On Rational Approximants of Multi-Valued Functions

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Notation

- f is the approximated analytic function
- A is a closed set on which f is approximated
- B is a closed set with connected complement B^c such that f is analytic in B^c and $A \subset B^c$ (B is for "boundary")
- $\mathcal{B}(f, A)$ is the collection of the sets B as above
- $\mathcal{R}_n(A)$ – all the rational functions of type (n, n) with poles in A^c

Theorem (Runge 1885)

Suppose A is compact and f is analytic on A. Given $\epsilon > 0$, there exists a rational function R with poles in A^c such that $|(f-R)(z)| < \epsilon$, $z \in A$.

There exists a rectifiable contour Γ such that

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in A.$$

By uniform continuity on $\Gamma \times A$, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left| \frac{f(\zeta)}{\zeta - z} - \frac{f(\zeta^*)}{\zeta^* - z} \right| < \frac{2\pi}{|\Gamma|}\epsilon, \quad z \in A, \quad \zeta, \zeta^* \in \Gamma, \quad |\zeta - \zeta^*| < \delta.$$

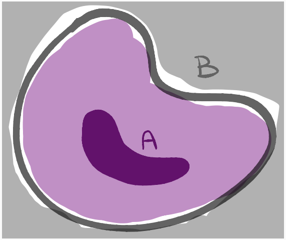
Let $\{\Gamma_i\}$ be a partition of Γ into Jordan arcs such that $|\Gamma_i| < \delta$, and $\zeta_i \in \Gamma_i$.

$$\left| f(z) - \frac{1}{2\pi \mathrm{i}} \sum_i \frac{f(\zeta_i)}{\zeta_i - z} \int_{\Gamma_i} d\zeta \right| \le \frac{1}{2\pi} \sum_i \int_{\Gamma_i} \left| \frac{f(\zeta)}{\zeta - z} - \frac{f(\zeta_i)}{\zeta_i - z} \right| |d\zeta| < \epsilon.$$

Limit Superior

$$\limsup_{n\to\infty} \rho_n^{1/n}(f, A) \leq ?, \quad \rho_n(f, A) := \inf_{R \in \mathcal{R}_n(A)} \|f - R\|_A.$$

In what follows, it will be convenient to think of A as compact.



A Computation

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Let f be analytic and bounded in $\{|z| < 1\} \supset A$, $B = \{|z| \geq 1\}$.

Let $z_1, \ldots, z_n \in A$. There exists $r_n \in \mathcal{R}_n(\{|z| \leq 1\})$ (with poles outside of the closed unit disk) such that $f(z_i) = r_n(z_i)$ and

$$|(f - r_n)(z)| \leq Cn^a |b_n(z)|$$

for some C, a independent of n, where

$$b_n(z) = \prod_{i=1}^{n} \frac{z - z_i}{1 - \overline{z}_i z}$$

is a rational function with zeros z_i and such that $|b_n(z)| \equiv 1$ on $\mathbb{T} = \{|z| = 1\}$.

A Computation

We also can write

$$|b_n(z)| = \prod_{i=1}^{n} \left| \frac{z - z_i}{1 - \overline{z}_i z} \right| = \exp\left\{ -\sum_{i=1}^{n} \log \left| \frac{1 - \overline{z}_i z}{z - z_i} \right| \right\}.$$

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Let $\nu_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{z_i}$ the normalized sum of point masses. Then

$$|f(z) - r_n(z)| \leq C n^a \exp\left\{ -n \int \log \left| \frac{1 - \overline{\zeta} z}{z - \zeta} \right| d\nu_n(\zeta) \right\}$$

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and therefore

$$\|f - r_n\|_A^{1/n} \le (C n^a)^{1/n} \exp\left\{ - \inf_{z \in A} \int \log \left| \frac{1 - \overline{\zeta} z}{z - \zeta} \right| d\nu_n(\zeta) \right\}.$$

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We have a complete freedom in choosing ν_n.

A Computation

Take a sequences such that $\nu_n \xrightarrow{*} \nu$ for some Borel measure ν on A:

$$\int h d\nu_n \to \int h d\nu$$

for any continuous function h on A. Then it holds that

$$\limsup_{n\to\infty} \|f - r_n\|_A^{1/n} \le \exp\left\{ - \inf_{z\in A} \int \log \left| \frac{1-\overline{\zeta} z}{z-\zeta} \right| d\nu(\zeta) \right\}.$$

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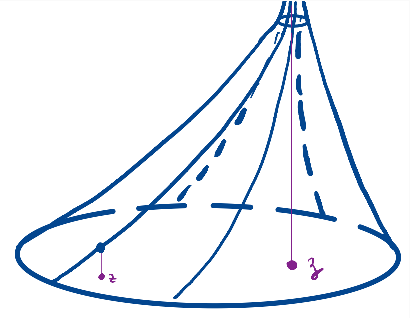
We still have a complete freedom in choosing ν. Therefore,

$$\limsup_{n\to\infty} \rho_n^{1/n}(f, A) \leq \inf_{|\nu|=1} \exp\left\{-\inf_{z\in A} \int \log\left|\frac{1-\overline{\zeta}z}{z-\zeta}\right| d\nu(\zeta)\right\}.$$

The function $\log\left|\frac{1-\overline{\zeta}z}{z-\zeta}\right|$ is known as the Green's function for the unit disk with pole at ζ.



It describes the work done in bringing a unit charge particle from the boundary (unit circle) to the point z in the presence of an electric field generated by a fixed unit charge at ζ.

Green's Functions

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Given a closed (non-polar) set B with connected complement B^c and $\zeta \in B^c$, there exists the unique function $g_B(z, \zeta)$, Green's function for B^c, such that

- $g_B(z, \zeta)$ is positive and harmonic in $B^c \setminus \{\zeta\}$;
- $g_B(z, \infty) - \log|z|$ is bounded near $\zeta = \infty$;
- $g_B(z, \zeta) + \log|z - \zeta|$ is bounded near $\zeta \neq \infty$;
- $g_B(z, \zeta) = 0$ for quasi every (up to a polar set) $z \in \partial B^c$.

Green's Potentials

The Green potential of a finite Borel measure ν supported in B^c is defined by

$$\boxed{g_B(z;\nu) := \int g_B(z,\zeta)\,d\nu(\zeta).}$$

The Green's energy of ν is defined by

$$\boxed{I_B[\nu] := \int\int g_B(z,\zeta)\,d\nu(\zeta)\,d\nu(z).}$$

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If $A \subset B^c$ is non-polar, then there exists the unique probability measure $\omega_{A,B}$ supported on A, the so-called Green equilibrium distribution on A, such that

$$\boxed{I_B[\omega_{A,B}] = \inf I_B[\nu],}$$

where the infimum is taken over all probability measures supported on A. The condenser capacity of A with respect to B is defined as

$$\boxed{C(A,B) := 1/I_B[\omega_{A,B}].}$$

Equilibrium Green's Potential

The measure $\omega_{A,B}$ describes the distribution of the unit charge that can freely move on A when it reaches the equilibrium (minimal energy) position.

The equilibrium potential $g_B(z;\omega_{A,B})$ is characterized by the property

$$\begin{aligned} g_B(z;\omega_{A,B}) &= 1/C(A,B), \quad && z \in A, \\ g_B(z;\omega_{A,B}) &= 0, && z \in \partial B^c, \end{aligned}$$

and it is harmonic in $B^c \setminus A$.

A Computation Revisited

In the previous computation we have shown that

$$\begin{aligned}\limsup_{n\to\infty} \rho_n^{1/n}(f, A) &\le \inf_{|\nu|=1} \exp\left\{-\inf_{z\in A} \int \log\left|\frac{1-\overline{\zeta}z}{z-\zeta}\right| d\nu(\zeta)\right\} \\ &= \inf_{|\nu|=1} \exp\left\{-\inf_{z\in A} g_B(z;\nu)\right\}.\end{aligned}$$

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$$= \inf_{|\nu|=1} \exp\left\{-\inf_{z\in A} g_B(z;\nu)\right\}.$$

It holds that

$$\inf_{z\in A} g_B(z;\nu) \le \int g_B(z;\nu)d\omega_{A,B}(z) = \int g_B(z;\omega_{A,B})d\nu(z) = 1/C(A,B).$$

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It holds that

$$\inf_{z\in A} g_B(z;\nu) \leq \int g_B(z;\nu)d\omega_{A,B}(z) = \int g_B(z;\omega_{A,B})d\nu(z) = 1/C(A,B).$$

Hence,

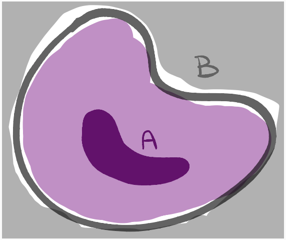
$$\boxed{\limsup_{n\to\infty} \rho_n^{1/n}(f, A) \leq \exp\left\{-\frac{1}{C(A,B)}\right\}.}$$

Theorem (Walsh 1934)

Let f be analytic in some neighborhood of a compact set A. Let $\mathcal{B}(f, A)$ be the collection of closed sets B such that $\infty \in B^\circ$, $A \subset B^c$ and f be analytic in B^c. Then

$$\limsup_{n\to\infty} \rho_n^{1/n}(f, A) \leq \inf_{B\in\mathcal{B}(f,A)} \exp\left\{-\frac{1}{C(A, B)}\right\}.$$

The bound is achieved by certain lacunary series (Levin and Tikhomirov 1967).



Limit Inferior

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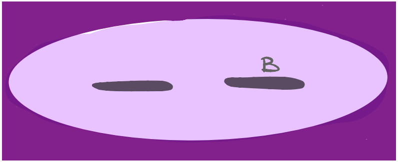

Gonchar's Conjecture

In 1978 (most likely earlier), Gonchar conjectured that

$$\boxed{\liminf_{n\to\infty} \rho_n^{1/n}(f, A) \leq \inf_{B\in\mathcal{B}(f,A)} \exp\left\{-\frac{2}{C(A,B)}\right\}.}$$

Adamyan-Arov-Krein Approximants

Take for now $A = \{|z| \geq 1\}$. Denote by H^∞ be space of bounded analytic functions in the unit disk. Set

$$H_n^\infty = H^\infty + \mathcal{R}_n(A),$$

which is the set of meromorphic functions with at most n poles in the unit disk and bounded traces on the unit circle $\mathbb{T}$.

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Consider the following approximation problem: *given a bounded function ϕ on the unit circle, find $M_n \in H_n^\infty$ such that*

$$\mathrm{dist}(\phi, H_n^\infty) = \inf_{M \in H_n^\infty} \|\phi - M\|_{\mathbb{T}} = \|\phi - M_n\|_{\mathbb{T}}.$$

When $n = 0$, this is known as the Nehari problem (1957).

If ϕ is Dini-continuous on $\mathbb{T}$, then such M_n exists and is continuous up to $\mathbb{T}$ (Carleson and Jacobs, 1972).

Hardy Spaces

The space of the square integrable functions on $\mathbb{T}$ can be defined as

$$L^2 = \left\{ \sum_{-\infty}^{\infty} a_n z^n \,:\, |z| = 1,\ \sum_{-\infty}^{\infty} |a_n|^2 < \infty \right\}.$$

The Hardy spaces H^2 and $H^2_- = L^2 \ominus H^2$ can be defined as

$$H^2 = \left\{ \sum_{0}^{\infty} a_n z^n \,:\, |z| = 1,\ \sum_{0}^{\infty} |a_n|^2 < \infty \right\}$$

and

$$H^2_- = \left\{ \sum_{-\infty}^{-1} a_n z^n \,:\, |z| = 1,\ \sum_{-\infty}^{-1} |a_n|^2 < \infty \right\}.$$

They can be identified with spaces of analytic functions in $\{|z| < 1\}$ and $\{|z| > 1\}$ that have L^2-traces on $\mathbb{T}$.

Let ϕ be a bounded function on $\mathbb{T}$. The Hankel operator Γ_ϕ with symbol ϕ is given by

$$\Gamma_\phi : H^2 \to H^2_-, \quad h \mapsto \mathbb{P}_-(h\phi),$$

where $\mathbb{P}_- : L^2 \to H^2_-$ is the orthogonal projection. When ϕ is continuous, Γ_ϕ is compact. Moreover,

$$(\Gamma_\phi h)(z) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}} \frac{(h\phi)(s)}{z - s} ds, \quad |z| > 1.$$

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Theorem (Adamyan-Arov-Krein 1971)

Let ϕ be a continuous function on $\mathbb{T}$. Then it holds that

$$\operatorname{dist}(\phi, H_n^\infty) = s_n(\Gamma_\phi),$$

where $s_n(\Gamma_\phi)$ is the n-th singular number of Γ_ϕ.

Rates of Rational and Meromorphic Approximation

Let f be analytic in $B^c \supset A = \{|z| \geq 1\}$ and M_n be the best meromorphic approximant of f in H_n^∞. Write

$$M_n = h_n + r_n,$$

where $h_n \in H^\infty$ and $r_n \in \mathcal{R}_n(A)$, $r_n(\infty) = f(\infty)$.

Let f be analytic in $B^c \supset A = \{|z| \geq 1\}$ and M_n be the best meromorphic approximant of f in H^∞_n. Write

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$$f(z) - r_n(z) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}} \frac{(f - M_n)(s)}{z - s} ds, \quad |z| > 1.$$

Therefore, we get for any $\delta > 0$ that

$$\|f - r_n\|_{\{|z| \geq 1+\delta\}} \leq \frac{\|f - M_n\|_{\mathbb{T}}}{2\pi\delta} = \frac{s_n(\Gamma_f)}{2\pi\delta}.$$

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Subsequently, it is enough to show that

$$\boxed{\liminf_{n\to\infty} s_n^{1/n}(\Gamma_f) \leq \exp\left\{-\frac{2}{C(A,B)}\right\}.}$$

Parfënov's Estimate

Assume that f is analytic in the closure of B^c, where ∂B is a smooth Jordan curve in the unit disk. Then

$$(\Gamma_f h)(z) = \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{T}} \frac{(fh)(\zeta)}{z-\zeta} d\zeta = \frac{1}{2\pi\mathrm{i}} \int_{\partial B} \frac{(fh)(\zeta)}{z-\zeta} d\zeta, \quad |z| > 1.$$

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Therefore, $\Gamma_f = \mathbb{E}_2 \circ \mathbb{P} \circ \mathbb{M}_f \circ \mathbb{E}_1$, where

- $\mathbb{E}_1$ is the embedding of H^2 into $L^2(\partial B)$
- $\mathbb{M}_f$ is the multiplication by f in $L^2(\partial B)$
- $\mathbb{P}$ is the projection from $L^2(\partial B)$ into Smirnov class $S^2(B^c)$
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It is known that $\mathbb{P}, \mathbb{M}_f$ are bounded operators and

$$\lim_{m\to\infty} s_m^{1/m}(\mathbb{E}_1) = \lim_{m\to\infty} s_m^{1/m}(\mathbb{E}_2) = \exp\{-1/C(B,A)\}$$

by (Zakharyuta-Skiba 1976) and (Fisher-Micchelli, 1980). The claim now follows from Horn-Weyl inequalities.

Theorem (Prokhorov 1993)

Let A, B be arbitrary disjoint closed sets. Let f be holomorphic in B^c. Then

$$\liminf_{n\to\infty} \rho_n^{1/n}(f, A) \leq \exp\left\{-\frac{2}{C(A,B)}\right\}.$$

Moreover,

$$\limsup_{n\to\infty} \rho_n^{1/n}(f, A) = \exp\left\{-\frac{1}{C(A,B)}\right\} \quad \Rightarrow \quad \liminf_{n\to\infty} \rho_n^{1/n}(f, A) = 0.$$

The case where A is a continuum was proved by Parfënov in 1986.

Prokhorov's proof relies on the generalization of the AAK theory to multiply connected domains (Prokhorov 1991).

Multi-Valued Functions

When is true that

$$\lim_{n\to\infty} \rho_n^{1/n}(f, A) = \inf_{B\in\mathcal{B}(f,A)} \exp\left\{-\frac{2}{C(A,B)}\right\}?$$

We say that a function f belongs to Stahl's class S if f is holomorphic and multi-valued outside of a compact polar set E_f.

That is, for any point $z_0 \notin E_f$ and any path γ starting at z_0 and avoiding E_f, f admits analytic continuation along γ. Moreover, there are paths with the same endpoints that lead to distinct continuations.

All algebraic functions (solutions of $p_n(z)f^n + p_{n-1}(z)f^{n-1} + \cdots + p_0(z) = 0$, where $p_k(z)$ are polynomials) are in this class as well as functions of the form

$$f(z) = \sum_{l=1}^{L} \prod_{i=1}^{I_l} (z - z_{l,i})^{\alpha_{l,i}},$$

where $\sum_{i=1}^{I_l} \alpha_{l,i}$ is an integer but some $\alpha_{l,i}$ are not. Logarithmic functions are in this class. All the above functions could be multiplied by factors like $e^{c/(z-z_*)}$ or any other single-valued function holomorphic outside of a polar set.

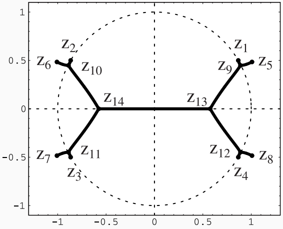
Minimal Capacity Contours

Theorem (Stahl 1985)

Given a branch of $f \in S$ analytic on a continuum $A \subset E_f^c$, there exists a compact set $\mathsf{B} \in \mathcal{B}(f, A)$ such that

$$C(A, \mathsf{B}) \leq C(A, B), \quad B \in \mathcal{B}(f, A).$$

B "essentially" consists of analytic arcs.



Minimal (logarithmic) capacity contour for $\sqrt{\sqrt{1 - z^{-2} + z^{-4}} - 0.4}$

Theorem of Gonchar-Rakhmanov

Theorem (Gonchar-Rakhmanov 1989)

Given $f \in S$ and a continuum $A \subset E_f^c$, there exists a sequence of rational interpolants R_n such that

$$\lim_{n\to\infty} \rho_n^{1/n}(f, A) = \lim_{n\to\infty} \|f - R_n\|_A^{1/n} = \exp\left\{-\frac{2}{C(A, \mathsf{B})}\right\}.$$

Moreover, the poles of these interpolants asymptotically distribute as $\omega_{\mathsf{B},A}$ (interpolation points asymptotically distribute as $\omega_{A,\mathsf{B}}$).

Multipoint Padé Approximants (proof of GR Theorem)

Let $\{z_{n,1}, \ldots, z_{n,2n}\} \subset A$ be a multiset of not necessarily distinct nor finite points and

$$V_n(z) = \prod_{|z_{n,i}|<\infty} (z - z_{n,i}).$$

The n-th diagonal multipoint Padé approximant is a rational function P_n/Q_n of type (n, n) such that

$$\frac{(Q_n f - P_n)(z)}{V_n(z)} = \mathcal{O}\left(z^{-n-1}\right) \quad \text{as} \quad z \to \infty$$

and is analytic on A. The above equation is in fact defines a linear system with one more unknown than equations. Hence, the rational function P_n/Q_n exists and happens to be unique.

Multipoint Padé Approximants (proof of GR Theorem)

Since B is a essentially a system of analytic arcs, it follows from the formula defining P_n/Q_n, Cauchy theorem and integral formula that

$$\int_{\mathsf{B}} t^k Q_n(t)(f_+ - f_-)(t) \frac{dt}{V_n(t)} = 0$$

for $k \in \{0, \ldots, n-1\}$, and

$$\frac{(Q_n f - P_n)(z)}{V_n(z)} = \frac{1}{2\pi \mathrm{i}} \int_{\mathsf{B}} \frac{Q_n(t)(f_+ - f_-)(t)}{z-t} \frac{dt}{V_n(t)}.$$

Stahl and then Gonchar-Rakhmanov had developed machinery how to use the above orthogonality relations and the minimality of B to get n-th root asymptotic behavior of error in the complex plane.

This gave the upper estimate for the limit superior.

Multipoint Padé Approximants (proof of GR Theorem)

If the limit inferior was smaller, there would exist rational functions $p_n/q_n \in \mathcal{R}_n(A)$ such that

$$\max_{z\in\gamma} |f(z) - p_n(z)/q_n(z)| < \min_{z\in\gamma} |f(z) - P_n(z)/Q_n(z)|$$

for some Jordan curve γ whose exterior domain, say D, lies in B^c and contains A.

Since P_n/Q_n interpolates f at $\{z_{n,1}, \ldots, z_{n,2n}, \infty\}$, $f - P_n/Q_n$ has $2n+1$ zeros in D. By Rouche's theorem,

$$\frac{p_n}{q_n} - \frac{P_n}{Q_n}$$

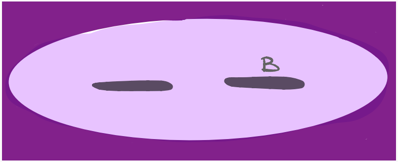
must have $2n+1$ zeros in D (including one at infinity), but it is impossible as its numerator has degree at most $2n-1$.

Behavior in A^c

In memory of Herbert Stahl (1942-2013).

What do the poles of best rational approximants do?

In what follows A is unbounded set whose boundary is a Jordan curve with Dini-continuous parametrization.



Distribution of Poles

We say that a sequence of rational approximants $R_n \in \mathcal{R}_n(A)$ is n-th root optimal if

$$\lim_{n\to\infty} \|f - R_n\|_A^{1/n} = \exp\left\{-\frac{2}{C(A,\mathsf{B})}\right\}.$$

Theorem (Baratchart-Stahl-Ya.)

There exists a class of functions $\mathcal{F}(A)$ analytic on A such that for every $f \in \mathcal{F}(A)$ if R_n are n-th root optimal rational approximants to f on A, then

$$\nu(R_n) \xrightarrow{*} \omega_{\mathsf{B},A},$$

where $\nu(R_n)$ is the normalized counting measure of poles of R_n.

Moreover, the functions R_n converge in capacity to f in $\mathsf{B}^c \setminus A$.

The same is true for n-th root optimal meromorphic approximants.

Approximated Functions

Recall that A is the closure of the unbounded component of the complement of a Jordan curve. Let D be the bounded component. The class $\mathcal{F}(A)$ consists of functions holomorphic on A with the following two properties:

- they can be continued into D along any path originating on ∂D which stays in $\overline{D}$ while avoiding a closed polar subset of D (that may depend on the function);
- they are not single-valued, but the number of distinct function elements lying above a point of D is uniformly bounded (the bound may depend on the function).