Advances in Asymptotics of Multiple Orthogonal Polynomials for Angelesco Systems

Maxim L. Yattselev

INDIANAPOLIS **SCHOOL OF SCIENCE**
Department of Mathematical Sciences

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References

This talk is based on

- M.Y., Strong asymptotics of Hermite–Padé approximants for Angelesco systems with complex weights, *Canad. J. Math.*, 2016

- A. Aptekarev, S. Denisov, and M.Y., Self-adjoint Jacobi matrices on trees and multiple orthogonal polynomials, *Trans. Amer. Math. Soc.*, 2020

- A. Aptekarev, S. Denisov, and M.Y., Jacobi matrices on trees generated by Angelesco systems: asymptotics of coefficients and essential spectrum, *J. Spectr. Theory*, 2021

- S. Denisov, and M.Y., Spectral theory of Jacobi matrices on trees whose coefficients are generated by multiple orthogonality. *Adv. Math.*, 2022

- A. Aptekarev, S. Denisov, and M.Y., Strong asymptotics of multiple orthogonal polynomials for Angelesco systems. Part I: non-marginal directions, *submitted*

- M.Y., Uniformity of strong asymptotics in Angelesco systems, *to be submitted*

Orthogonal Polynomials

Let μ be a probability measure with bounded infinite support on the real line. Define $P_n(x)$ to be the monic orthogonal polynomial of degree n, i.e.,

$$\int P_n(x)x^k d\mu(x) = 0, \quad k = \overline{0, n-1}.$$

When $d\mu(x)$ is the arcsine distribution on $[-1, 1]$, i.e., $d\omega(x) = \dfrac{dx}{\pi\sqrt{1-x^2}}$, it holds that

$$P_n(z) = T_n^\star(z) = 2^{-n}\left(\Phi^n(z) + \Phi^{-n}(z)\right), \quad \Phi(z) = z + \sqrt{z^2-1},$$

is the monic Chebyshëv polynomial of the second kind.

It holds that

$$\frac{1}{n}\log|T_n^\star(z)| \sim \log\left|\frac{z+\sqrt{z^2-1}}{2}\right| = -V^\omega(z),$$

where $V^\sigma(z) = -\int \log|x-y|d\sigma(y)$ is the logarithmic potential of σ.

The arcsine distribution is the logarithmic equilibrium distribution on $[-1,1]$:

$$\int \log\frac{1}{|x-y|}d\omega(x)d\omega(y) \leq \int \log\frac{1}{|x-y|}d\sigma(x)d\sigma(y)$$

for any probability Borel measure σ on $[-1,1]$. Equivalently, it is the unique measure such that

$$V^\sigma(z) \leq C \quad z\in\mathbb{C}, \quad V^\sigma(z) \geq C \quad \text{q.e.} \quad z \in \operatorname{supp}(\sigma),$$

for some constant C.

Weak Asymptotics

Theorem (Stahl - Totik (book))

If the measure μ is UST-regular, then

$$\frac{1}{n} \log |P_n(z)| \sim -V^{\omega_{\text{supp}}(\mu)}(z),$$

where $\omega_{\text{supp}}(\mu)$ is the logarithmic equilibrium distribution of $\text{supp}(\mu)$.

Let $\Delta(\mu)$ be the convex hull of $\text{supp}(\mu)$. If

$$\liminf_{r \to 0} \log \mu([x - r, x + r]) \geq 0$$

for almost every $x \in \Delta(\mu)$, then μ is UST-regular.

Recurrence Relations

The monic orthogonal polynomials satisfy the three-term recurrence relations:

$$xP_n(x) = P_{n+1}(x) + b_n P_n(x) + a_{n-1} P_{n-1}(x)$$

with $P_{-1} := 0$, $P_0 = 1$, and $a_n > 0$. In the case of Chebyshëv polynomials it holds that

$$xT_n^\star(x) = T_{n+1}^\star(x) + (1/4)T_{n-1}^\star(x).$$

Ratio Asymptotics

Theorem (Nevai)

If $\text{supp}(\mu) = [-1, 1]$, then

$$\frac{P_{n+1}(z)}{P_n(z)} \sim \frac{z + \sqrt{z^2 - 1}}{2} \quad \Leftrightarrow \quad b_n \to 0,\ a_n \to 1/4.$$

The above conditions are satisfied if

$$d\mu(x) = \mu'(x)dx + d\mu^s(x),$$

and $\mu'(x) > 0$ almost everywhere on $[-1, 1]$.

Jacobi Operators

Let $c_n = \sqrt{a_n}$. The Jacobi matrix $\mathcal{J}$, defined by

$$\mathcal{J} := \begin{bmatrix} b_0 & c_0 & 0 & \dots \\ c_0 & b_1 & c_1 & \dots \\ 0 & c_1 & b_2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

is symmetric in $\ell^2(\mathbb{Z}_+)$. Since the sequences $\{a_n\}$ and $\{b_n\}$ are both bounded, the operator $\mathcal{J}$ is bounded and self-adjoint.

$\ell^2(\mathbb{Z}_+)$ is equal to the cyclic subspace of $(1, 0, 0, \dots)$. The spectral measure of this element is μ.

Theorem (Blumenthal-Weyl)

Let parameters $\{a_n, b_n\}$ be such that $a_n \to 1/4$ and $b_n \to 0$. Then the corresponding Jacobi operator satisfies

$$\sigma_{\text{ess}}(\mathcal{J}) = [-1, 1].$$

L^2 Minimality

For any polynomial $Q_{n-1}(x)$ of degree $n-1$ it holds that

$$\begin{aligned}\int (P_n + Q_{n-1})^2 d\mu &= \int P_n^2 d\mu + 2\int P_n Q_{n-1} d\mu + \int Q_{n-1}^2 d\mu \\ &= \int P_n^2 d\mu + \int Q_{n-1}^2 d\mu \geq \int P_n^2 d\mu.\end{aligned}$$

If we expect that

$$P_n(z) \sim \frac{\Phi^n(z)}{2^n} \frac{G(\infty)}{G(z)}$$

for some non-vanishing analytic function off $[-1, 1]$, then

$$|G(\infty)|^2 \int |G_\pm(x)|^{-2} d\mu(x)$$

should be minimal in some suitable space of non-vanishing holomorphic functions.

Szegő Asymptotics

Theorem (Szegő (book))

If $\text{supp}(\mu) = [-1, 1]$, $d\mu(x) = v(x)d\omega(x) + d\mu^s(x)$, and $v \in L^1(\omega)$, then

$$P_n(z) \sim \frac{\Phi^n(z)}{2^n} \frac{G_\mu(\infty)}{G_\mu(z)},$$

where

$$G_\mu(z) = \exp\left\{ \frac{\sqrt{z^2-1}}{2} \int \log v(x) \frac{d\omega(x)}{z-x} \right\}$$

(this is an outer function off $[-1, 1]$ such that $|G_{\mu\pm}(x)|^2 = v(x)$).

In what follows, the measures with the above property will be called Szegő measures.

Multiple Orthogonal Polynomials

Let μ_i, $i \in I_d = \{1, 2, \ldots, d\}$, be compactly supported Borel measures on the real line such that

$$\Delta(\mu_i) < \Delta(\mu_j), \quad i < j,$$

where $\Delta(\mu_i)$ is the convex hull of the support of μ_i. This is an Angelesco system of d measures.

Type II MOP corresponding to a multi-index $\vec{n} = (n_1, n_2, \ldots, n_d)$ is defined as the unique monic polynomial of degree $|\vec{n}| := n_1 + n_2 + \cdots + n_d$ such that

$$\int x^k P_{\vec{n}}(x) d\mu_i(x) = 0, \quad k = \overline{0, n_i - 1}.$$

This polynomial has n_i zeros on $\Delta(\mu_i)$. We can write $P_{\vec{n}}(z) = P_{\vec{n},1}(z) \cdots P_{\vec{n},d}(z)$.

Weak Asymptotics

Tools developed to understand OPs yield that the asymptotic behavior of MOPs for Angelesco systems is governed by the following potential theoretic extremal problem:

If $\vec{n}/|\vec{n}| \to \vec{c} \in (0,1)^d$, $c_1 + c_2 + \cdots + c_d = |\vec{c}| = 1$, one needs to find measures $\omega_{\vec{c},1}, \omega_{\vec{c},2}, \ldots, \omega_{\vec{c},d}$ such that

$$\begin{cases} \operatorname{supp} \omega_{\vec{c},i} \subseteq \Delta(\mu_i), & |\omega_{\vec{c},i}| = c_i, \\ V^{\omega_{\vec{c},i} + \sum \omega_{\vec{c},j}} = \ell_i, & \operatorname{supp} \omega_{\vec{c},i}, \\ V^{\omega_{\vec{c},i} + \sum \omega_{\vec{c},j}} < \ell_i, & x \in \Delta(\mu_i) \setminus \operatorname{supp} \omega_{\vec{c},i}, \end{cases}$$

for some constants ℓ_i, $i \in I_d$.

Weak Asymptotics

Theorem (Gonchar-Rakhmanov 1981)

For any $\vec{c} \in (0,1)^d$, $|\vec{c}| = 1$, the vector-measure $(\omega_{\vec{c},1}, \omega_{\vec{c},2}, \ldots, \omega_{\vec{c},d})$ exists, is unique, and it holds that

$$\operatorname{supp} \omega_{\vec{c},i} = \Delta_{\vec{c},i} = [\alpha_{\vec{c},i}, \beta_{\vec{c},i}].$$

If each μ_i is absolutely continuous w.r.t. Lebesgue measure and $\mu_i'(x) > 0$ a.e. on $\Delta(\mu_i)$, then for each $i \in I_d$ it holds that

$$\frac{n_i}{|\vec{n}|} \log |P_{\vec{n},i}(z)| \sim -V^{\omega_{\vec{c},i}}(z)$$

and the zero counting measures of $P_{\vec{n},i}$ converge weak* to $\omega_{\vec{c},i}$ as $\vec{n}/|\vec{n}| \to \vec{c}$.

Nearest-Neighbor Recurrence Relations

Theorem (Van Assche 2011)

It holds for any $j \in I_d$ that

$$xP_{\vec{n}}(x) = P_{\vec{n}+\vec{e}_j}(x) + b_{\vec{n},j}P_{\vec{n}}(x) + \sum_{i \in I_d} a_{\vec{n},i}P_{\vec{n}-\vec{e}_i}(x)$$

where $\vec{e}_i$ is the i-th coordinate vector.

Ratio Asymptotics

Theorem (Van Assche 2015)

If $a_{\vec{n},i}, b_{\vec{n},i}$ have limits $A_{\vec{c},i}, B_{\vec{c},i}$ as $\vec{n}/|\vec{n}| \to \vec{c} \in (0,1)^d$, $|\vec{c}| = 1$, then

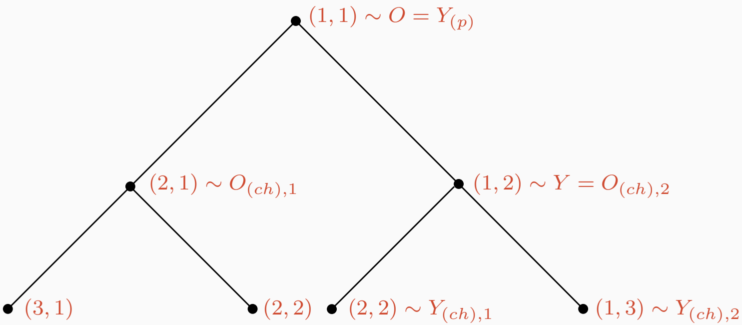
$$\lim \frac{P_{\vec{n}+\vec{e}_j}(x)}{P_{\vec{n}}(x)} = z(x) - B_{\vec{c},j}$$

for each $j \in I_d$, where $z(x) - x \to 0$ as $x \to \infty$ and

$$x = z + \sum_i \frac{A_{\vec{c},i}}{z - B_{\vec{c},i}}.$$

Homogeneous Rooted Tree

Let $\mathcal{T}$ be the rooted tree of all possible paths on $\mathbb{N}^d$ starting at $(1, 1, \dots, 1)$.



We denote the set of all vertices of $\mathcal{T}$ by $\mathcal{V}$. We let

$$\ell : \mathcal{V} \to I_d, \quad Y \mapsto \ell_Y \text{ such that } \Pi(Y) = \Pi(Y_{(p)}) + \vec{e}_{\ell_Y},$$

where Π is the natural projection of $\mathcal{V}$ onto $\mathbb{N}^d$.

Jacobi Operators

Define two interaction functions $A, B : \mathcal{V} \to \mathbb{R}$ by

$$\begin{cases} A_Y := a_{\Pi(Y_{(p)}),\ell_Y}, \quad B_Y := b_{\Pi(Y_{(p)}),\ell_Y}, & Y \neq O, \\ A_O := 1, \quad B_O := \sum_i \kappa_i b_{\vec{1}-\vec{e}_i,1}, & Y = O. \end{cases}$$

The corresponding Jacobi operator can be defined as

$$\begin{cases} (\mathcal{J}_{\vec{\kappa}} f)_Y & := A_Y^{1/2} f_{Y_{(p)}} + (Bf)_Y + \sum_i (A^{1/2} f)_{Y_{(ch),i}}, \\ (\mathcal{J}_{\vec{\kappa}} f)_O & := (Bf)_O + \sum_i (A^{1/2} f)_{O_{(ch),i}}. \end{cases}$$

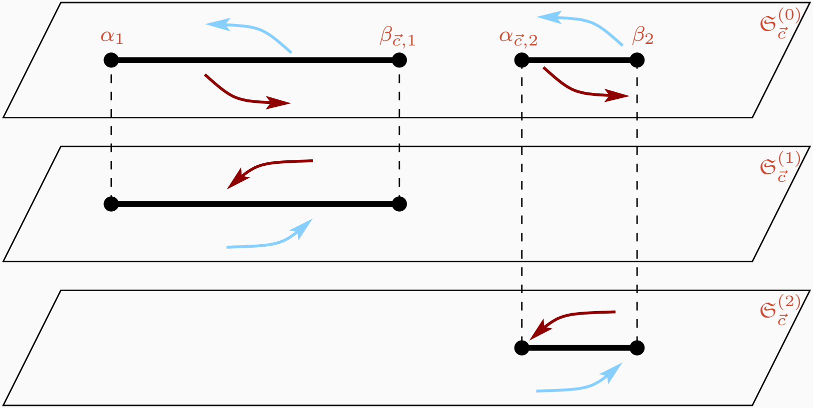
$\mathcal{J}_{\vec{\kappa}}$ is a bounded and self-adjoint operator on $\ell^2(\mathcal{V})$.

Riemann Surface

Fix $\vec{c} \in (0,1)^d$, $|\vec{c}| = 1$, and let $\{\Delta_{\vec{c},i}\}$ be the supports of the vector-equilibrium measure corresponding to $\vec{c}$. Set

$$\mathfrak{S}_{\vec{c}}^{(0)} = \overline{\mathbb{C}} \setminus \cup_i \Delta_{\vec{c},i} \quad \text{and} \quad \mathfrak{S}_{\vec{c}}^{(j)} = \overline{\mathbb{C}} \setminus \Delta_{\vec{c},j}, \quad j \in I_d.$$

Glue them to each other cross-wise in a standard manner.



Asymptotics of the Recurrence Coefficients

Theorem (Aptekarev-Denisov-Ya. 2020)

Let all the measures μ_i be absolutely continuous with respect the Lebesgue measure and $\mu_i'(x)$ be Fisher-Hartwig perturbations of analytic and non-vanishing densities. Then

$$a_{\vec{n},i} \to A_{\vec{c},i} \quad \text{and} \quad b_{\vec{n},i} \to B_{\vec{c},i}$$

as $\vec{n}/|\vec{n}| \to \vec{c} \in (0,1)^d$, $|\vec{c}| = 1$, where $\chi_{\vec{c}} : \mathfrak{S}_{\vec{c}} \to \overline{\mathbb{C}}$ is conformal and such that

$$\chi_{\vec{c}}^{(0)}(z) = z + O\left(\frac{1}{z}\right) \quad \text{and} \quad \chi_{\vec{c}}^{(i)}(z) =: B_{\vec{c},i} + \frac{A_{\vec{c},i}}{z} + O\left(\frac{1}{z^2}\right).$$

Asymptotics of the Recurrence Coefficients

Theorem (Aptekarev-Denisov-Ya. 2021)

Let $d = 2$ and $\mu_i'(x)$ be analytic and positive. Then

$$a_{\vec{n},i} \to A_{c,i} \quad \text{and} \quad b_{\vec{n},i} \to B_{c,i}$$

as $\vec{n}/|\vec{n}| \to (c, 1-c) \in [0,1]^2$, where

$$A_{0,2} = \left(\frac{\beta_2 - \alpha_2}{4}\right)^2, \quad B_{0,2} = \frac{\beta_2 + \alpha_2}{2}, \quad A_{0,1} = 0, \quad B_{0,1} = B_{0,2} + \varphi_2(\alpha_1),$$

and $\varphi_2(z) = z + O(1)$ is the conformal map of the complement of $\Delta(\mu_2) = [\alpha_2, \beta_2]$ to the complement of a disk.

Essential Spectrum

Theorem (Aptekarev-Denisov-Ya. 2021)

Let $A_{c,1}, A_{c,2}, B_{c,1}, B_{c,2}$ be as in the previous theorem (coming from some intervals $\Delta_1 < \Delta_2$). Further, let $\mathcal{J}_{\vec{\kappa}}$ be a Jacobi operator constructed as before for *some* constants $\{a_{\vec{n},1}, a_{\vec{n},2}, b_{\vec{n},1}, b_{\vec{n},2}\}$. If

$$a_{\vec{n},i} \to A_{c,i} \quad \text{and} \quad b_{\vec{n},i} \to B_{c,i}$$

as $\vec{n}/|\vec{n}| \to (c, 1-c) \in [0,1]^2$, then $\sigma_{\text{ess}}(\mathcal{J}_{\vec{\kappa}}) = \Delta_1 \cup \Delta_2$.

Spectrum

Theorem (Denisov-Ya. 2022)

Let $\mathcal{J}_{\vec{\kappa}}$ be a Jacobi operator of an Angelesco system on $\Delta(\mu_1), \Delta(\mu_2)$. Then $\ell^2(\mathcal{V})$ admits a decomposition into an orthogonal sum of cyclic subspaces of $\mathcal{J}_{\vec{\kappa}}$ and there is a semi-explicit representation for the spectral measure of generating element of each cyclic subspace. In particular,

$$\sigma(\mathcal{J}_{\vec{\kappa}}) \subseteq \Delta(\mu_1) \cup \Delta(\mu_2) \cup E_{\vec{\kappa}},$$

where $E_{\vec{\kappa}}$ is either one point or empty. If each $\operatorname{supp} \mu_i = \Delta(\mu_i)$, then $\subseteq$ is $=$.

Strong Asymptotics

Theorem (Aptekarev 1988)

Let $d = 2$, $\vec{n} = (n, n)$, μ_i be absolutely continuous and Szegő. Further, let (ω_1, ω_2) be the vector-equilibrium measure for $\Delta(\mu_1), \Delta(\mu_2)$ corresponding to $\vec{c} = (1/2, 1/2)$. Then it holds that

$$P_{n,i}(z) \sim \exp\left\{2n \int \log(z - x) d\omega_i(x)\right\} \frac{S_i(\infty)}{S_i(z)}$$

for $i = 1, 2$, where the Szegő functions $S_i(z)$ can be constructed as a unique solution of a simultaneous norm minimization problem in Hardy spaces among non-vanishing functions.

Strong Asymptotics

Theorem (Ya. 2016)

Let $\mu_i'(x)$, $i \in I_d$, be Fisher-Hartwig perturbations of analytic and non-vanishing densities. Then

$$P_{\vec{n},i}(z) \sim \exp\left\{|\vec{n}| \int \log(z-x)d\omega_{\vec{n},i}(x)\right\} \frac{S_{\vec{c},i}(\infty)}{S_{\vec{c},i}(z)}$$

as $\vec{n}/|\vec{n}| \to \vec{c} \in (0,1)^d$, $i \in I_d$, where the function $S_{\vec{c},i}(z)$ were constructed using explicit integral representation via discontinuous Cauchy kernel on $\mathfrak{S}_{\vec{c}}$.

The construction of Szegő functions using integrals of analytic densities against a discontinuous Cauchy kernel was proposed by Aptekarev and Lysov while studying graph generated MOPs systems along the main diagonal.

Strong Asymptotics

Theorem (Aptekarev-Denisov-Ya. 2021)

Let $d = 2$ and $\mu_i'(x)$ be analytic and positive. If $\min\{n_1, n_2\} \to \infty$, strong asymptotic formuale extend to marginal cases $\vec{c} = (1, 0)$ and $\vec{c} = (0, 1)$.

Theorem (Ya. to be submitted)

Let $d = 2$ and $\mu_i'(x)$ be analytic and positive. All formulae of strong asymptotics hold uniformly in $|\vec{n}|$ as long as

$$\varepsilon_{\vec{n}} := 1/\min\{n_1, n_2\} \to 0.$$

The error terms are (uniform in $\vec{n}$ and) of order $\varepsilon_{\vec{n}}^{1/3}$. In particular,

$$a_{\vec{n},i} = A_{\vec{n},i} + O\left(\varepsilon_{\vec{n}}^{1/3}\right) \quad \text{and} \quad b_{\vec{n},i} = B_{\vec{n},i} + O\left(\varepsilon_{\vec{n}}^{1/3}\right).$$

Szegő Functions

Proposition (Aptekarev-Denisov-Ya. submitted)

Let $\{\Delta_i\}_{i\in I_d}$ be a collection of pairwise disjoint closed intervals and $\{\mu_i\}_{i\in I_d}$ be positive Borel measures such that μ_i is Szegő on Δ_i for each $i \in I_d$. There exists a unique collection of functions $\{S_i(z)\}_{i\in I_d}$ such that each $S_i(z)$ is a conjugate-symmetric outer function in $H^2(\overline{\mathbb{C}} \setminus \Delta_i)$ with $S_i(\infty) > 0$ and it holds that

$$|S_{i\pm}(x)|^2 \prod_{j\in I_d, j\neq i} S_j(x) = v_i(x) \quad \text{for a.e.} \quad x \in \Delta_i, \quad i \in I_d,$$

where $d\mu_i = v_i d\omega_{\Delta_i} + d\mu^s$ and μ^s is singular to the Lebesgue measure.

Szegő functions are constructed as usual (one-dimensional) Szegő functions, but for densities that are obtained by inverting some harmonic extension operator.

Szegő Functions

Let $2\vec{v} = (v_1, v_2, \ldots, v_d)$. The densities $\vec{s} = (s_1, s_2, \ldots, s_d)$ are the solutions of

$$(\mathcal{I} - \mathcal{H})\vec{s} = \vec{v},$$

where

$$\mathcal{H} = -\frac{1}{2}\begin{bmatrix} 0 & H_{\Delta_2 \to \Delta_1} & \cdots & H_{\Delta_n \to \Delta_1} \\ H_{\Delta_1 \to \Delta_2} & 0 & \cdots & H_{\Delta_n \to \Delta_2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{\Delta_1 \to \Delta_n} & H_{\Delta_2 \to \Delta_n} & \cdots & 0 \end{bmatrix}$$

and

$$(H_{\Delta_i \to \Delta_j} u)(z) = \Re\left(\sqrt{(z-\alpha_i)(z-\beta_i)} \int_\Delta u(x) \frac{d\omega_{\Delta_i}(x)}{z-x}\right)_{|\Delta_j}.$$

Strongly Szegő Measures

We say that a measure μ is *uniformly Szegő* on a closed interval Δ if $\mu_{|\Delta}$ is a Szegő measure and for any sequence of closed intervals $\{\Delta_n\}$ such that $\Delta_n \subseteq \Delta(\mu)$ and $\Delta_n \to \Delta$ as $n \to \infty$, it holds that

$$\lim_{n\to\infty} \int |\log \mu'(x) - \log \mu'(l_{\Delta\to\Delta_n}(x))| d\omega_\Delta(x) \to 0 \,,$$

where $l_{\Delta\to\Delta_n}(x)$ is a linear function that maps Δ onto Δ_n.

Szegő Functions

Proposition (Aptekarev-Denisov-Ya. submitted)

Let $\{\Delta_i\}_{i\in I_d}$ be a collection of pairwise disjoint closed intervals and $\{\mu_i\}_{i\in I_d}$ be a collection of positive Borel measures such that μ_i is uniformly Szegő on Δ_i for each $i \in I_d$. Further, let $\{\Delta_{n,i}\}_{i\in I_d}$, $n \in \mathbb{N}$, be collections of pairwise disjoint closed intervals such that $\Delta_{n,i} \subseteq \Delta(\mu_i)$ and $\Delta_{n,i} \to \Delta_i$ as $n \to \infty$ for each $i \in I_d$. Then

$$S_{n,i}(z) \to S_i(z) \quad \text{as} \quad n \to \infty$$

locally uniformly in $\overline{\mathbb{C}} \setminus \Delta_i$ for each $i \in I_d$.

Strong Asymptotics

Theorem (Aptekarev-Denisov-Ya. submitted)

Let $\{\mu_i\}$ be an Angelesco system of measures, $\vec{c} \in (0,1)^d$, $|\vec{c}| = 1$, $\{\omega_{\vec{c},i}\}$ be the corresponding vector-equilibrium measure. Assume that each μ_i is uniformly Szegő on $\Delta_{\vec{c},i} = \operatorname{supp}(\omega_{\vec{c},i})$. Then it holds that

$$P_{\vec{n},i}(z) \sim \exp\left\{ |\vec{n}| \int \log(z-x)\,d\omega_{\vec{n},i}(x) \right\} \frac{S_{\vec{c},i}(\infty)}{S_{\vec{c},i}(z)}$$

for each $i \in I_d$ as $\vec{n}/|\vec{n}| \to \vec{c}$.

Varying Weights

The proof of the previous result uses in its core the Schauder-Tychonoff fixed-point theorem and the two following theorems (no pushing and pushing regimes).

Theorem (Aptekarev-Denisov-Ya. submitted)

Let $\{(\mu_n, h_n, \omega_n)\}$ be a sequence of triples, where μ_n, ω_n are measures on an interval Δ and h_n is a continuous function on Δ. Write $\theta_n = 2nV^{\omega_n} + h_n$. Then

$$T_n\left(e^{\theta_n}\mu_n\right)(z) \sim \exp\left\{n\int \log(z-x)d\omega_n(x)\right\}\frac{G(e^{h_n}\mu_n,\infty)}{G(e^{h_n}\mu_n,z)},$$

where $T_n\left(e^{\theta_n}\mu_n\right)$ is the n-th monic orthogonal polynomial w.r.t. $e^{\theta_n}\mu_n$.

This theorem generalizes a result by Totik: it replaces a single absolutely continuous measure μ with a sequence of not necessarily absolutely continuous measures $e^{h_n}\mu_n$.

Varying Weights

Theorem (Aptekarev-Denisov-Ya. submitted)

We assume that

- there exists μ on Δ such that for any non-negative function $f \in C(\Delta)$ it holds that $\limsup_{n\to\infty} \int f d\mu_n \leq \int f d\mu$;
- $\| \log v_n - \log v \|_{L^1(\omega_\Delta)} \to 0$ as $n \to \infty$, where $d\mu_n = v_n d\omega_\Delta + d\mu_n^s$;
- the functions h_n belong to $\mathcal{K}$, a fixed compact subset of $C(\Delta)$;
- $d\omega_n(x) = \omega_n'(x)dx$ are probability measures on $\Delta = [\alpha, \beta]$ such that ω_n' form a uniformly equicontinuous family on each compact subset of (α, β) and satisfy growth/decay conditions around α, β.

Varying Weights with Pushing

Theorem (Aptekarev-Denisov-Ya. submitted)

Let μ be a compactly supported positive Borel measure and $\Delta_n \subseteq \Delta(\mu)$ be intervals that converge to some interval $\Delta = [\alpha, \beta]$. Assume that μ is uniformly Szegő on Δ. Further, let $\{(\kappa_n, h_n, \omega_n)\}$ be a sequence of triples, where κ_n, h_n are continuous functions on $\Delta(\mu)$ and ω_n are measures on $\Delta(\mu)$ such that

- the functions κ_n are such that $\kappa_n \leq 0$ on $\Delta(\mu)$, $\kappa_n \equiv 0$ on Δ_n, and on $\Delta(\mu) \setminus \Delta_n$ κ_n satisfy uniform growth/decay condition;
- the functions h_n belong to $\mathcal{K}$, a fixed compact subset of $C(\Delta(\mu))$;
- $d\omega_n(x) = \omega_n'(x)dx$ are probability measures such that $\operatorname{supp} \omega_n = \Delta_n$, ω_n' form a uniformly equicontinuous family on each compact subset of (α, β) and satisfy growth/decay conditions around α, β.

Set $\theta_n := 2n(V^{\omega_n} + \kappa_n) + h_n$. Then

$$T_n\left(e^{\theta_n}\mu\right)(z) \sim \exp\left\{n \int \log(z - x)d\omega_n(x)\right\} \frac{G(e^{h_n}\mu_{|\Delta}, \infty)}{G(e^{h_n}\mu_{|\Delta}, z)}.$$

Theorem (Aptekarev-Denisov-Ya. submitted)

Let $\{(\sigma_n, g_n, W_n)\}$ be triples of a finite positive Borel measure on $\mathbb{T}$, a continuous real-valued function on $\mathbb{T}$, and a monic polynomial of degree n with all its zeros inside the unit disk. Assume that

- there exists σ such that $\limsup_{n\to\infty} \int f d\sigma_n \le \int f d\sigma$, $0 \le f \in C(\mathbb{T})$;
- $\|\log v_n - \log v\|_{L^1(\mathbb{T})} \to 0$ as $n \to \infty$, $d\sigma_n = v_n dm + d\sigma_n^s$;
- the functions g_n belong to a fixed compact subset of $C(\mathbb{T})$;
- the zeros $\{b_{n,j}\}$ of W_n satisfy $\sum_{j=1}^n (1 - |b_{n,j}|) \to \infty$ as $n \to \infty$.

Let $q^*(z) = z^n \overline{q(1/\bar{z})}$ and

$$\int \phi_n(\xi)\bar{\xi}^k \frac{e^{g_n(\xi)} d\sigma_n(\xi)}{|W_n(\xi)|^2} = 0, \quad k = \overline{0, n-1}.$$

Then

$$\frac{\phi_n^*(z)}{W_n^*(z)} D_n(z) = 1 + o_{\mathcal{E}}(1), \quad \frac{\phi_n(z)}{\phi_n^*(z)} = o_{\mathcal{E}}(1)$$

locally uniformly in $\mathbb{D}$, where D_n is the Szegő function of $e^{g_n}\sigma_n$.

This theorem generalizes Stahl 2000 and de la Calle Ysern – López Lagomasino 1998.