Convergent interpolation to Cauchy integrals over analytic arcs with Jacobi-type weights

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Let Δ be a smooth arc with endpoints ± 1 and $D := \overline{\mathbb{C}} \setminus \Delta$. Set

$$w(z) := \sqrt{z^2 - 1}, \quad w(z)/z \to 1 \quad \text{as} \quad z \to \infty,$$

where holomorphic in D branch is selected. Define

$$\varphi(z) := z + w(z), \quad z \in D.$$

Then

$$w^+ = -w^- \quad \text{and} \quad \varphi^+ \varphi^- = 1 \quad \text{on} \quad \Delta,$$

where Δ is assumed to be oriented from -1 to 1 and $w^\pm$ and $\varphi^\pm$ are the (unrestricted) boundary values on w and φ.

Let μ be given by

$$d\mu(t) = (hw_{\alpha,\beta})(t)\frac{idt}{\pi},$$

where h is a non-vanishing function on Δ with ``some smoothness'' and

$$w_{\alpha,\beta}(z) := (1-z)^{\alpha}(1+z)^{\beta}, \quad \alpha, \beta > -1,$$

is analytic across Δ°.

Define

$$f_{\mu}(z) := \int \frac{d\mu(t)}{z-t} = \int \frac{(hw_{\alpha,\beta})(t)}{z-t}\frac{idt}{\pi},$$

Let $\mathscr{E} := \{E_n\}$ be an interpolation scheme on $E \subset D$, that is,

$E_n \subset E$ consists of $2n$ not necessarily distinct nor finite points.

Denote by v_n the monic polynomial that vanishes at finite points of E_n according to their multiplicity.

The n-th diagonal multipoint Padé approximant to f_μ associated with $\mathscr{E}$ is the unique rational function $\Pi_n = p_n/q_n$ satisfying:

- $\deg p_n \leq n$, $\deg q_n \leq n$, and $q_n \not\equiv 0$;
- $\left(q_n(z)f_\mu(z) - p_n(z)\right)/v_n(z)$ is analytic on E;
- $\left(q_n(z)f_\mu(z) - p_n(z)\right)/v_n(z) = O\left(1/z^{n+1}\right)$ as $z \to \infty$.

Let $\mathscr{E} = \{E_n\}$ be an interpolations scheme in D. Associate to each E_n a function

$$r_n(z) := \prod_{e \in E_n} \frac{\varphi(z) - \varphi(e)}{1 - \varphi(z)\varphi(e)}, \quad z \in D.$$

Then

- r_n is holomorphic in D;
- r_n vanishes at each $e \in E_n$;
- $r_n^+ r_n^- = 1$ on Δ.

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Definition (BY)

We say that Δ is symmetric w.r.t. an interpolation scheme $\mathscr{E}$ if $r_n = o(1)$ locally uniformly in D and $|r_n^\pm| = O(1)$ uniformly on Δ.

Theorem (BY)

Let Δ be a rectifiable Jordan arc with an additional condition near ± 1 (below). Then the following are equivalent:

- $\exists$ an interpolation scheme $\mathscr{E}$, $\cap_n \overline{\cup_{k\geq n} E_k} =: \mathrm{supp}(\mathscr{E}) \subset D$, such that Δ is symmetric with respect to $\mathscr{E}$;
- $\exists$ a positive Borel measure ν, $\mathrm{supp}(\nu) \subset D$, such that Δ is symmetric with respect to ν (in the sense of Stahl);
- Δ is an analytic Jordan arc.

It is assumed that such that for $x = \pm 1$ and all $t \in \Delta$ sufficiently close to x it holds that $|\Delta_{t,x}| \leq \mathrm{const.}|x - t|^{\beta}$, $\beta > 1/2$.

It follows easily from the very definition of $\Pi_n = p_n/q_n$ that q_n are Non-Hermitian orthogonal polynomials:

$$\int_\Delta t^j q_n(t) w_n(t) dt = 0, \quad j = 0, \ldots, n-1,$$

where

$$w_n := w_{\alpha,\beta} h / v_n, \quad \deg(v_n) \leq 2n.$$

Functions of the second kind:

$$R_n(z) := \int_\Delta \frac{q_n(t) w_n(t)}{t - z} \frac{dt}{\pi i}, \quad z \in \overline{\mathbb{C}} \setminus \Delta.$$

Then

$$(R_n w)^+ + (R_n w)^- = 2 q_n w_n w^+ \quad \text{on} \quad \Delta.$$

Set

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathscr{Y} := \begin{pmatrix} q_n & R_n \\ m_n q^*_{n-1} & m_n R^*_{n-1} \end{pmatrix},$$

where q^*_{n-1} are polynomials satisfying

$$\int_\Delta t^j q^*_{n-1}(t) w_n(t) dt = 0, \quad j \in \{0, \ldots, n-2\},$$

and R^*_{n-1} are their functions of the second kind.

For simplicity, we put $\alpha = \beta = 0$.

$\mathscr{Y}$ is the unique solution of the following RH-problem:

(a) $\mathscr{Y}$ is analytic in $\mathbb{C} \setminus \Delta$ and

$$\lim_{z\to\infty} \mathscr{Y}(z) z^{-n\sigma_3} = \mathscr{I},$$

where $\mathscr{I}$ is the identity matrix;

(b) $\mathscr{Y}$ has continuous traces, $\mathscr{Y}_\pm$, on Δ° and

$$\mathscr{Y}_+ = \mathscr{Y}_- \begin{pmatrix} 1 & 2w_n \\ 0 & 1 \end{pmatrix};$$

(c) $\mathscr{Y}$ has the following behavior near $z = \pm 1$:

$$\mathscr{Y} = O\begin{pmatrix} 1 & \log|1 \mp z| \\ 1 & \log|1 \mp z| \end{pmatrix},$$

as $D \ni z \to \pm 1$.

After proper renormalization, we obtain matrix function $\mathscr{T}$ that solves the following RH-problem:

(a) $\mathscr{T}$ is analytic in D and $\mathscr{T}(\infty) = \mathscr{I}$;

(b) $\mathscr{T}$ has continuous traces, $\mathscr{T}_\pm$, on Δ° and

$$\mathscr{T}_+ = \mathscr{T}_- \begin{pmatrix} (r_n c)^+ & w_{a,\beta} \\ 0 & (r_n c)^- \end{pmatrix},$$

where

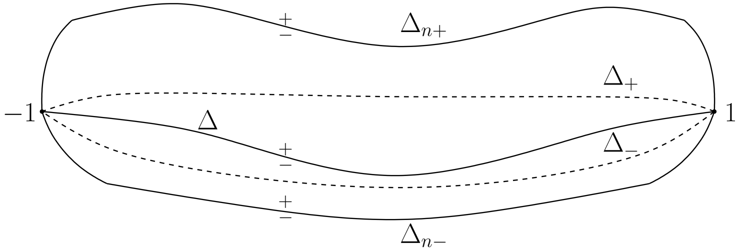
$$c^\pm(\tau) = \exp\left\{ \frac{w^\pm(\tau)}{\pi i} \int_\Delta \frac{\partial(t)}{w^+(t)} \frac{dt}{t-\tau} \right\};$$

(c) $\mathscr{T}$ has the following behavior near $z = \pm 1$:

$$\mathscr{T} = O\begin{pmatrix} 1 & \log|1 \mp z| \\ 1 & \log|1 \mp z| \end{pmatrix},$$

as $D \ni z \to \pm 1$.

$$\begin{pmatrix} (r_n c)^+ & w_{\alpha,\beta} \\ 0 & (r_n c)^- \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ (r_n c)^-/w_{\alpha,\beta} & 1 \end{pmatrix} \begin{pmatrix} 0 & w_{\alpha,\beta} \\ -1/w_{\alpha,\beta} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ (r_n c)^+/w_{\alpha,\beta} & 1 \end{pmatrix}.$$



The contour $\Sigma_n := \Delta_{n+} \cup \Delta \cup \Delta_{n-}$ (solid lines). The extension contour $\Sigma_{ext} := \Delta_+ \cup \Delta \cup \Delta_-$ (dashed lines and Δ).

Definition

A function ∂ belongs to the Sobolev class $W_p^{1-1/p}$, $p \in (2,\infty)$, if

$$\iint_{\Delta\times\Delta} \left|\frac{\partial(x)-\partial(y)}{x-y}\right|^p |dx||dy| < \infty.$$

Lemma (BY)

Let $h = \exp\{\partial\}$, $\partial \in W_p^{1-1/p}$, $p \in (2,\infty)$. Then there exists a continuous in $\mathbb{C}\setminus\Delta$ and up to $\Delta^\pm$ function c satisfying

$$c_{|\Delta^\pm} = c^\pm, \quad c_{|\Delta_\pm} = \exp\{w\ell\}, \quad \text{and} \quad \bar\partial c = cf,$$

where $\deg(\ell) \le 1$ and $f \in L^p(\Omega_\pm)$.

Define

$$\mathcal{S}\phi(\tau) := \frac{1}{\pi i}\int_{\Delta}\frac{\phi(t)}{t-\tau}dt, \quad \tau \in \Delta^{\circ}.$$

Then the first step to prove the previous lemma is to show the following.

Lemma (BY)

Let $\partial \in W_p^{1-1/p}$, $p \in (2,\infty)$. Then

$$w^{\pm}\mathcal{S}(\partial/w^{+}) = \pm d + w^{\pm}\ell, \quad d(\pm 1) = 0,$$

where $d \in W_q^{1-1/q}$ for any $q \in (2,p)$ and $\deg(\ell) \le 1$.

The second step is to use the trace theorems for Sobolev spaces on domains with corners.

Matrix function $\mathscr{T}$ is transformed into the matrix function $\mathscr{S}$ that solves the following RH$\bar\partial$-problem:

(a) $\mathscr{S}$ is continuous in $\overline{\mathbb{C}} \setminus \Sigma_n$ and $\mathscr{S}(\infty) = \mathscr{I}$;

(b) $\mathscr{S}$ has traces, $\mathscr{S}_\pm$, on $\Sigma_n^\circ := \Sigma_n \setminus \{\pm 1\}$ and

$$\begin{aligned} \mathscr{S}_+ &= \mathscr{S}_- \begin{pmatrix} 1 & 0 \\ r_n c/w_{a,\beta} & 1 \end{pmatrix} \quad \text{on} \quad \Delta_{n+}^\circ \cup \Delta_{n-}^\circ, \\ \mathscr{S}_+ &= \mathscr{S}_- \begin{pmatrix} 0 & w_{a,\beta} \\ -1/w_{a,\beta} & 0 \end{pmatrix} \quad \text{on} \quad \Delta^\circ; \end{aligned}$$

(c) $\mathscr{S}$ has the following behavior near $z = \pm 1$:

$$\mathscr{S}(z) = O\begin{pmatrix} \log|1 \mp z| & \log|1 \mp z| \\ \log|1 \mp z| & \log|1 \mp z| \end{pmatrix} \text{as} \quad \mathbb{C} \setminus \Sigma_n \ni z \to \pm 1;$$

(d) $\mathscr{S}$ deviate from an analytic matrix function as $\bar\partial \mathscr{S} = \mathscr{S}\mathscr{W}_0$, where the support of $\mathscr{W}_0$ is contained within the extension lens.

Now, we seek the solution for the following RH-problem:

(a) $\mathscr{A}$ is a holomorphic matrix function in $\overline{\mathbb{C}} \setminus \Sigma_n$ and $\mathscr{A}(\infty) = \mathscr{I}$;

(b) $\mathscr{A}$ has continuous traces, $\mathscr{A}_\pm$, on Σ_n° that satisfy the same relations as $\mathscr{S}$;

(c) the behavior of $\mathscr{A}$ near ± 1 is identical to the behavior of $\mathscr{S}$.

Observe that we may assume $c = \exp\{w\ell\}$.

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Observe that we may assume $c = \exp\{w\ell\}$.

This problem was solved by Kuijlaars, McLaughlin, Van Assche, and Vanlessen for $\Delta = [-1, 1]$ and no polynomial weight. With some technical challenges the proof can be adapted to the present situation.

Hence, the problem for $\mathscr{A}$ is indeed solvable.

We seek the solution of the following $\bar{\partial}$-problem:

(a) $\mathscr{D}$ is a continuous matrix function in $\overline{\mathbb{C}}$ and $\mathscr{D}(\infty) = \mathscr{I}$;

(b) $\mathscr{D}$ satisfies $\bar{\partial}\mathscr{D} = \mathscr{D}\mathscr{W}$ with $\mathscr{W} := \mathscr{A}\mathscr{W}_0\mathscr{A}^{-1}$.

We seek the solution of the following $\bar\partial$-problem:

(a) $\mathscr{D}$ is a continuous matrix function in $\overline{\mathbb{C}}$ and $\mathscr{D}(\infty) = \mathscr{I}$;
(b) $\mathscr{D}$ satisfies $\bar\partial\mathscr{D} = \mathscr{D}\mathscr{W}$ with $\mathscr{W} := \mathscr{A}\mathscr{W}_0\mathscr{A}^{-1}$.

This problem is solvable if and only if there exists a solution of

$$\mathscr{I} = (\mathcal{I} - \mathcal{K}_{\mathscr{W}})\mathscr{D},$$

where

$$\mathcal{K}_{\mathscr{W}}\mathscr{D}(z) := \frac{1}{2\pi i}\iint_{\Omega} \frac{(\mathscr{W}\mathscr{D})(\zeta)}{\zeta - z} d\zeta \wedge d\bar\zeta.$$

Matrix functions $\mathscr{D}$ exist since

$$||\mathcal{K}_{\mathscr{W}}|| \to 0 \quad \text{as} \quad n \to \infty.$$

Theorem (BY)

Let Δ be an analytic Jordan arc, which is symmetric with respect to $\mathscr{E}$, and

$$f_\mu(z) = \int \frac{(hw_{\alpha,\beta})(t)}{z-t}\frac{idt}{\pi},$$

where $h = e^{\partial}$, $\partial \in W_p^{1-1/p}$, $p \in (2,\infty)$, and $\alpha, \beta \in (\frac{2}{p}-1, 1-\frac{2}{p})$.

If $\{\Pi_n\}$ is the sequence of diagonal multipoint Padé approximants to f_μ associated to $\mathscr{E}$, then

$$(f_\mu - \Pi_n)w = [2G_{\dot{\mu}} + o(1)]S^2\, r_n$$

locally uniformly in D.

The theorem also holds under the condition

$$\alpha, \beta \in (-s, s) \cap (-1, \infty),$$

where

$$s := \begin{cases} 2\varsigma - 1, & \text{if} \quad \partial \in C^{0,\varsigma}, \quad \varsigma \in (\frac{1}{2}, 1], \\ m + \varsigma, & \text{if} \quad \partial \in C^{m,\varsigma}, \quad m \in \mathbb{N}, \quad \varsigma \in (0, 1], \end{cases}$$

and

a function ∂ belongs to the class $C^{m,\varsigma}$, $m \in \mathbb{Z}_+$, $\varsigma \in (0, 1]$, if ∂ is *m*-times continuously differentiable on Δ and its *m*-th derivative is uniformly Hölder continuous with exponent ς, i.e.,

$$|\partial^{(m)}(t_1) - \partial^{(m)}(t_2)| \leq \text{const.}|t_1 - t_2|^{\varsigma}, \quad t_1, t_2 \in \Delta,$$

where const. $< \infty$ depends only on ∂.