Uniformity of Strong Asymptotics in Angelesco Systems

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Angelesco Systems

Let μ_1, μ_2 be Borel measures on the real line and $\Delta_i = [\alpha_i, \beta_i]$ be the convex hull of the support of μ_i ($\beta_1 < \alpha_2$).

Let $\vec{n} = (n_1, n_2)$ be a multi-index of non-negative integers and $|\vec{n}| = n_1 + n_2$.

Type II multiple orthogonal polynomial $P_{\vec{n}}(x)$ is defined as the minimal degree monic polynomial satisfying

$$\int x^k P_{\vec{n}}(x) d\mu_i(x) = 0, \quad k = \overline{0, n_i - 1}.$$

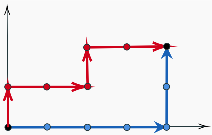
The index $\vec{n}$ is called *normal* if $\deg P_{\vec{n}} = |\vec{n}|$.

Recurrence Relations and Compatibility Conditions (Van Assche)

If $\vec{n}$ and $\vec{n}+\vec{e}_i$ are normal, then

$$xP_{\vec{n}}(x) = P_{\vec{n}+\vec{e}_i}(x) + b_{\vec{n},i}P_{\vec{n}}(x) + a_{\vec{n},1}P_{\vec{n}-\vec{e}_1}(x) + a_{\vec{n},2}P_{\vec{n}-\vec{e}_2}(x).$$

Recurrence relations imply that $P_{\vec{n}}(x)$ can be built in many different ways:



This, in particular, means that the recurrence coefficients cannot be arbitrary. It can be shown that they must satisfy

$$b_{\vec{n}+\vec{e}_1,2} - b_{\vec{n}+\vec{e}_2,1} = b_{\vec{n},2} - b_{\vec{n},1},$$

$$\sum_{k=1}^{2} a_{\vec{n}+\vec{e}_j,k} - \sum_{k=1}^{2} a_{\vec{n}+\vec{e}_i,k} = b_{\vec{n}+\vec{e}_j,i}b_{\vec{n},j} - b_{\vec{n}+\vec{e}_i,j}b_{\vec{n},i},$$

$$a_{\vec{n},i}(b_{\vec{n},j} - b_{\vec{n},i}) = a_{\vec{n}+\vec{e}_j,i}(b_{\vec{n}-\vec{e}_i,j} - b_{\vec{n}-\vec{e}_i,i}).$$

Vector Equilibrium Measures

If μ_i are positive measures, then $P_{\vec{n}}(x) = P_{\vec{n},1}(x)P_{\vec{n},2}(x)$, where $P_{\vec{n},i}(x)$ has all of its zeros on Δ_i. Moreover,

$$\int P^2_{\vec{n},i}(x)|P_{\vec{n},3-i}(x)|d\mu_i(x) = \min_{P(x)=x^{n_i}+\cdots} \int P^2(x)|P_{\vec{n},3-i}(x)|d\mu_i(x).$$

Let $V^{\omega}(x) = -\int \log|x-y|d\omega(y)$. Then,

$$P_{\vec{n},i}(x) = \exp\{-|\vec{n}|V^{\nu_{\vec{n},i}}(x)\},$$

where $\nu_{\vec{n},i}$ is the normalized the counting measure of zeros of $P_{\vec{n},i}(x)$.

If we replace the L^1-norms with the supremum norms, then

$$\min_{\Delta_i} V^{2\nu_{\vec{n},i}+\nu_{\vec{n},3-i}}(x) = \max_{\nu(P)} \min_{\Delta_i} V^{2\nu(P)+\nu_{\vec{n},3-i}}(x).$$

Vector Equilibrium Measures

Let $c \in (0,1)$ and $\mathcal{N}_c$ be a ray-sequence of indices such that

$$n_1 = (c + o(1))|\vec{n}| \quad \text{and} \quad n_2 = (1 - c + o(1))|\vec{n}|.$$

In the limit, the potential-theoretic problem becomes:

find $\omega_{c,i}$, $\operatorname{supp} \omega_i \subseteq \Delta_i$, *such that* $|\omega_{c,1}| = c$, $|\omega_{c,2}| = 1 - c$, *and*

$$\min_{\Delta_i} V^{2\omega_{c,i} + \omega_{c,3-i}}(x) = \max_{\omega} \min_{\Delta_i} V^{2\omega + \omega_{c,3-i}}(x).$$

Weak Asymptotics

Theorem (Gonchar-Rakhmanov)

For any $c \in (0,1)$, the pair $(\omega_{c,1}, \omega_{c,2})$ exists, is unique, and it holds that $\operatorname{supp} \omega_{c,1} = [\alpha_1, \beta_{c,1}] =: \Delta_{c,1}$ and $\operatorname{supp} \omega_{c,2} = [\alpha_{c,2}, \beta_2] =: \Delta_{c,2}$.

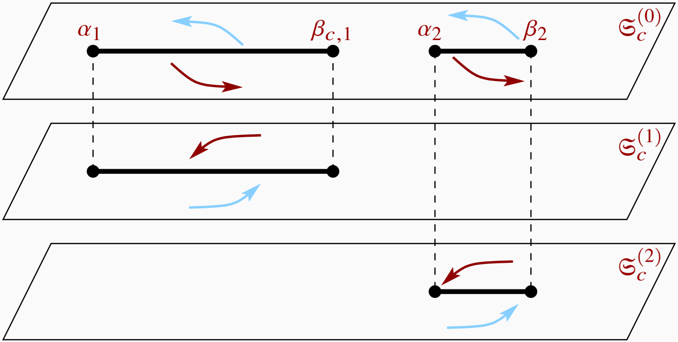
Theorem (Gonchar-Rakhmanov)

Assume that each μ_i is UST-regular on Δ_i. Then it holds that

$$\lim_{\mathcal{N}_c} \frac{n_i}{|\vec{n}|} \log |P_{\vec{n},i}(z)| = -V^{\omega_{c,i}}(z)$$

locally uniformly in $\mathbb{C} \setminus \Delta_i$. Moreover, $n_i \nu_{\vec{n},i} \xrightarrow{*} \omega_{c,i}$ along $\mathcal{N}_c$.

Riemann Surface



Conformal Map

Consider a conformal map $\chi_c : \mathfrak{S}_c \to \overline{\mathbb{C}}$ such that

$$\chi_c^{(0)}(z) = z + \mathcal{O}(1/z) \quad \text{as} \quad z \to \infty.$$

Define

$$\chi_c^{(i)}(z) =: B_{c,i} + A_{c,i}/z + \mathcal{O}(1/z^2) \quad \text{as} \quad z \to \infty.$$

In fact, it holds that

$$z = \chi_c + \frac{A_{c,1}}{\chi_c - B_{c,1}} + \frac{A_{c,2}}{\chi_c - B_{c,2}}.$$

Vector Equilibrium Measures

Let

$$h_c := \chi_c' \left(\frac{c}{\chi_c - B_{c,1}} + \frac{1-c}{\chi_c - B_{c,2}} \right).$$

Then,

$$h_c^{(i)}(z) = \int \frac{d\omega_{c,i}(x)}{x-z}, \quad h_c^{(0)} + h_c^{(1)} + h_c^{(2)} \equiv 0,$$

and respectively

$$d\omega_{c,i}(x) = (h_{c+}^{(i)}(x) - h_{c-}^{(i)}(x)) \frac{dx}{2\pi \mathrm{i}}.$$

Parametrization by c

Proposition (Aptekarev-Denisov-Ya.)

There exist $0 < c^* < c^{**} < 1$ such that $\beta_{c,1}$ is strictly increasing function of $c \in (0, c^*]$ with $\beta_{0+,1} = \alpha_1$, $\beta_{c^*,1} = \beta_1$ and otherwise it is constant; $\alpha_{c,2}$ is strictly increasing function of $c \in [c^{**}, 1)$ with $\alpha_{c^{**},2} = \alpha_2$, $\alpha_{1-,2} = \beta_2$, and otherwise it is constant.

Proposition (Ya.)

There exists $z_c \in (\alpha_1, \beta_2)$ such that

$$(h_c) = \infty^{(0)} + \infty^{(1)} + \infty^{(2)} + z_c^{(0)} - \alpha_1^{(0)} - \beta_{c,1}^{(0)} - \alpha_{c,2}^{(0)} - \beta_2^{(0)}.$$

Moreover, $z_c = \beta_{c,1}$, $c \in (0, c^*]$ and $z_c = \alpha_{c,2}$, $c \in [c^{**}, 1)$.

Parametrization by z_*

Proposition (Denisov-Ya.)

Let $z_\star \in (\alpha_1, \beta_2)$. Define

$$\beta_{\star,1} = \min\{z_\star, \beta_1\} \quad \text{and} \quad \alpha_{\star,2} := \max\{\alpha_2, z_\star\}.$$

Define $h_\star$ to be the rational function on $\mathfrak{S}_\star$ such that

$$h_\star^{(0)}(z) = 1/z + \mathcal{O}(1/z^2) \quad \text{as} \quad z \to \infty$$

and

$$(h_\star) = \infty^{(0)} + \infty^{(1)} + \infty^{(2)} + z_\star^{(0)} - \alpha_1^{(0)} - \beta_{\star,1}^{(0)} - \alpha_{\star,2}^{(0)} - \beta_2^{(0)}.$$

Put

$$h_\star^{(1)}(z) =: -c_\star/z + \mathcal{O}(1/z^2) \quad \text{as} \quad z \to \infty.$$

Then

$$h_\star = h_{c_\star}.$$

Set $c(\vec{n}) := \vec{n}/|\vec{n}|$. Let $\Phi_{\vec{n}}$ be a rational function on $\mathfrak{S}_{c(\vec{n})}$ with the divisor

$$n_1\infty^{(1)} + n_2\infty^{(2)} - |\vec{n}|\infty^{(0)}$$

whose branches multiply to 1. We can write

$$\Phi_{\vec{n}} = \tau_{\vec{n}}(\chi_{c(\vec{n})} - B_{c(\vec{n}),1})^{n_1}(\chi_{c(\vec{n})} - B_{c(\vec{n}),2})^{n_2},$$

where $\tau_{\vec{n}}$ is the normalization constant. Also,

$$\Phi_{\vec{n}}(z) = \exp\left\{|\vec{n}|\left(\int_{\beta_2^{(0)}}^{z} - \int_{\beta_2^{(0)}}^{\beta_2^{(1)}}\right) h_{c(\vec{n})}(x)dx\right\}.$$

It is also true that

$$\log|\Phi_{\vec{n}}^{(i)}(z)| = |\vec{n}|V^{\omega_{c(\vec{n}),i}}(z) + l_{\vec{n},i}.$$

for some constants $l_{\vec{n},i}$.

Szegő Functions

Let $\rho_i(x)$ be a non-vanishing function analytic around Δ_i.

Denote by C_z the discontinuous Cauchy kernel on $\mathfrak{S}_c$: third kind differential with three simple poles, located at z and the other two points with the same natural projection, and the residues 2 at z and -1 at the other points.

$$S_c(z) = \exp\left\{ \frac{1}{6\pi\mathrm{i}} \sum_{i=1}^{2} \int_{\partial \mathfrak{S}_c^{(i)}} \log(\rho_i w_{c,i+}) dC_z \right\},$$

where $w_{c,i}(z) = \sqrt{(z-\alpha_{c,i})(z-\beta_{c,i})}$. Then,

$$S_{c\pm}^{(i)}(x) = S_{c\mp}^{(0)}(x)(\rho_i w_{c,i+})(x), \quad x \in (\alpha_{c,i}, \beta_{c,i}),$$

and it holds that the product of all branches is identically 1. Hence,

$$S_{c+}^{(i)}(x) S_{c-}^{(i)}(x) S_c^{(3-i)}(x) = (\rho_i w_{c,i+})(x), \quad x \in (\alpha_{c,i}, \beta_{c,i}).$$

Proposition (Aptekarev-Denisov-Ya.)

If $(\rho_i w_{c,i+})(x) > 0$ is an integrable function with integrable logarithm on $\Delta_{c,i}$ for each i, then there exist a unique pair of conjugate-symmetric outer functions $S_{c,1}(z), S_{c,2}(z)$ such that

$$|S_{c\pm}^{(i)}(x)|^2 S_c^{(3-i)}(x) = (\rho_i w_{c,i+})(x), \quad x \in (\alpha_{c,i}, \beta_{c,i}).$$

Functions $S_{c,i}(z)$ are standard Szegő functions of densities $s_{c,i}(x)$:

$$\begin{bmatrix} s_{c,1} \\ s_{c,2} \end{bmatrix} = \frac{1}{2}\left(\mathcal{I} + \frac{1}{2}\begin{bmatrix} 0 & H_{\Delta_{c,2}\to\Delta_{c,1}} \\ H_{\Delta_{c,1}\to\Delta_{c,2}} & 0 \end{bmatrix}\right)^{-1}\begin{bmatrix} \log(\rho_1 w_{c,1+}) \\ \log(\rho_2 w_{c,2+}) \end{bmatrix},$$

where $H_{\Delta_{c,i}\to\Delta_{c,j}}$ is the operator of harmonic extension of a function on $\Delta_{c,i}$ followed by the restriction to $\Delta_{c,j}$.

Theorem

Let $d\mu_i(x) = -\rho_i(x)\frac{dx}{2\pi \mathrm{i}}$ where $\rho_i(x)$ is a non-vanishing analytic function around Δ_i. Then,

$$P_{\vec{n},i}(z) = (1 + o(1))\frac{\gamma_{\vec{n},i}}{(S_{c(\vec{n})}\Phi_{\vec{n}})^{(i)}(z)}$$

uniformly in $\mathrm{dist}\{z, (\Delta_{\vec{n}/|\vec{n}|,i})\} \geqslant d$, where $\gamma_{\vec{n},i}$ is a normalization constant. Moreover,

$$\begin{cases} a_{\vec{n},i} & = (1 + o(1))A_{c(\vec{n}),i} \\ b_{\vec{n},i} & = (1 + o(1))B_{c(\vec{n}+\vec{e}_i),i}. \end{cases}$$

The error terms are of order

$$\mathcal{O}\left(\min\{n_1, n_2\}^{-1/3}\right).$$

If indices $c(\vec{n})$ are separated from c^*, c^{**}, then the power $1/3$ can be removed.

Strong Asymptotics

$$P_{\vec{n}}(z) = \gamma_{\vec{n}} \left(1 + \mathcal{O}\left(\min\{n_1, n_2\}^{-1/3}\right)\right) \left(S_{c(\vec{n})} \Phi_{\vec{n}}\right)^{(0)}(z)$$

Ya, 16: along $\mathcal{N}_c$ with $c \in (0,1)$, analytic + Fisher-Hartwig, any size

A-D-Ya, 21: along $\mathcal{N}_c$ with $c \in [0,1]$, analytic + positivity, two measures

A-D-Ya, submitted: along $\mathcal{N}_c$ with $c \in (0,1)$, Szegő, any size

D-Ya, in preparation: along $\mathcal{N}_c$ with $c \in [0,1]$, Szegő, any size

Theorem

Set $R(c) := (c/(1-c))^2(A_{c,2}/A_{c,1})$. Then

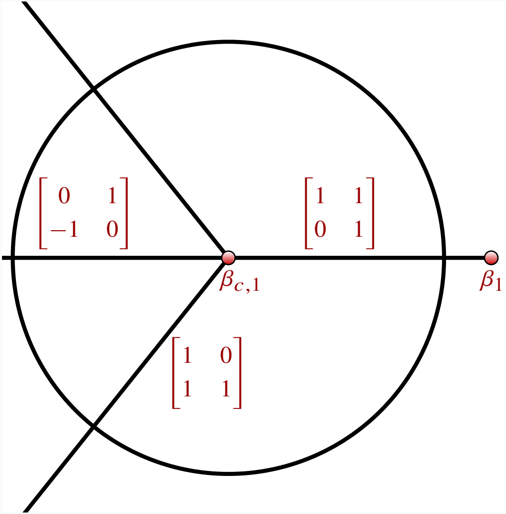
$$R'(c) = \frac{6R(c)(1+R(c))}{1-c^2+c(2-c)R(c)}$$

on $(0, c^*) \cup (c^{**}, 1)$. Moreover, we have that

$$\begin{cases} \dfrac{B'_c}{B_c} = -\dfrac{c}{1-c}\dfrac{A'_{c,1}}{A_{c,1}} = -\dfrac{1-c}{c}\dfrac{A'_{c,2}}{A_{c,2}} = -2\dfrac{1-c-cR(c)}{1-c^2+c(2-c)R(c)}, \\ cB'_{c,1} + (1-c)B'_{c,2} = 0 \Leftrightarrow B'_{c,2} = cB'_c \Leftrightarrow B'_{c,1} = -(1-c)B'_c. \end{cases}$$

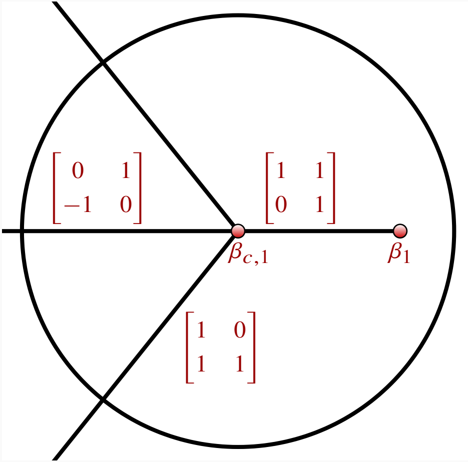
Aptekarev-Kozhan in 2020 derived these equations (in any size) out of compatibility conditions under the assumption on the speed of convergence of the recurrence coefficients (which has not been proven yet).

Soft Edge (Airy)



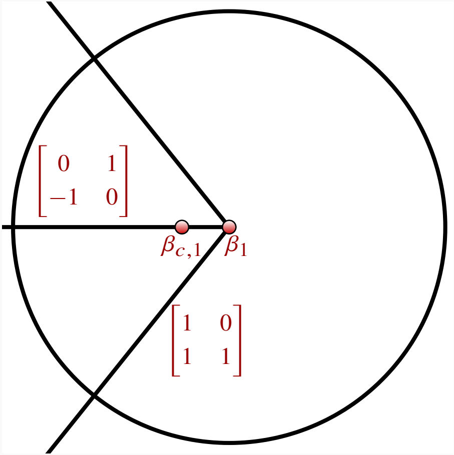
$$\text{(prefactor)} \left(I + \mathcal{O}\left(\frac{1}{\zeta^{3/2}} \right) \right) \exp\left\{ -\frac{2}{3} \zeta^{3/2} \sigma_3 \right\}.$$

Sliding Soft Edge (Perturbed Airy)



$$(\text{prefactor})\left(I + \mathcal{O}\left(\frac{1}{|\zeta|^{1/2}\min\{|\zeta|, \tau\}}\right)\right)\exp\left\{-\frac{2}{3}\zeta^{3/2}\sigma_3\right\}.$$

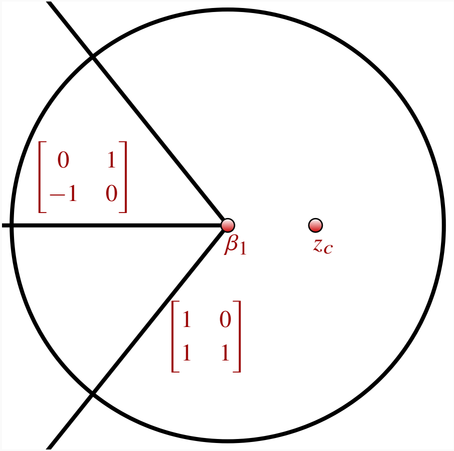
Critical Soft Edge I



$$\text{(prefactor)}\left(I+\mathcal{O}\left(\frac{1}{\zeta^{1/2}}\right)\right)\exp\left\{-\frac{2}{3}(\zeta+s)^{3/2}\sigma_3\right\}$$

locally uniformly in s.

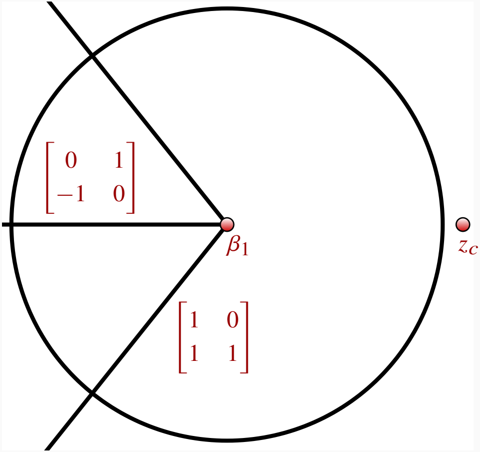
Critical Soft Edge II



$$(\text{prefactor})\left(I + \mathcal{O}\left(\frac{1}{(1-s)\zeta^{1/2}}\right)\right)\exp\left\{-\frac{2}{3}\left(\zeta^{3/2} + s\zeta^{1/2}\right)\sigma_3\right\}$$

uniformly in $s \leqslant 0$.

Hard Edge (Bessel)



$$\text{(prefactor)} \left(I + \mathcal{O}\left(\frac{1}{\zeta^{1/2}} \right) \right) \exp\left\{ 2\zeta^{1/2}\sigma_3 \right\}.$$