Ratios of Norms for Polynomials and Connected *n*-width Problems

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Let

- G be a bounded simply connected domain, $\Gamma := \partial G$;
- E be a regular compact set with connected complement D;
- H^∞ be the Hardy space of bounded analytic functions in G;
- $C(E)$ be the space of continuous functions on E;
- A^∞ be the unit ball of H^∞ restricted to E.

[1] Rational approximation and n-dimensional diameter, J. Approx. Theory, 5: 343--361, 1972

[2] The n-width of analytic functions, Duke Math. J., 47(4): 789--801, 1980

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It was obtained by Widom[1] that

$$\lim_{k\to\infty}\left(\frac{1}{k}\log d_k(A^\infty; C(E))\right) = -\frac{1}{\operatorname{cap}(E,\Gamma)}.$$

Later, Fisher and Micchelli[2] showed that

$$d_k(A^\infty; C(E)) = \inf_{z_1,\ldots,z_k} \sup\left\{\|h\|_E : \ h\in A^\infty,\ h(z_j)=0\right\}.$$

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Set

$$A_n^\infty := A^\infty \cap \mathcal{P}_n,$$

where $\mathcal{P}_n$ is the space of polynomials of degree at most n.

We are interested in the asymptotic behavior of

$$d_{k_n}(A_n^\infty; C(E)) \quad \text{and} \quad \chi_n := \inf_{p\in\mathcal{P}_{k_n}} \sup_{q\in\mathcal{P}_{n-k_n}} \frac{\|pq\|_E}{\|pq\|_\Gamma},$$

when

$$\lim_{n\to\infty} \frac{k_n}{n} = \partial, \quad \partial \in [0, 1].$$

For a positive Borel measure σ, set

$$M(\sigma) := \min_{\Gamma} V^{\sigma} - \min_{E} V^{\sigma},$$

where $V^{\sigma}(z) = -\int \log|z-t|d\sigma(t)$. Then

$$\frac{1}{n}\log\left(\frac{\|pq\|_E}{\|pq\|_\Gamma}\right) = M\left(\upsilon(p) + \upsilon(q)\right),$$

where

$$\upsilon(p) := \frac{1}{n}\sum_{p(z)=0}\delta_z \quad \text{and} \quad \upsilon(q) := \frac{1}{n}\sum_{q(z)=0}\delta_z.$$

Hence,

$$\frac{1}{n}\log \chi_n = \inf_{p\in\mathcal{P}_{k_n}} \sup_{q\in\mathcal{P}_{n-k_n}} M\left(\upsilon(p)+\upsilon(q)\right).$$

Set $\Lambda_\epsilon(K)$ to be the set of positive Borel measures supported on K of mass at most ϵ. Consider

$$\inf_{\mu\in\Lambda_\partial(E)} \sup_{\lambda\in\Lambda_{1-\partial}(\Gamma)} M(\mu+\lambda).$$

There uniquely exists $\lambda_\alpha \in \Lambda_{1-\alpha}(\Gamma)$ such that

$$J_\alpha(\lambda_\alpha) = \min_{\lambda \in \Lambda_{1-\alpha}(\Gamma)} J_\alpha(\lambda),$$

where

$$J_\alpha(\lambda) := \iint g_D(z,t)\,d\lambda(t)\,d\lambda(z) - 2\int g_D(t,\infty)\,d\lambda(t)$$

and $g_D(z,t)$ is the Green function for D.

It holds that

$$V_D^{\lambda_\theta}(z) - g_D(z,\infty) = m_\theta, \quad z \in S_\theta := \operatorname{supp}(\lambda_\theta) \subseteq \Gamma,$$

where

$$V_D^{\lambda}(z) = \int g_D(z,t)\,d\lambda(t),$$

and

$$V_D^{\lambda_\theta}(z) - g_D(z,\infty) \geq m_\theta, \quad z \in \Gamma.$$

Constant m_∂ can be expressed as

$$m_\partial := \frac{1}{1-\partial}\left(J_\partial(\lambda_\partial) + \int g_D(t,\infty)d\lambda_\partial(t)\right).$$

Set $m_1 = -\max_\Gamma g_D(\cdot,\infty)$. Then m_∂ is a continuous and strictly decreasing function of $\partial \in [0,1]$. In particular, $m_0 = 0$.

Moreover, S_∂ is a decreasing family of sets, $S_0 = \Gamma$, such that

$$S_\partial \subseteq \cap_{0\le\tau<\partial} S_\tau = \{z \in \Gamma : V_D^{\lambda_\partial}(z) - g(z,\infty) = m_\partial\}$$

and

$$S_1 := \cap_{0\le\tau<1} S_\tau = \{z \in \Gamma : g(z,\infty) = -m_1\}.$$

There uniquely exists $\mu_{\ni} \in \Lambda_{\ni}(E)$ such that

$$I_{\ni}(\mu_{\ni}) = \min_{\mu \in \Lambda_{\ni}(E)} I_{\ni}(\mu),$$

where

$$I_{\ni}(\mu) := -\iint \log|z-t|\,d\mu(t)\,d\mu(z) + 2\int V^{\lambda_{\ni}}(t)\,d\mu(t).$$

Theorem 1 (PSY)

For each $\theta \in [0,1]$ we have

$$m_\theta = \inf_{\mu\in\Lambda_\theta(E)} \sup_{\lambda\in\Lambda_{1-\theta}(\Gamma)} M(\mu+\lambda) = \sup_{\lambda\in\Lambda_{1-\theta}(\Gamma)} \inf_{\mu\in\Lambda_\theta(E)} M(\mu+\lambda).$$

Moreover, if μ^*, $|\mu^*| \le \theta$, and λ^*, $|\lambda^*| \le 1-\theta$, are compactly supported positive Borel measures such that

$$m_\theta = M(\mu^*+\lambda^*) = \sup_{\lambda\in\Lambda_{1-\theta}(\Gamma)} M(\mu^*+\lambda)$$

then $\mathrm{supp}(\mu^*) \subseteq E$, $\widehat{\mu^*} = \mu_\theta$, and $\lambda^* = \lambda_\theta$ when $S_\theta \ne \Gamma$ and $\mathrm{supp}(\lambda^*) \subset \mathbb{C}\setminus G$, $\widetilde{\lambda^*} = \lambda_\theta - (1-\theta-|\lambda^*|)\omega_\Gamma$, otherwise.

Recall

$$\chi_n = \inf_{p \in \mathcal{P}_{k_n}} \sup_{q \in \mathcal{P}_{n-k_n}} \frac{\|pq\|_E}{\|pq\|_\Gamma}.$$

Theorem 2 (PSY)

Let $k_n/n \to \theta \in [0, 1]$. Then

$$\lim_{n\to\infty} \left(\frac{1}{n} \log \chi_n \right) = m_\theta.$$

Let $k_n \to \infty$ and $k_n = o(n)$ as $n \to \infty$. Then

$$\lim_{n\to\infty} \left(\frac{1}{k_n} \log \chi_n \right) = -\frac{1}{\operatorname{cap}(E, \Gamma)}.$$

Set

$$G_{\vartheta} := \left\{ z \in \mathbb{C} : \; V_D^{\mu_\vartheta}(z) - g_D(z, \infty) > m_\vartheta \right\}.$$

Then the following theorem takes place.

Theorem 3 (PSY)

Let $k_n/n \to \vartheta \in [0, 1]$, G' be a simply connected domain such that $G \subseteq G' \subseteq G_\vartheta$, and $A_n^\infty = A_n^\infty(G')$. Then

$$\lim_{n\to\infty} \left(\frac{1}{n} \log d_{k_n}(A_n^\infty; C(E)) \right) = m_\vartheta.$$

In particular, when $\vartheta = 0$ and $k_n \to \infty$ as $n \to \infty$, we have that

$$\lim_{n\to\infty} \left(\frac{1}{k_n} \log d_{k_n}(A_n^\infty; C(E)) \right) = -\frac{1}{\operatorname{cap}(E, \Gamma)}.$$