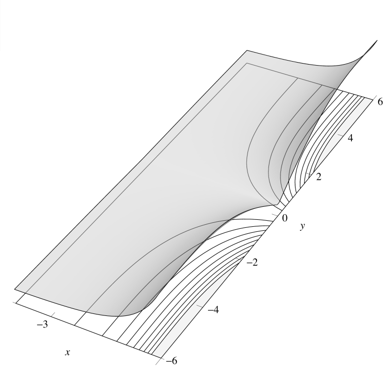
where the convergence is locally uniform in $\mathbb{C} \times \mathbb{C}$,

$$\kappa_\xi(u,v) = \omega(u\xi)\,\omega(v\xi)\,\frac{\xi}{4}\int_0^1 \tau(1-\lambda\tau)\Big(M'(u\xi\tau)M(v\xi\tau) - M(u\xi\tau)M'(v\xi\tau)\Big)\mathrm{d}\tau,$$

and $M(z) = {}_1F_1(3/2, 1; z)$, i.e., $zM''(z) + (1-z)M'(z) - \frac{3}{2}M(z) = 0$.

Expected Number of Zeros Around ± 1





The scaled intensity of complex roots near 1, for $\lambda = 1$ (left) and $\lambda = 0$ (right). Note how the roots tend to accumulate near the unit disk (the y-axis here) and repel from the real axis.

Theorem

Assuming that $\lambda := \lim_{N\to\infty} Ns^{-1} \in [0,1]$ *exists, it holds that*

$$\lim_{N\to\infty} K_N(z,w) = \frac{1}{\pi}\frac{1}{(1-z\overline{w})^2}$$

and

$$\lim_{N\to\infty} \kappa_N(u,v) = \frac{1}{4\pi}\int_{\mathbb{T}} \frac{\left(v\sqrt{-\tau} - u\sqrt{-\overline{\tau}}\right)|\mathrm{d}\tau|}{\left(1-u^2\overline{\tau}\right)^{3/2}\left(1-v^2\tau\right)^{3/2}}$$

locally uniform in $\mathbb{D}\times\mathbb{D}$, *where* $\sqrt{-\tau}$ *is the branch defined by* $-\frac{2}{\pi}\sum_{-\infty}^{\infty}\frac{\tau^m}{2m-1}$.

Expected Number of Zeros on Bounded Subsets of $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$

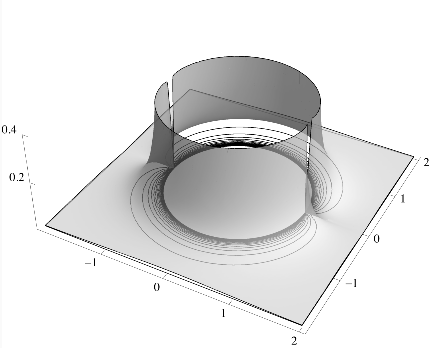
Theorem

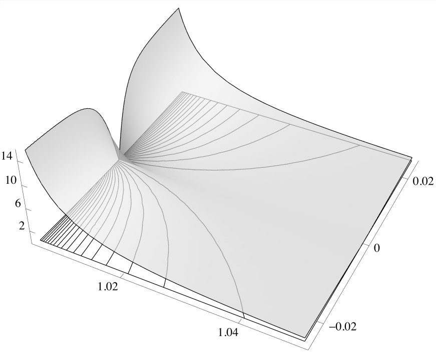
Assuming that $\lambda := \lim_{N\to\infty} Ns^{-1} \in [0,1]$ *amd* $c := \lim_{N\to\infty}(s-N) \in [0,\infty]$ *exist, it holds that*

$$\lim_{N\to\infty} \frac{|z\overline{w}|^s}{(z\overline{w})^N} \frac{K_N(z,w)}{s-N} = \frac{\lambda}{\pi} \frac{1}{z\overline{w}-1} \left[1 + \frac{c^{-1}}{z\overline{w}-1}\right]$$

and

$$\lim_{N\to\infty} \frac{|uv|^s}{(uv)^N} \frac{\kappa_N(u,v)}{s-N} = \frac{\lambda}{\pi} \frac{1}{uv-1} \left[1 + \frac{c^{-1}}{uv-1}\right] \frac{v-u}{\sqrt{u^2-1}\sqrt{v^2-1}}.$$





The limiting intensity of complex roots outside the disk, with a close up view near $z = 1$, for the Mahler measure ($c = 1$) case.