Hermite-Padé Approximation of Markov Functions

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Criterion

A number α is transcendental if for any $m \in \mathbb{N}$ and any $\varepsilon > 0$ there exist $m+1$ linearly independent vectors of integers $\left(q_j, p_j^{(1)}, \ldots, p_j^{(m)}\right)$ such that $\left|q_j \alpha^k - p_j^{(k)}\right| \leqslant \varepsilon, \forall k$.

If α is algebraic, then $\exists\, m \in \mathbb{N},\ a_k \in \mathbb{Z}$, $k = \overline{0, m}$, such that $\sum_{k=0}^m a_k \alpha^k = 0$. Hence,

$$\sum_{k=1}^{m} a_k \left(q_j \alpha^k - p_j^{(k)}\right) + a_0 q_j + \sum_{k=1}^{m} a_k p_j^{(k)} = 0.$$

Since vectors $\left(q_j, p_j^{(1)}, \ldots, p_j^{(m)}\right)$ are linearly independent, there exists j_0 such that

$$0 \neq a_0 q_{j_0} + \sum_{k=1}^{m} a_k p_{j_0}^{(k)} \in \mathbb{Z}.$$

Then, it holds that

$$1 \leqslant \left| \sum_{k=1}^{m} a_k \left(q_{j_0} \alpha^k - p_{j_0}^{(k)}\right) \right| \leqslant \varepsilon \sum_{k=1}^{m} |a_k|.$$

In 1873, Hermite proved that e is transcendental in the following way.

Let $n_0, n_1, \dots, n_m$ be non-negative integers. Set $N := n_0 + \cdots + n_m$ and consider the following system:

$$Q(z)e^{kz} - P_k(z) = \mathcal{O}\left(z^{N+1}\right),$$

where $\deg(Q) \leqslant N - n_0$ and $\deg(P_k) \leqslant N - n_k$.

Hermite proceeded to **explicitly** construct these polynomials, which as it turned out have **integer coefficients**. By evaluating them at **1** and **varying** the parameters $n_0, n_1, \dots, n_m$ he succeeded in applying the above criterion.

Let $\vec{f} = (f_1, \ldots, f_m)$ be a vector of functions holomorphic and vanishing at infinity:

$$f_i(z) = \frac{f_{i1}}{z} + \frac{f_{i2}}{z^2} + \cdots + \frac{f_{in}}{z^n} + \cdots .$$

Further, let $\vec{n} \in \mathbb{N}^m$ be a multi-index, while $P_{\vec{n}}^{(1)}, \ldots, P_{\vec{n}}^{(m)}$ and $Q_{\vec{n}}$ be polynomials such that

$$\deg(Q_{\vec{n}}) \leqslant |\vec{n}| := n_1 + \cdots + n_m$$

and

$$R_n^{(i)}(z) := \left(Q_{\vec{n}} f_i - P_{\vec{n}}^{(i)}\right)(z) = \mathcal{O}\left(z^{-n_i-1}\right) \quad \text{as} \quad z \to \infty.$$

The vector of rational functions

$$\left(P_{\vec{n}}^{(1)}/Q_{\vec{n}}, \ldots, P_{\vec{n}}^{(m)}/Q_{\vec{n}}\right)$$

is called the **type II Hermite-Padé approximant** to $\vec{f}$ corresponding to $\vec{n}$.

It follows from Cauchy integral formula that

$$f_i(z) = \int \frac{d\mu_i(x)}{z-x}$$

for some compactly supported Borel generally speaking complex measure μ_i. Since $R_{\vec{n}}^{(i)}(z) = \mathcal{O}\left(z^{-n_i-1}\right)$, it holds that

$$0 = \int_\Gamma z^k R_{\vec{n}}^{(i)}(z)dz = \int_\Gamma z^k Q_{\vec{n}}(z) f_i(z) dz = \int x^k Q_{\vec{n}}(x) d\mu_i(x)$$

for $k = \overline{0, n_i - 1}$, where Γ is any Jordan curve encircling the support of μ_i. In what follows, it assumed that $Q_{\vec{n}}$ is the **monic polynomial of minimal degree**.

The goal is to understand the asymptotic behavior of $Q_{\vec{n}}$ and $R_{\vec{n}}^{(i)}$ for a “large” class of measures μ_i.

Let μ be a positive Borel measure compactly supported on the real line. Then

$$f(z) = \int \frac{d\mu(x)}{z - x}$$

is called a **Markov function**. The n-th Padé approximant is defined by the condition

$$R_n(z) = \big(Q_n f - P_n\big)(z) = \mathcal{O}\big(z^{-n-1}\big).$$

In this case it holds that

$$\int x^k Q_n(x) d\mu(x) = 0, \quad k = \overline{0, n-1}.$$

That is, Q_n is the n-th **orthogonal polynomial** with respect to the measure μ.

Notice that all the zeros of Q_n are **distinct** and belong to the **convex hull** of $\mathrm{supp}(\mu)$. Indeed, otherwise, set P to be a polynomial vanishing at the odd multiplicity zeros of Q_n on the convex hull. Then $\deg(P) \leqslant n-1$ and

$$\text{orthogonality} \quad \Rightarrow \quad 0 = \int P(x)Q_n(x)d\mu(x) > 0 \quad \Leftarrow \quad \text{positivity.}$$

Denote by σ_n the normalized **counting measure of zeros** of Q_n. That is,

$$\sigma_n := \frac{1}{n}\sum_{i=1}^{n}\delta(x_i), \quad Q_n(x) = \prod_{i=1}^{n}(x-x_i),$$

where $\delta(x_i)$ is the Dirac δ-distribution with mass at x_i. Recall that a sequence of measures converges weak*, $\nu_n \xrightarrow{*} \nu$, if $\int h d\nu_n \to \int h d\nu$ for any continuous function h.

Theorem

If $\mathrm{supp}(\mu) = [-1,1]$ and $\mu' > 0$ a.e. on $[-1,1]$, then $\sigma_n \xrightarrow{*} \omega$, where $d\omega(x) = \dfrac{dx}{\pi\sqrt{1-x^2}}$.

The "simplest" Markov function is

$$f(z) = \frac{1}{2\pi}\int_{[-1,1]} \frac{1}{z-x}\frac{dx}{\sqrt{1-x^2}} = \frac{1}{2\sqrt{z^2-1}}.$$

Write $w(z) = \sqrt{z^2-1}$. The polynomials

$$\begin{cases} T_n(z) & := \left(z+\sqrt{z^2-1}\right)^n + \left(z-\sqrt{z^2-1}\right)^n, \\ w(z)U_{n-1}(z) & := \left(z+\sqrt{z^2-1}\right)^n - \left(z-\sqrt{z^2-1}\right)^n, \end{cases}$$

are the **Chebyshëv polynomial of the first and second kind**. Then

$$T_n(z)f(z) - \frac{1}{2}U_{n-1}(z) = \frac{T_n(z)-w(z)U_{n-1}(z)}{2w(z)} = \frac{\left(z-\sqrt{z^2-1}\right)^n}{w(z)}.$$

Define

$$\Phi(z) := z + \sqrt{z^2 - 1} \quad \Leftrightarrow \quad \Phi^{-1}(z) = z - \sqrt{z^2 - 1}.$$

In fact, $\Phi(z)$ and $\Phi^{-1}(z)$ are the inverse functions of the **Zhoukovsky transformation** $J(z) = (z + z^{-1})/2$. In particular, $\Phi : \overline{\mathbb{C}} \setminus [-1, 1] \to \{|z| > 1\}$ is the **conformal map** such that $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$. Hence,

$$T_n(z)f(z) - \frac{1}{2}U_{n-1}(z) = \frac{\left(z - \sqrt{z^2 - 1}\right)^n}{w(z)} = \frac{1}{w(z)\Phi^n(z)} = \mathcal{O}\left(z^{-n-1}\right).$$

That is, the n-th Padé approximant to $1/2w$ is given by $U_{n-1}/2T_n$ and

$$\begin{cases} Q_n(z) & := \quad \Phi^n(z) + \Phi^{-n}(z), \\ (wR_n)(z) & := \quad \Phi^{-n}(z). \end{cases}$$

Theorem (Szegő, 30's)

Let ρ be a non-negative function satisfying $\int_{[-1,1]} \log \rho d\omega > -\infty$. Set

$$f(z) := \frac{1}{2\pi} \int_{[-1,1]} \frac{1}{z-x} \frac{\rho(x)dx}{\sqrt{1-x^2}}.$$

Then it holds locally uniformly in $\overline{\mathbb{C}} \setminus [-1,1]$ that

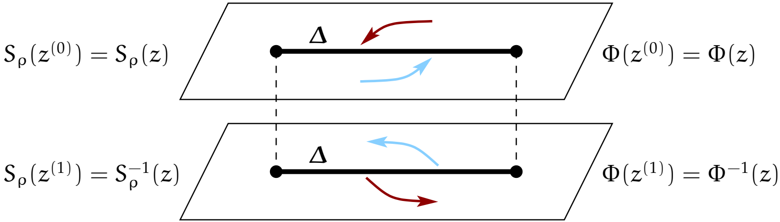
$$\begin{cases} Q_n(z) & \cong & \left(\Phi^n S_\rho\right)(z), \\ (wR_n)(z) & \cong & \left(\Phi^n S_\rho\right)^{-1}(z), \end{cases}$$

where $w(z) = \sqrt{z^2-1}$ and S_ρ is the **Szegő function** of ρ, i.e.,

$$S_\rho(z) := \exp\left\{ \frac{w(z)}{2\pi i} \int_{[-1,1]} \frac{\log \rho(x)}{z-x} \frac{dx}{w^+(x)} \right\}$$

is the unique non-vanishing holomorphic function off $[-1,1]$ such that $S_\rho^+ S_\rho^- = 1/\rho$.

Let $\mathfrak{R}$ be the Riemann surface of $w^2 = z^2 - 1$.



Then Φ^n is a rational function with the divisor $n\infty^{(1)} - n\infty^{(0)}$ and S_ρ is holomorphic and non-vanishing off $\boldsymbol{\Delta}$ that satisfies $S_\rho^+ = (\rho \circ \pi) S_\rho^-$. Then

$$\begin{cases} Q_n(z) & \cong & (\Phi^n S_\rho)\big(z^{(0)}\big), \\ (wR_n)(z) & \cong & (\Phi^n S_\rho)\big(z^{(1)}\big). \end{cases}$$

We shall say that a vector function $\vec{f} = (f_1, \dots, f_m)$ forms an **Angelesco system** if

$$f_i(z) = \int \frac{d\mu_i(x)}{z - x}, \quad \mu_i > 0, \quad \operatorname{supp}(\mu_i) = [a_i, b_i], \quad [a_i, b_i] \cap [a_j, b_j] = \varnothing.$$

Given a multi-index $\vec{n} = (n_1, \dots, n_m)$, $|\vec{n}| := n_1 + \cdots + n_m$, we can write

$$\int x^k Q_{\vec{n}}(x) d\mu_i(x) = 0, \quad k = \overline{0, n_i - 1}.$$

Hence, $Q_{\vec{n}}$ has n_i simple zeros on $[a_i, b_i]$. We denote by $\sigma_{\vec{n},i}$ their counting measure normalized by $|\vec{n}|$. That is, $|\sigma_{\vec{n},i}| = n_i/|\vec{n}|$.

Theorem (Gonchar-Rakhmanov, 81)

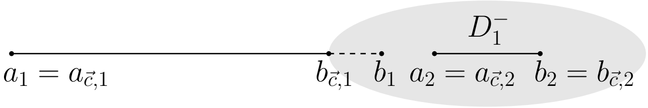
Assume that $\mu_i' > 0$ a.e. on $[a_i, b_i]$. Let $\{\vec{n}\}$ be a sequence of multi-indices such that

$$\frac{\vec{n}}{|\vec{n}|} \to \vec{c} \in (0,1)^m, \quad (|\vec{c}| = 1).$$

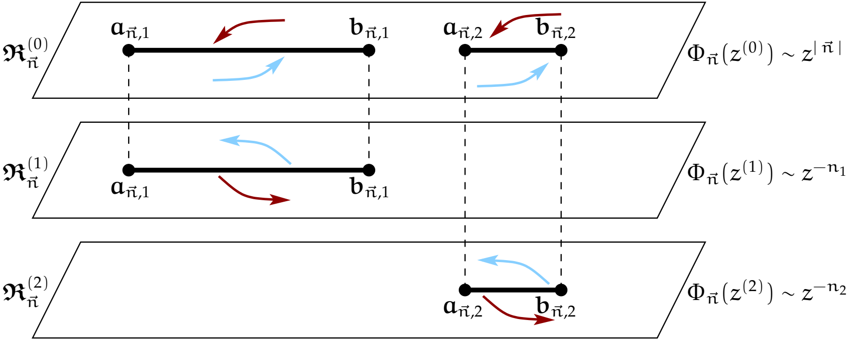
Then there exists a vector equilibrium measure $\vec{\omega}_{\vec{c}} = (\omega_{\vec{c},1}, \dots, \omega_{\vec{c},m})$ such that

$$\sigma_{\vec{n},i} \xrightarrow{*} \omega_{\vec{c},i}.$$

Moreover, it holds that $\operatorname{supp}(\omega_{\vec{c},i}) = [a_{\vec{c},i}, b_{\vec{c},i}] \subseteq [a_i, b_i]$.



Let $\vec{\omega}_{\vec{n}}$ be the vector equilibrium measure for $\vec{n}/|\vec{n}|$. Define $\mathfrak{R}_{\vec{n}}$ w.r.t. $\vec{\omega}_{\vec{n}}$ by



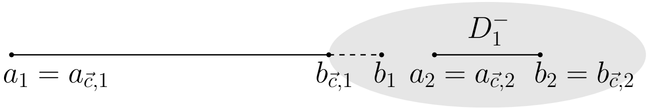
The surface $\mathfrak{R}_{\vec{n}}$ has genus 0. Given a multi-index $\vec{n}$, let $\Phi_{\vec{n}}$ be the rational function on $\mathfrak{R}_{\vec{n}}$ with the divisor and normalization given by

$$(\Phi_{\vec{n}}) = n_1\infty^{(1)} + \cdots + n_m\infty^{(m)} - |\vec{n}|\infty^{(0)}, \quad \prod \Phi_{\vec{n}}(z^{(k)}) \equiv 1.$$

Define

$$\begin{cases} D^+_{\vec{n},i} & := & \left\{z : \left|\Phi_{\vec{n}}\left(z^{(0)}\right)\right| > \left|\Phi_{\vec{n}}\left(z^{(i)}\right)\right|\right\}, \\ D^-_{\vec{n},i} & := & \left\{z : \left|\Phi_{\vec{n}}\left(z^{(0)}\right)\right| < \left|\Phi_{\vec{n}}\left(z^{(i)}\right)\right|\right\}. \end{cases}$$

It might happen that $D^-_{\vec{n},i} \neq \varnothing$.



As the following theorem shows, D^-_i is the divergence domain for $P^{(i)}_{\vec{n}}/Q_{\vec{n}}$.

Theorem (Y., 16)

Let ρ_i be a Fisher-Hartwig perturbation of a non-vanishing holomorphic function on $[a_i, b_i]$ and

$$f_i(z) := \frac{1}{2\pi i}\int_{[a_i,b_i]} \frac{\rho_i(x)dx}{x-z}.$$

Further, let $\{\vec n\}$ be a sequence of multi-indices such that $\vec n/|\vec n| \to \vec c \in (0,1)^m$. Then

$$\begin{cases} Q_{\vec n}(z) & \cong & \big(\Phi_{\vec n}S\big)\big(z^{(0)}\big), \\ \big(w_i R_{\vec n}^{(i)}\big)(z) & \cong & \big(\Phi_{\vec n}S\big)\big(z^{(i)}\big), \end{cases}$$

where $w_i(z) := \sqrt{(z-a_{\vec c,i})(z-b_{\vec c,i})}$ and S is a Szegő-type function on $\mathfrak{R}_{\vec c}$.

- Kalyagin, 79: $[-1,0]$ and $[0,1]$ + Jacobi weights
- Aptekarev, 88: two functions + Szegő weights + diagonal multi-indices
- Aptekarev–Lysov, 10: m functions + analytic weights + diagonal multi-indices

We shall say that a vector function $\vec{f} = (f_1, f_2)$ forms a **symmetric Stahl system** if

$$f_i \leftrightarrow \mu_i, \quad \mathrm{supp}(\mu_1) = [-1, a], \quad \mathrm{supp}(\mu_2) = [-a, 1], \quad a \in (0, 1).$$

Further, let h be an algebraic function given by

$$A(z)h^3 - 3B_2(z)h - 2B_1(z) = 0,$$

where

$$\begin{cases} A(z) & := & (z^2 - 1)(z^2 - a^2), \\ B_2(z) & := & z^2 - p^2, \\ B_1(z) & := & z, \end{cases}$$

for some parameter $p > 0$.

Denote by $\mathfrak{R}$ the Riemann surface of h. We are looking for the surface such that

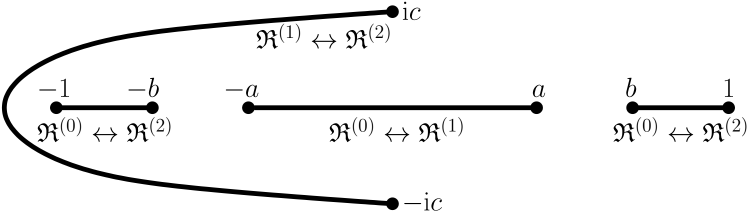
$$N(z) := \mathrm{Re}\left(\int^z h(t)dt\right) \quad \textbf{single-valued} \text{ harmonic function on } \mathfrak{R}. \tag{1}$$

Theorem (Aptekarev–Van Assche–Y.)

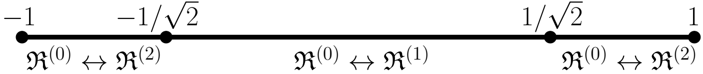
(I) If $a \in (0, 1/\sqrt{2})$, then there exists $p \in \left(a, \sqrt{(1+a^2)/3}\right)$ such that condition (1) is fulfilled. In this case $\mathfrak{R}$ has 8 ramification points whose projections are $\{\pm 1, \pm a\}$ and $\{\pm b, \pm ic\}$ for some uniquely determined $b \in (a, p)$ and $c > 0$.

(II) If $a = 1/\sqrt{2}$, then condition (1) is fulfilled for $p = 1/\sqrt{2}$. In this case $\mathfrak{R}$ has 4 ramification points whose projections are $\{\pm 1, \pm 1/\sqrt{2}\}$.

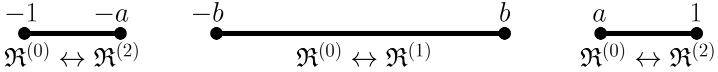
(III) If $a \in (1/\sqrt{2}, 1)$, then condition (1) is fulfilled for $p = \sqrt{(1+a^2)/3}$. In this case $\mathfrak{R}$ has 6 ramification points whose projections are $\{\pm 1, \pm a\}$ and $\{\pm b\}$, $b \in (p, a)$.



(a) Case I



(b) Case II



(c) Case III

Let $\Phi(z) := \exp\left\{\int^z h(t)dt\right\}$. It is a multiplicatively multi-valued function on $\mathfrak{R}$ with the divisor $\infty^{(1)} + \infty^{(2)} - 2\infty^{(0)}$ and normalized so that $\Phi\left(z^{(0)}\right)\Phi\left(z^{(1)}\right)\Phi\left(z^{(2)}\right) \equiv 1$.

Let ρ_1 and ρ_2 be functions holomorphic and non-vanishing in a neighborhood of $[-1,1]$. In Case I, assume also that the ratio ρ_1/ρ_2 holomorphically extends to a non-vanishing function in a neighborhood of $\mathfrak{R}^{(1)} \cap \mathfrak{R}^{(2)}$. Then $\Psi_n \leftrightarrow \Phi^n$, where

$$\begin{cases} \left(\Psi_n^{(1)}\right)^{\pm} = \pm\left(\Psi_n^{(0)}\right)^{\mp}\rho_1 & \text{on} \quad \Delta_1^\circ, \\ \left(\Psi_n^{(2)}\right)^{\pm} = \mp\left(\Psi_n^{(0)}\right)^{\mp}\rho_2 & \text{on} \quad \Delta_{21}^\circ, \\ \left(\Psi_n^{(2)}\right)^{\pm} = \pm\left(\Psi_n^{(0)}\right)^{\mp}\rho_2 & \text{on} \quad \Delta_{22}^\circ, \\ \left(\Psi_n^{(2)}\right)^{\pm} = \pm\left(\Psi_n^{(1)}\right)^{\mp}(\rho_2/\rho_1) & \text{on} \quad \Delta_0^\circ, \end{cases}$$

Ψ_n has a wandering zero (2 in Case I) and there exists a subsequence $\mathbb{N}_*$ such that

- $|\Psi_n| \leqslant C(\mathbb{N}_*)\,|\Phi^n|$ uniformly away from the branch points of $\mathfrak{R}$
- $|\Psi_n| \geqslant C(\mathbb{N}_*)^{-1}\,|\Phi^n|$ uniformly in a neighborhood of $\infty^{(0)}$

Theorem (Aptekarev-Van Assche-Y.)

Let

$$f_i(z) := \frac{1}{2\pi i}\int \frac{\rho_i(x)dx}{x-z},$$

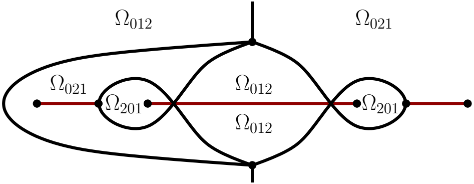
where ρ_1 and ρ_2 are as before and we assume in addition that the ratio ρ_2/ρ_1 extends from $(-a,a)$ to a holomorphic and non-vanishing function

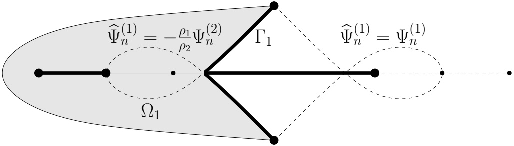
- in a domain that contains in its interior the closure of all the bounded components of the regions Ω_{ijk} in Case I;
- in a domain whose complement is compact and belongs to the right-hand component of Ω_{021} in Cases II and IIIa;
- in the extended complex plane, i.e., the ratio is a non-zero constant, in Case IIIb.

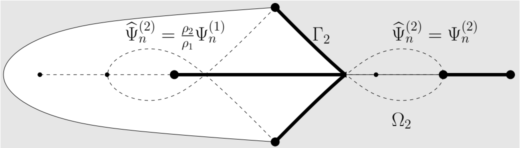
Then for multi-indices $\vec{n} = (n,n)$ it holds that

$$\begin{cases} Q_{\vec{n}} & \text{“}\cong\text{”} \quad C_n\Psi_n^{(0)}, \\ R_{\vec{n}}^{(i)} & \text{“}\cong\text{”} \quad C_n\widehat{\Psi}_n^{(i)}, \end{cases} \qquad n \in \mathbb{N}_*.$$

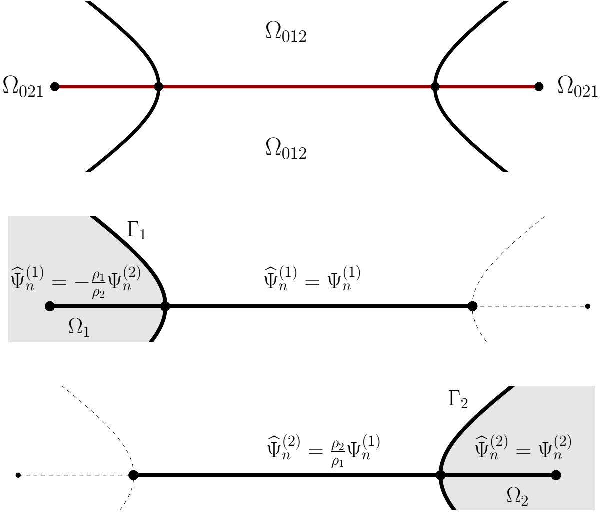
The ratio ρ_2/ρ_1 extends from $(-a, a)$ to a holomorphic and non-vanishing function in a domain that contains in its interior the closure of the bounded components of Ω_{ijk}.







The ratio ρ_2/ρ_1 extends from $(-a, a)$ to a holomorphic and non-vanishing function in a domain whose complement compactly belongs to the right-hand component of Ω_{021}.



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