On Symmetric Contours in Rational Interpolation

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Orthogonal Polynomials and Applications
Conference in Honor of Walter Van Assche
June 9th, 2023

Padé Approximants

Let $f(z)$ be an analytic function at infinity. Then

$$f(z) = f_0 + \frac{f_1}{z} + \frac{f_2}{z^2} + \cdots + \frac{f_{2n}}{z^{2n}} + O\left(\frac{1}{z^{2n+1}}\right).$$

We are looking for a rational function $r_n(z)$ of type (n, n) such that

$$r_n(z) = f_0 + \frac{f_1}{z} + \frac{f_2}{z^2} + \cdots + \frac{f_{2n}}{z^{2n}} + O\left(\frac{1}{z^{2n+1}}\right).$$

Such a rational function might not exist. However, there always exist polynomials $p_n(z)$ and $q_n(z)$ of degree at most n such that

$$q_n(z) f(z) - p_n(z) = O\left(\frac{1}{z^{n+1}}\right).$$

The rational function $p_n(z)/q_n(z)$ is unique in its reduced form and is called diagonal Padé approximant of $f(z)$. We take $q_n(z)$ to be the smallest degree solution.

Orthogonality

Let Γ be a curve in the exterior domain of which $f(z)$ is analytic. Then

$$0 = \int_\Gamma z^k (q_n(z) f(z) - p_n(z)) dz = \int_\Gamma z^k q_n(z) f(z) dz$$

for each $k = \overline{0, n-1}$. If there exists a system of Jordan arcs that does not disconnect the plane such that

$$f(z) = \int_L \frac{(f_+ - f_-)(s)}{s - z} \frac{ds}{2\pi \mathrm{i}},$$

then the orthogonality relations can be rewritten as

$$\int_L s^k q_n(s) (f_+ - f_-)(s) ds, \quad k = \overline{0, n-1}.$$

Markov Functions

If $f(z)$ is a Markov function, that is,

$$f_\mu(z) = \int_{-1}^{1} \frac{d\mu(x)}{x - z}$$

for some measure μ on $[-1, 1]$, then

$$\int_{-1}^{1} x^k q_n(x) d\mu(x), \quad k = \overline{0, n-1}.$$

If $d\mu(x) = \frac{1}{\pi}(1 - x^2)^{-1/2} dx$, the $q_n(x)$ is simply the Chebyshëv polynomial of the first kind, i.e.,

$$q_n(z) = z^n + z^{-n}, \quad z = J(z) = \frac{z + z^{-1}}{2}.$$

More generally, for nice enough measures μ it holds that

$$q_n(z) \sim z^n \quad \text{and} \quad f_\mu(z) - \frac{p_n(z)}{q_n(z)} \sim \frac{1}{z^{2n+1}}, \quad |z| \geq 1$$

(that is, $z = z + \sqrt{z^2 - 1} \sim 2z$ as $z \to \infty$).

Riemann Surfaces and Symmetry

- The Joukovsky map $J(z)$ provides 2 to 1 ramified cover of the Riemann sphere by itself. It maps the unit circle onto the interval $[-1,1]$, where the poles of approximants are contained.

- The unit circle is the 0-level line of $\log|z^n|$, which is a harmonic function except for polar singularities, at 0 and ∞ of opposite signs.

- The function $g(z) = \log|z|$, $z = J(z)$, $|z| > 1$, is harmonic in $\mathbb{C} \setminus [-1,1]$, is zero on $[-1,1]$, and behaves like $\log|z|$ as $z \to \infty$. That is, $g(z)$ is the Green's function for $\overline{\mathbb{C}} \setminus [-1,1]$ with pole at infinity.

- The symmetry $z \mapsto 1/z$ of the Joukovsky map implies that $-g(z)$ is the harmonic continuation of $g(z)$ across $[-1,1]$. This is equivalent to saying that

$$\frac{\partial g}{\partial n_+} = \frac{\partial g}{\partial n_-}$$

on $(-1,1)$, where $n_\pm$ are one-sided normal derivatives, which is a definition of a symmetric contour (S-curve).

Hyperelliptic Functions

For hyperelliptic functions, like $f(z) = \left[(z-a_1)(z-a_2)\cdots(z-a_{2g+2})\right]^{-1/2}$, one needs to consider their Riemann surface

$$\mathfrak{S} = \left\{\boldsymbol{z} = (z, w) : w^2 = (z-a_1)(z-a_2)\cdots(z-a_{2g+2})\right\}.$$

Let $\pi(\boldsymbol{z}) = z$ and $\boldsymbol{z}^* = (z, -w)$ for $\boldsymbol{z} = (z, w)$ and $\boldsymbol{\infty}$ be such that $w(\boldsymbol{z}) \sim z^{g+1}$ as $\boldsymbol{z} \to \boldsymbol{\infty}$.

Let $g(\boldsymbol{z})$ be harmonic on $\mathfrak{S}$ and such that $g(\boldsymbol{z}^*) = -g(\boldsymbol{z})$ and $g(\boldsymbol{z}) = \log|z|$ around $\boldsymbol{\infty}$.

Theorem (Nuttall-Singh 1977)

The poles of the Padé approximants to $f(z)$ accumulate on a symmetric contour

$$\Delta = \pi(\Gamma), \quad \Gamma = \{\boldsymbol{z} : g(\boldsymbol{z}) = 0\}.$$

Moreover, it holds in the strong sense (discussed later) that

$$|q_n(z)| = e^{ng(\boldsymbol{z})} \quad \text{and} \quad \left|f(z) - \frac{p_n(z)}{q_n(z)}\right| = e^{-(2n+1)g(\boldsymbol{z})},$$

where $z \in \overline{\mathbb{C}} \setminus \Delta$, $g(\boldsymbol{z}) > 0$, and $f(z)$ is the branch holomorphic outside Δ.

Logarithmic Capacity

For a compact set K it holds that either the unbounded component of its complement is too large (the set is too small) and cannot support a Green's function or it can.

In the former case its is said that K is polar and we set $\mathrm{cap}(K) = 0$ and in the latter we set

$$\mathrm{cap}(K) := \exp\left\{\lim_{z\to\infty} \log|z| - g(z)\right\},$$

where $g(z)$ is the Green's function for unbounded component of its complement of K with pole at infinity.

One can readily compute that in the case $K = \{z : |z| \le R\}$, it holds that

$$g(z) = \log|z| - \log R, \quad |z| > R \quad \Rightarrow \quad \mathrm{cap}(K) = R.$$

Stahl's Class

Let $\mathcal{S}$ be the class of functions holomorphic at infinity, which can be continued along any curve in $\mathbb{C} \setminus A$ starting at infinity, where $\mathrm{cap}(A) = 0$ and some paths do lead to distinct continuation.

Denote by $\mathcal{K}_f$ the collection of compact sets K that do not disconnect the plane and such that f has a single-valued analytic continuation in to $\mathbb{C} \setminus K$.

Theorem (Stahl 1985 (3) + 1997)

For any function $f \in \mathcal{S}$ there exists $\Delta \in K_f$ such that

$$\mathrm{cap}(\Delta) = \min \left\{ \mathrm{cap}(K) : K \in \mathcal{K}_f \right\}.$$

Δ is a symmetric contour in the sense that

$$\frac{\partial g}{\partial n_+} = \frac{\partial g}{\partial n_-}, \quad s \in \Delta,$$

where $g(z)$ is the Green's function for the complement of Δ. Moreover,

$$\frac{1}{n} \log |q_n(z)| \sim g(z) \quad \text{and} \quad \frac{1}{2n} \log \left| f(z) - \frac{p_n(z)}{q_n(z)} \right| \sim -g(z).$$

Multipoint Padé Approximants

Let $f(z)$ be holomorphic at infinity and D be a subdomain of the extended complex plane $\overline{\mathbb{C}}$ into which $f(z)$ admits a single-valued holomorphic continuation.

Let E_n be a multi-set of $2n$ not necessarily distinct nor finite points $e_{n,i}$ in D. Let $p_n(z), q_n(z)$ be polynomials of degree at most n such that

$$\frac{(q_n f - p_n)(z)}{\prod_{|e_{n,i}|<\infty}(z - e_{n,i})} = O\left(z^{-n-1}\right)$$

and is analytic in D. In the reduced form $p_n(z)/q_n(z)$ is unique. We take $q_n(z)$ to be the solution of minimal degree.

$p_n(z)/q_n(z)$ is called the diagonal multipoint Padé approximant of f of order n associated with E_n.

Green's Potentials

Let D be a domain with non-polar boundary. Green's function with pole at $w \in D$, $|w| < \infty$, say $g(z, w)$, is the unique harmonic function in $D \setminus \{w\}$ that is zero quasi everywhere (up to a polar set) on ∂D and such that $g(z, w) + \log|z - w|$ is bounded around w.

We also write $g(z, \infty)$ for the Green's function with pole at infinity.

Let ν be a Borel measure in D. Then the Green's potential of ν is defined as

$$g_\nu(z) := \int g(z, w) d\nu(w).$$

Theorem (Gonchar-Rakhmanov 1987)

Let $f \in \mathcal{S}$. Assume that the interpolation sets $E_n = \{e_{n,i}\}$ are such that

$$\frac{1}{2n}\sum \delta(e_{n,i}) \xrightarrow{*} \nu$$

for some measure ν. Assume that there exists $\Delta \in \mathcal{K}_f$ such that $\mathrm{supp}(\nu) \subset \overline{\mathbb{C}}\setminus\Delta$ and

$$\frac{\partial g_\nu}{\partial n_+} = \frac{\partial g_\nu}{\partial n_-}, \quad s \in \Delta.$$

Then it holds in the weak sense for the corresponding multipoint Padé approximants that

$$\frac{1}{2n}\log\left|f(z) - \frac{p_n(z)}{q_n(z)}\right| \sim -g_\nu(z).$$

Symmetric Contours that Separate the Plane

Theorem (Buslaev 2013 (two-point) + 2015 (several-point))

Let $f_0(z)$ and $f_\infty(z)$ be analytic around 0 and ∞, be continuable along any arc that does not pass through a finite sets of points, and continuations are multi-valued. Then there exists a compact set Δ such that

- $\overline{\mathbb{C}} \setminus \Delta = D_0 \cup D_\infty$, where $0 \in D_0$ and $\infty \in D_\infty$ are either disjoint or coincide and $f_e(z)$ has an analytic continuation into D_e, $e \in \{0, \infty\}$;
- Δ consists of open analytic arcs and their endpoints and

$$\frac{\partial(g(s,0)+g(s,\infty))}{\partial n_+} = \frac{\partial(g(s,0)+g(s,\infty))}{\partial n_-}, \quad s \in \Delta,$$

  where $g(z,e)$ is the Green's function with pole at $e \in D_e$, $e \in \{0, \infty\}$;
- it holds that

$$\frac{1}{2n} \log\left| f(z) - \frac{p_n(z)}{q_n(z)} \right| \sim -g(z,0) - g(z,\infty),$$

  where $p_n(z)/q_n(z)$ is a multipoint Padé approximant with interpolation conditions asymptotically equally split between 0 and ∞.

Symmetric Contours that Separate the Plane

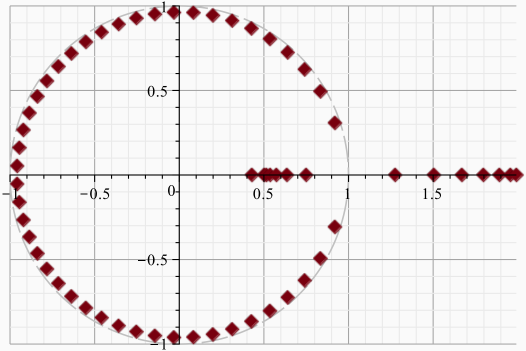
Theorem (Ya. 2021)

Given $|a| < 1$, let $w(z) = \sqrt{(z-a)(z-1/a)}$. If

$$f_0(z) = \frac{c_0}{w(z)} \quad \text{and} \quad f_\infty(z) = \frac{c_\infty}{w(z)},$$

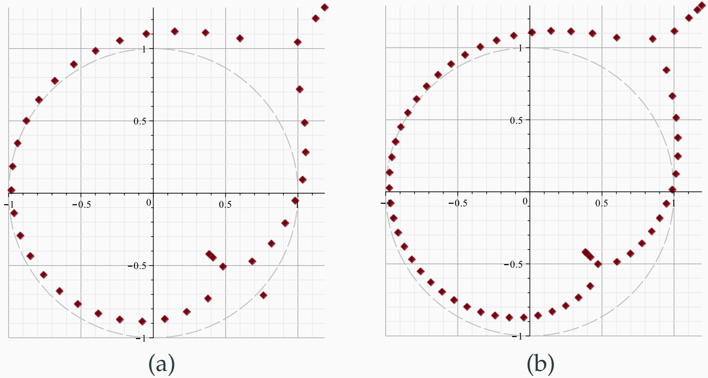
or similarly nice pairs of functions, strong asymptotics of the multipoint Padé approximants can be derived.

Symmetric Contours that Separate the Plane



Zeros of the denominator polynomial $q_{60}(z)$ when the approximated pair is given by $f_0(z) = \log\left(\frac{z-1}{z-1/a}\right)$ and $f_\infty(z) = \log\left(\frac{z-a}{z-1}\right)$ for $a = 2$.

Symmetric Contours that Separate the Plane



Zeros of the denominator polynomial (a) $q_{40}(z)$ and (b) $q_{60}(z)$ when the approximated pair is given by $f_0(z) = \log\left(\frac{z-1}{z-1/a}\right)$ and $f_\infty(z) = \log\left(\frac{z-a}{z-1}\right)$ for $a = 1.2 + 1.3\mathrm{i}$.

Proposition

Let L be a smooth Jordan arc joining -1 and 1 and $w_L(z) := \sqrt{z^2-1}$ be the branch holomorphic in $\mathbb{C} \setminus L$. Set $T = J^{-1}(L)$, where $J(z)$ is the Joukovsky transformation. Let U be the interior domain of T and

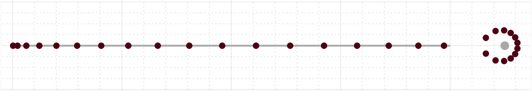
$$\Psi_n(z) = z^n \prod_{i=1}^{2n} \left(1 - \frac{e_{n,i}}{z}\right),$$

for some interpolation set $E_n = \{e_{n,i}\}$, $e_{n,i} = J(\boldsymbol{e}_{n,i})$, $\boldsymbol{e}_{n,i} \in U$. Then

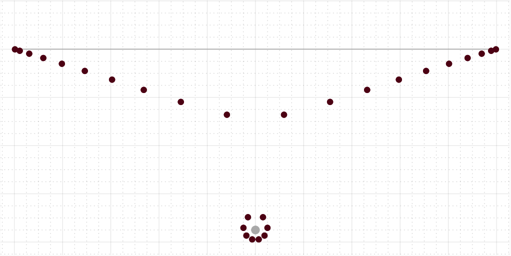
$$q_n(z) = \Psi_n(\boldsymbol{z}) + \Psi_n(1/\boldsymbol{z}) \quad \text{and} \quad \frac{1}{w_L(z)} - \frac{p_n(z)}{q_n(z)} = \frac{2}{w_L(z)} \frac{\Psi_n(\boldsymbol{z})}{\Psi_n(\boldsymbol{z}) + \Psi_n(1/\boldsymbol{z})},$$

where $\boldsymbol{z} \in U$, $z = J(\boldsymbol{z})$, and $p_n(z)/q_n(z)$ is the multipoint Padé approximant of $1/w_L(z)$ associated with E_n.

Bernstein-Szegő Case



(c)



(d)

Approximated function is $1/w_L(z)$ where panel (c): L is an arc connecting -1 to some $x_* > 5/4$ through the upper half-plane and then x_* to 1 through the lower half-plane and there are 48 interpolation conditions at infinity and 10 conditions at $5/4$; panel (d): L is a lower unit semi-circle and there are 48 interpolation conditions at infinity and 8 conditions at $-3i/4$.

Bernstein-Szegő Case

$$\frac{1}{w_L(z)} - \frac{p_n(z)}{q_n(z)} = \frac{2}{w_L(z)} \frac{\Psi_n(\boldsymbol{z})}{\Psi_n(\boldsymbol{z}) + \Psi_n(1/\boldsymbol{z})} = \frac{2}{w_L(z)} \frac{B_n(\boldsymbol{z})}{1 + B_n(\boldsymbol{z})},$$

for $\boldsymbol{z} \in U$ and $z = J(\boldsymbol{z})$, where

$$B_n(\boldsymbol{z}) := \prod_{i=1}^{2n} \frac{\boldsymbol{z} - \boldsymbol{e}_{n,i}}{1 - \boldsymbol{e}_{n,i}\boldsymbol{z}}.$$

If in some subdomain of U the function $B_n(\boldsymbol{z})$ is very small, then

$$\frac{p_n(z)}{q_n(z)} \sim \frac{1}{w_L(z)},$$

and if $B_n(\boldsymbol{z})$ is very large, then

$$\frac{p_n(z)}{q_n(z)} \sim -\frac{1}{w_L(z)}.$$

Notice that $-w_L(z)$ is the analytic continuation of $w_L(z)$ across L.

Symmetric Contours

Let L be a Jordan curve oriented from -1 to 1 and $\{E_n\}$ be an interpolation scheme, $E_n = \{e_{n,i}\}$ from the complement of L. Set $T = J^{-1}(L)$ and U be the interior of T. Define $e_{n,i} = J(\boldsymbol{e}_{n,i})$, $\boldsymbol{e}_{n,i} \in U$, and

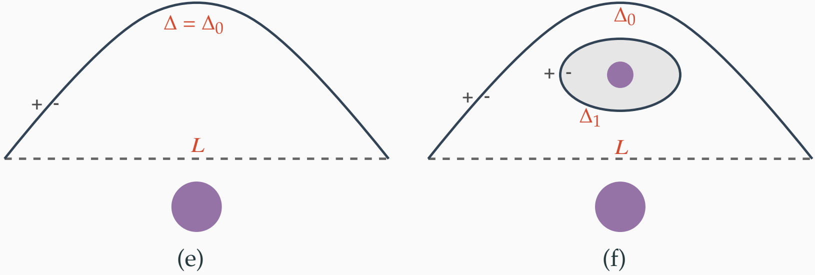
$$B_n(z) = \prod_{i=1}^{2n} \frac{z - \boldsymbol{e}_{n,i}}{1 - \boldsymbol{e}_{n,i} z}.$$

Assume that there exists a contour Γ such that

- $M^{-1} \leq |B_n(\boldsymbol{s})| \leq M$ for $\boldsymbol{s} \in \Gamma$;
- in each connected component of $\mathbb{C} \setminus \Gamma$ either $B_n(z) \to 0$ or $B_n(z) \to \infty$.

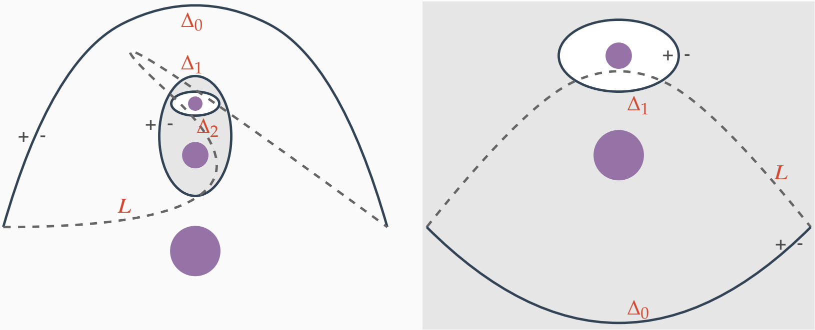
We shall call $\Delta = J(\Gamma)$ a symmetric contour associated with L and $\{E_n\}$.

Symmetric Contours



Darker filled circles represent interpolation points (bigger circle represents more interpolation conditions at the point), dashed lines represent L, solid lines represent Δ, and lightly shaded regions represent D_Δ^∞.
Panel (e): interpolation points create an external field that pushes L up to Δ.
Panel (f): interpolation points below L push it up, interpolation points above L push it down, but create weaker external field resulting in L going through them while simultaneously forming a barrier Δ_1.

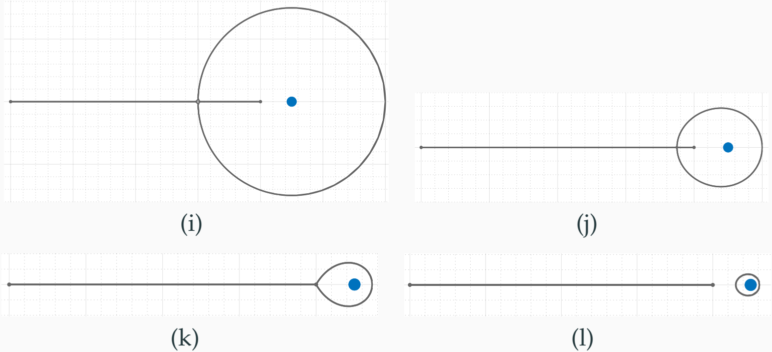
Symmetric Contours



(g) (h)

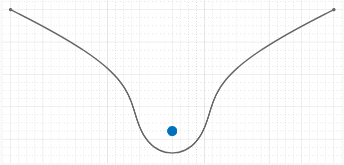
Panel (g): top and bottom groups of interpolation points create an external field that pushes L up while the middle group pushes L down, due to different strength of the components of the external field generated by these groups, two barriers are created. Panel (h): interpolation points below L create an external field that pushes L up all the way through ∞ to the displayed position of Δ_0, interpolation points above L create a weaker external field that results in a barrier Δ_1.

Symmetric Contours

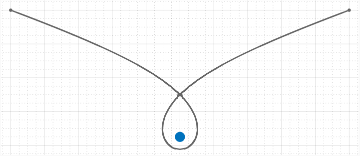


Symmetric contours Δ that correspond to L that connects -1 to some $x_* > 5/4$ through the upper half-plane and then x_* to 1 through the lower half-plane and interpolation schemes where the interpolation conditions are equally distributed between between ∞ and $5/4$ (i) or there are twice (j), three times (k), or four times (l) more interpolation conditions at ∞ than at $5/4$.

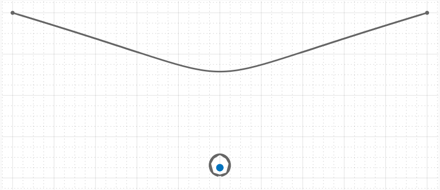
Symmetric Contours



(m)



(n)



(o)

Symmetric contours Δ that correspond to L being a lower unit semi-circle and interpolation schemes where there are four (m), five (n), or six (o) times more interpolation conditions at ∞ than at $-3i/4$.

Strong Asymptotics

Theorem (Baratchart-Ya. 2009 + 2010 + Ya. 2021 + in progress)

Let L, $\{E_n\}$, and Δ be as above. Let

$$f_L(z) := \frac{1}{2\pi \mathrm{i}} \int_L \frac{\rho(s)}{s - z} \frac{ds}{w_{L+}(s)},$$

where $\rho(s)$ is analytic and non-vanishing in a "large enough" domain. Then

$$\frac{p_n(z)}{q_n(z)} \to f_\Delta(z),$$

where $p_n(z)/q_n(z)$ is the multipoint Padé approximants associated with E_n and $f_\Delta(z)$ is the analytic continuation of $f_L(z)$ into $\overline{\mathbb{C}} \setminus \Delta$ that coincides with $f_L(z)$ at the interpolation points.

Bernstein-Szegő Case

Let now

$$w_L(z) = \sqrt{(z - a_1)(z - a_2)\cdots(z - a_{2g+1})}$$

be the branch holomorphic outside some contour L, $w_L(z) \sim z^{g+1}$.

Let $\{E_n\}$ be an interpolation scheme, $E_n = \{e_{n,i}\}_{i=1}^{2n-g}$ from the complement of L ($g+1$ interpolation conditions are automatically placed at infinity).

Define

$$\mathfrak{S} = \left\{\boldsymbol{z} = (z, w) : w^2 = (z - a_1)(z - a_2)\cdots(z - a_{2g+2})\right\}.$$

Let $\pi(\boldsymbol{z}) = z$ and $\boldsymbol{z}^* = (z, -w)$ for $\boldsymbol{z} = (z, w)$ and $\boldsymbol{\infty}$ be such that $w(\boldsymbol{z}) \sim z^{g+1}$ as $\boldsymbol{z} \to \boldsymbol{\infty}$.

Denote by U the connected component of $\mathfrak{S}$ in which $w(\boldsymbol{z}) = -w_L(z)$ and define $e_{n,i} = \pi(\boldsymbol{e}_{n,i}), \boldsymbol{e}_{n,i} \in U$.

Proposition

Let $\sigma : \{a_1, \dots, a_{2g+1}\} \to \{0,1\}$ and

$$f(z) = \frac{u_\sigma(z)}{w_L(z)} - l_\sigma(z), \quad u_\sigma(z) = \prod_{i=1}^{2g+2} (z - a_i)^{\sigma(a_i)},$$

where $l_\sigma(z)$ is a polynomial such that $f(\infty) = 0$. Let $\Psi_n(\boldsymbol{z})$ be the rational function on $\mathfrak{S}$ with the zero/pole divisor

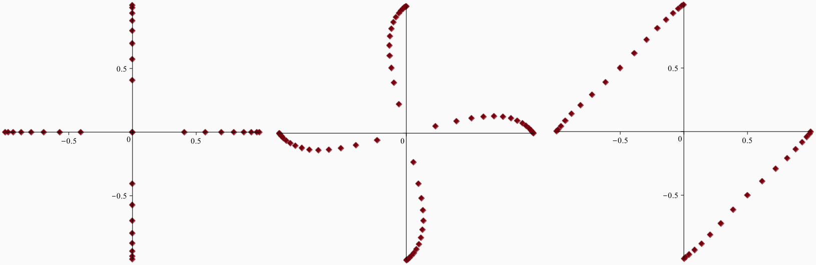
$$\sum_{i=1}^{g} \boldsymbol{z}_{n,i} + \sum_{i=1}^{2n-g} \boldsymbol{e}_{n,i} - \sum_{i=1}^{2g+2} \sigma(a_i)\boldsymbol{a}_i - n\infty - (n - |\sigma|)\infty^*,$$

where $\boldsymbol{a}_i$ are ramification points, $|\sigma| = \sum \sigma(a_i)$, and $\boldsymbol{z}_{n,i}$ are determined from the Jacobi inversion problem. Then

$$q_n(z) = \Psi_n(\boldsymbol{z}) + \Psi_n(\boldsymbol{z}^*) \quad \text{and} \quad f(z) - \frac{p_n(z)}{q_n(z)} = 2\frac{u_\sigma(z)}{w_L(z)} \frac{\Psi_n(\boldsymbol{z})}{\Psi_n(\boldsymbol{z}) + \Psi_n(\boldsymbol{z}^*)}$$

for $\boldsymbol{z} \in U$.

Bernstein-Szegő Case



Zeros of q_{36}, q_{60}, and q_{34} to $(z^4 - 1)^{-1/2}$ corresponding to the interpolation schemes $\{\pm 1 \pm \mathrm{i}\}$, $\{1/4 + \mathrm{i}, -1/4 - \mathrm{i}, 1 - \mathrm{i}/4, -1 + \mathrm{i}/4\}$, and $\{1 + \mathrm{i}, -1 - \mathrm{i}\}$.

Symmetric Contours

Let L and $E_n = \{e_{n,i}\}$ be as before. For each non-ramification point $\boldsymbol{e} \in \mathfrak{S}$ there exists a function $g(z,\boldsymbol{e})$ that is harmonic in $\mathfrak{S} \setminus \{\boldsymbol{e}, \boldsymbol{e}^*\}$, satisfies $g(z,\boldsymbol{e}) = -g(z^*,\boldsymbol{e})$, and blows up like a logarithm at $\boldsymbol{e}$. Set

$$g_n(z) = \sum_{i=1}^{2n-g} g(z, \boldsymbol{e}_{n,i}).$$

Assume that there exists a collection of cycles Γ such that

- $M^{-1} \leq |g_n(\boldsymbol{s})| \leq M$ for $\boldsymbol{s} \in \Gamma$;
- in each connected component of $\mathfrak{S} \setminus \Gamma$ either $g_n(z) \to \infty$ or $g_n(z) \to -\infty$.

We shall call $\Delta = J(\Gamma)$ a symmetric contour associated with L and $\{E_n\}$.

Strong-type Asymptotics

Theorem (Ya. 2015 + 2018 + in progress)

Let L, $\{E_n\}$, and Δ be as above. Let

$$f_L(z) := \frac{1}{2\pi\mathrm{i}} \int_L \frac{\rho(s)}{s-z} \frac{ds}{w_{L+}(s)},$$

where $\rho(s)$ is analytic and non-vanishing in a "large enough" domain. Then

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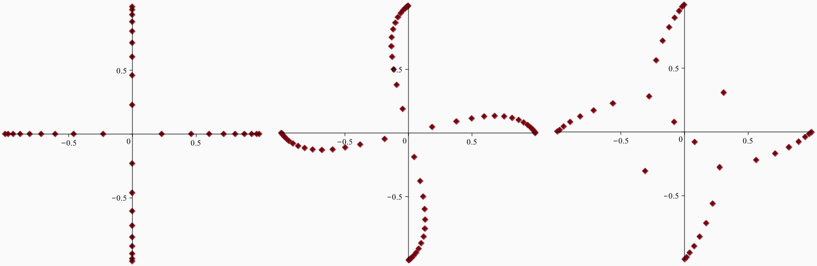
Strong-type Asymptotics



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