Convergent Interpolation to Cauchy Integrals

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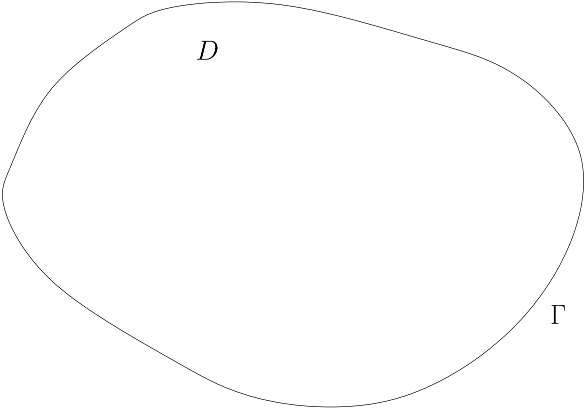
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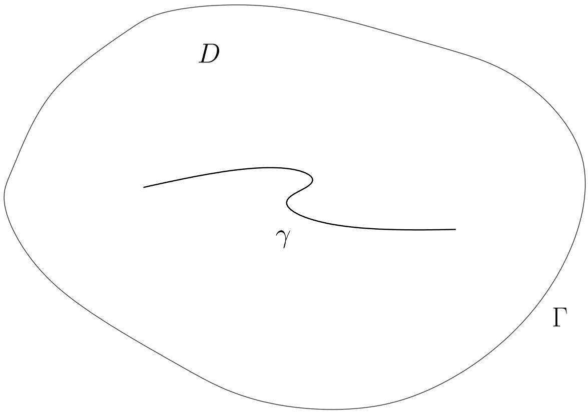
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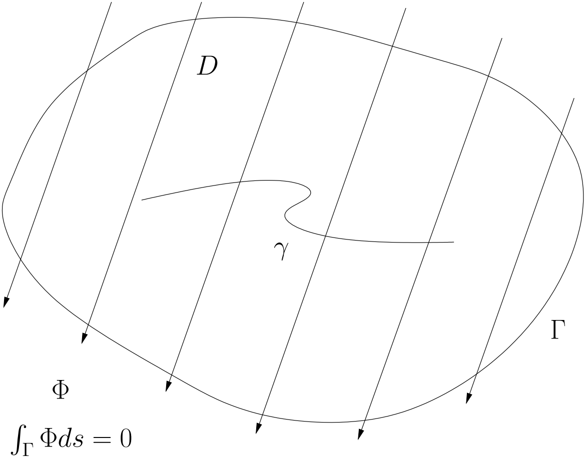
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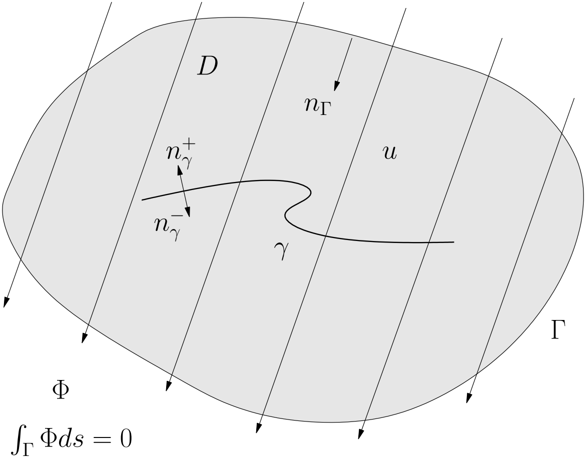
$\int_\Gamma \Phi ds = 0$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$\int_\Gamma \Phi ds = 0$

Let u be the equilibrium distribution of heat or current. Then

$$\begin{cases} \Delta u = 0 & \text{in } D \setminus \gamma \\ \dfrac{\partial u}{\partial n_\Gamma} = \Phi & \text{on } \Gamma := \partial D \\ \dfrac{\partial u^\pm}{\partial n_\gamma^\pm} = 0 & \text{on } \gamma \setminus \{\gamma_0, \gamma_1\} \end{cases},$$

where Δu is the Laplacian of u.

u has well-defined conjugate in $D \setminus \gamma$ and

$$F(\xi) = u(\xi) - i \int_{\xi_0}^{\xi} \Phi \mathrm{d}s, \quad \xi \in \partial D.$$

Further,

$$F(z) = h(z) + \frac{1}{2\pi i} \int_{\gamma} \frac{(F^- - F^+)(t)}{z - t} \mathrm{d}t, \quad z \in D \setminus \gamma,$$

where h is analytic in D and continuous in $\overline{D}$.

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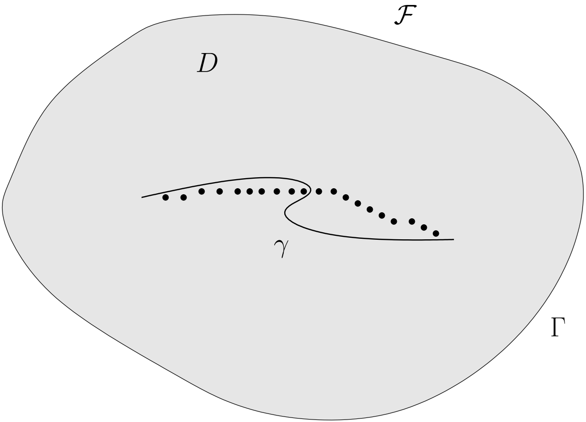
One approximates F on Γ by rational functions with poles in D and observes the asymptotic behavior of their poles as the number of poles grows large.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Let

$$f(z) = \sum_{j=1}^{\infty} \frac{f_j}{z^j}$$

be holomorphic at infinity. A rational function $\pi_n = \frac{p_n}{q_n}$ of type (n, n) is called a diagonal Padé approximant to f of order n if

$$(q_n f - p_n)(z) = O\left(\frac{1}{z^{n+1}}\right) \quad \text{as} \quad z \to \infty.$$

Polynomials q_n and p_n may not be unique, but π_n is. It is characterized by the property

$$(f - \pi_n)(z) = O\left(\frac{1}{z^{2n+1}}\right) \quad \text{as} \quad z \to \infty.$$

π_n has the highest order of tangency with f at infinity.

In 1961, Baker, Gammel, and Willes[1] conjectured that

> if f is meromorphic outside of the unit disk, then
>
> $$\pi_n \to f, \quad n \in \mathbb{N}_1 \subset \mathbb{N},$$
>
> locally uniformly in $\{|z| > 1\} \setminus \{\text{poles of } f\}$.

The conjecture was disproved by Lubinsky[2]. Another, simpler counterexample was constructed by Buslaev[3] who considered a special hyperelliptic function of genus 2.

[1] An investigation of the applicability of the Padé approximant method, J. Math. Anal. Appl. 2, 4005–418, 1961

[2] Rogers-Ramanujan and the Baker-Gammel-Wills (Padé) conjecture, Ann. of Math. 157(3), 847–889, 2003

[3] On the Baker-Gammel-Willes conjecture in the theory of Padé approximants, Math. Sb. 193:6, 25–38, 2002

A polar set is a set that cannot support a single positive Borel measure with finite logarithmic energy. Polar sets are totally disconnected and have area measure zero.

It is said that a property holds quasi everywhere (q.e.) if it holds everywhere except on a polar set.

Let D be an unbounded domain with non-polar boundary. The Green function for D with pole at infinity, $g_D(\cdot, \infty)$, is the unique function such that

(i) $g_D(z, \infty)$ is a positive harmonic function in $D \setminus \{\infty\}$;

(ii) $g_D(z, \infty) - \log|z|$ is bounded near ∞;

(iii) $\lim_{z \to \xi,\, z \in D} g_D(z, \infty) = 0$ for q.e. $\xi \in \partial D$.

Let D be an unbounded domain with non-polar boundary. The Green function for D with pole at infinity, $g_D(\cdot,\infty)$, is the unique function such that

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(ii) $g_D(z,\infty)-\log|z|$ is bounded near ∞;

(iii) $\displaystyle\lim_{z\to\xi,\ z\in D} g_D(z,\infty)=0$ for q.e. $\xi\in\partial D$.

Let F be a non-polar set and D be the unbounded component of the complement of F. The logarithmic capacity of F is defined as

$$\mathrm{cap}(F) := \exp\left\{\lim_{z\to\infty}\left(log|z| - g_D(z,\infty)\right)\right\}.$$

The following result is due to Nuttall[4] and Pommerenke[5].

Theorem

Let f be a meromorphic and single-valued function in $D = \overline{\mathbb{C}} \setminus F$ with F compact and $\mathrm{cap}(F) = 0$. Then for any set $E \subset \mathbb{C}$ and $\epsilon > 0$ we have

$$\lim_{n\to\infty} \mathrm{cap}\left\{z \in E : \; |(f - \pi_n)(z)|^{1/2n} > \epsilon\right\} = 0.$$

In other words, the diagonal Padé approximants π_n converge in capacity to f.

[4] The convergence of Padé approximants of meromorphic functions, J. Math. Anal. Appl. 31, 129–140, 1970

[5] Padé approximants and convergence in capacity, J. Math. Anal. Appl. 41, 775–780, 1973

A tremendous step forward in the investigation of the behavior of Padé approximants was done by Stahl[6].

Theorem

Let f be holomorphic at infinity with all its singularities contained in a compact set F, $\mathrm{cap}(F) = 0$, f is multiple-valued outside of F.

[6] The convergence of Padé approximants to functions with branch points, J. Approx. Theory, 91, 139–204, 1997

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Theorem

Let f be holomorphic at infinity with all its singularities contained in a compact set F, $\mathrm{cap}(F)=0$, f is multiple-valued outside of F. Then

- there exists a domain D_f, unique up to a polar set, such that the sequence $\{\pi_n\}$ converges in capacity to f in D_f;
- if $\widetilde{D}\supset D_f$, $\mathrm{cap}(\widetilde{D}\setminus D)>0$, then $\{\pi_n\}$ does not converge in capacity to f in the whole domain $\widetilde{D}$;
- it holds that

$$|(f-\pi_n)(z)|^{1/2n}\overset{\mathrm{cap}}{\to}\exp\{-g_{D_f}(z,\infty)\}.$$

[6] The convergence of Padé approximants to functions with branch points, J. Approx. Theory, 91, 139–204, 1997

Theorem (Stahl)

D_f is uniquely characterized by the properties:

(i) f is single-valued in D_f;

(ii) $\text{cap}(\partial D_f) \leq \text{cap}(\partial D)$ for any domain D satisfying (i);

(iii) $D_f \supseteq D$ for any domain D satisfying (i) and (ii).

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Observe that

$$|(f - \pi_n)(z)|^{1/2n} \sim \exp\{-g_{D_f}(z,\infty)\} \sim \frac{\mathrm{cap}(\partial D_f)}{|z|}$$

near infinity.

Another fascinating part of Stahl's work is the description of the structure of the extremal domain[7] D_f.

Structure Theorem

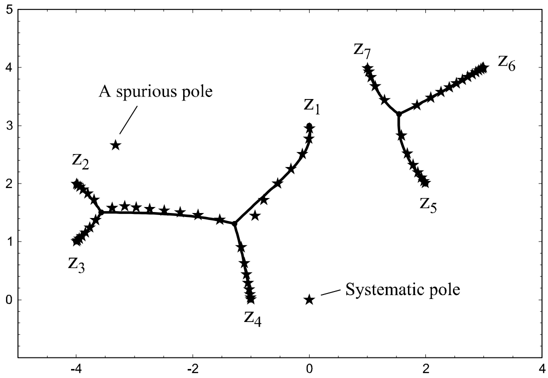
Let $\Delta := \overline{\mathbb{C}} \setminus D_f$. Then Δ has empty interior and

$$\Delta = F_0 \cup \bigcup \Delta_j,$$

where F_0 is a compact polar set, $F_0 \setminus F$ consists of isolated points, and Δ_j are open analytic arc. Moreover, the Green function for D_f possesses the following symmetry property

$$\frac{\partial g_{D_f}(z,\infty)}{\partial n_+} = \frac{\partial g_{D_f}(z,\infty)}{\partial n_-}, \quad z \in \Delta_j.$$

[7] The structure of extremal domains associated with an analytic function, Complex Variables Theory Appl. 4, 339–354, 1985



The poles[8] of Padé approximant π_{63} to function

$$f(z) = \sqrt[4]{\prod_{k=1}^{4}(1 - z_k/z)} + \sqrt[3]{\prod_{k=5}^{7}(1 - z_k/z)}.$$

[8] The picture is taken from H. Stahl, Sets of Minimal Capacity and Extremal Domains, 2006

Let h be an integrable function with compact support. Set

$$f_h(z) := \int \frac{h(t)\mathrm{d}t}{z-t}.$$

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Theorem (Stahl)

Let Δ be as in Structure Theorem and h be a q.e. non-vanishing function on Δ. Then

$$|(f_h - \pi_n)(z)|^{1/2n} \overset{\text{cap}}{\to} \exp\{-g_D(z,\infty)\}$$

in $D := \overline{\mathbb{C}} \setminus \Delta$.

Let D be an unbounded domain and f be a function holomorphic in D. Let also $\mathscr{E} := \{E_n\}$ be an interpolation scheme in D, i.e.,

$E_n \subset D$ consists of $2n$ not necessarily distinct nor finite points.

Denote by v_n the monic polynomial that vanishes at finite points of E_n according to their multiplicity and by ν_n the normalized counting measure of points in E_n.

The n-th diagonal multipoint Padé approximant to f associated with $\mathscr{E}$ is the unique rational function $\Pi_n = p_n/q_n$ satisfying:

- $\deg p_n \leq n$, $\deg q_n \leq n$, and $q_n \not\equiv 0$;
- $(q_n(z)f(z) - p_n(z))/v_n(z)$ is analytic in D;
- $(q_n(z)f(z) - p_n(z))/v_n(z) = O\left(1/z^{n+1}\right)$ as $z \to \infty$.

Let D be a domain with non-polar boundary. The Green function for D with pole at finite $u \in D$, $g_D(\cdot, u)$, is the unique function such that

(i) $g_D(z, u)$ is a positive harmonic function in $D \setminus \{u\}$;

(ii) $g_D(z, \infty) + \log|z - u|$ is bounded near u;

(iii) $\lim_{z \to \xi,\, z \in D} g_D(z, u) = 0$ for q.e. $\xi \in \partial D$.

Let D be a domain with non-polar boundary. The Green function for D with pole at finite $u \in D$, $g_D(\cdot, u)$, is the unique function such that

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Let ν be a probability Borel measure supported in D. The Green potential of ν is given by

$$V_D^\nu(z) := \int g_D(z,u)\mathrm{d}\nu(u).$$

Let compact Δ have connected complement D and be of the form

$$\Delta = F_0 \cup \bigcup \Delta_j,$$

where $\mathrm{cap}(F_0) = 0$ and Δ_j are open analytic arcs.

We say that Δ possesses the symmetry property in the field generated by ν, $\mathrm{supp}(\nu) \subset D$, if

$$\frac{\partial V_D^\nu(z)}{\partial n_+} = \frac{\partial V_D^\nu(z)}{\partial n_-}, \quad z \in \Delta_j.$$

Building on the work of Stahl, Gonchar and Rakhmanov[9] obtained the following result.

Theorem

Assume that

- Δ, as above, possesses the symmetry property in the field generated by ν;
- interpolation scheme $\mathscr{E} = \{E_n\}$ is such that $\nu_n \xrightarrow{*} \nu$;
- h is non-vanishing q.e. on Δ.

Then

$$|f_h - \Pi_n|^{1/2n} \overset{\text{cap}}{\to} \exp\{-V_D^\nu(z)\}.$$

[9] Equilibrium distributions and degree of rational approximation of analytic functions, Math. Sb. 134(176), 306–352, 1987

If you have "correctly shaped" Δ and you happen to know a measure ν that makes it symmetric, then multipoint Padé approximants to Cauchy integrals of non-vanishing densities will converge in capacity outside of Δ.

Two questions:

- Given f_h is there an interpolation scheme $\mathscr{E}$ such that the corresponding multipoint Padé approximants converge?
- Can this convergence be made uniform?

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In what follows, we restrict ourselves to the case of a single arc.

The following work is joint with Laurent Baratchart.

Let Δ be a smooth arc with endpoints ± 1 and $D := \overline{\mathbb{C}} \setminus \Delta$. Set

$$w(z) := \sqrt{z^2 - 1}, \quad w(z)/z \to 1 \quad \text{as} \quad z \to \infty,$$

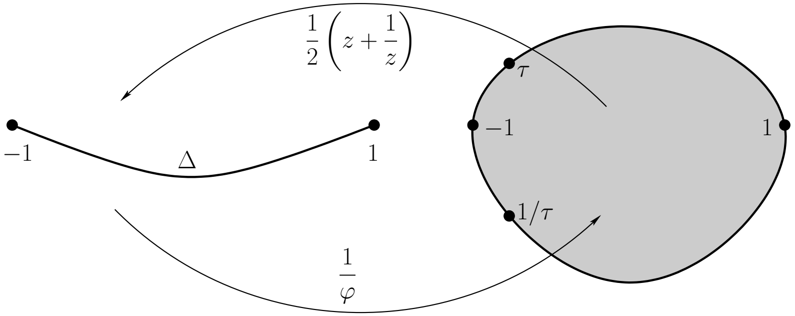
where holomorphic in D branch is selected. Define

$$\varphi(z) := z + w(z), \quad z \in D.$$

Then

$$w^+ = -w^- \quad \text{and} \quad \varphi^+\varphi^- = 1 \quad \text{on} \quad \Delta,$$

where Δ is assumed to be oriented from -1 to 1 and $w^\pm$ and $\varphi^\pm$ are the (unrestricted) boundary values on w and φ.


$\frac{1}{2}\left(z+\frac{1}{z}\right)$
τ
-1
1
-1
Δ
1
$1/\tau$
$\frac{1}{\varphi}$

Let $\mathscr{E} = \{E_n\}$ be an interpolations scheme in D. Associate to each E_n a function

$$r_n(z) := \prod_{e \in E_n} \frac{\varphi(z) - \varphi(e)}{1 - \varphi(z)\varphi(e)}, \quad z \in D.$$

Then

- r_n is holomorphic in D;
- r_n vanishes at each $e \in E_n$;
- $r_n^+ r_n^- = 1$ on Δ.

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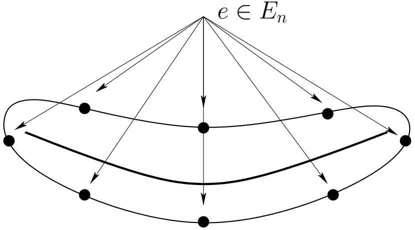
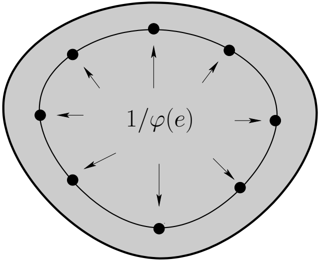
Definition (BY)

We say that Δ is symmetric w.r.t. an interpolation scheme $\mathscr{E}$ if $r_n = o(1)$ locally uniformly in D and $|r_n^\pm| = O(1)$ uniformly on Δ.


$e \in E_n$


$1/\varphi(e)$

Theorem (BY)

Let Δ be a rectifiable Jordan arc with an additional condition near ± 1 (below). Then the following are equivalent:

- $\exists$ an interpolation scheme $\mathscr{E}$, $\cap_n \overline{\cup_{k\geq n} E_k} =: \mathrm{supp}(\mathscr{E}) \subset D$, such that Δ is symmetric with respect to $\mathscr{E}$;
- $\exists$ a positive Borel measure ν, $\mathrm{supp}(\nu) \subset D$, such that Δ is symmetric with respect to ν (in the sense of Stahl);
- Δ is an analytic Jordan arc.

It is assumed that such that for $x = \pm 1$ and all $t \in \Delta$ sufficiently close to x it holds that $|\Delta_{t,x}| \leq \mathrm{const.}|x - t|^{\beta}$, $\beta > 1/2$.

Remarks

- The above theorem covers only the case where $\operatorname{supp}(\mathscr{E})$ is disjoint with Δ;
- The proof of this theorem is constructive. In other words, for a given analytic arc Δ, suitable (not unique) measure ν and scheme $\mathscr{E}$ can be explicitly written in terms of the function Ξ that analytically parametrizes Δ.

Let measure μ be given by

$$d\mu(t) = \frac{h(t)}{w^+(t)} \frac{idt}{\pi}, \quad t \in \Delta.$$

For a non-vanishing Dini-continuous complex-valued function h there exists a constant G_h, called the geometric mean of h, and a function S_h, called the Szegő function of h, such that S_h is analytic and non-vanishing in D, $S_h(\infty) = 1$, and

$$h = G_h S_h^+ S_h^-.$$

Theorem (BY)

Let Δ be a closed analytic Jordan arc symmetric with respect to $\mathscr{E}$ and

$$f_\mu(z) = \int \frac{1}{z-t} \frac{h(t)}{w^+(t)} \frac{idt}{\pi},$$

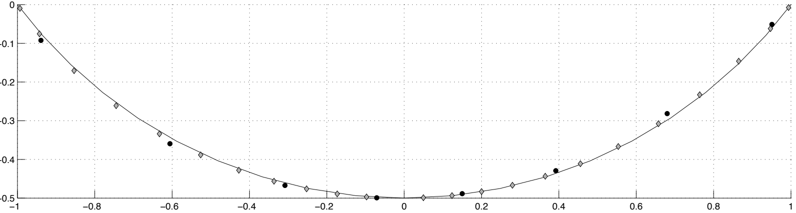
where h is non-vanishing and Dini-continuous on Δ.

If $\{\Pi_n\}$ is the sequence of multipoint Padé approximants to f_μ associated to $\mathscr{E}$, then

$$(f_\mu - \Pi_n)w = [2G_h + o(1)]\ S_h^2\ r_n$$

locally uniformly in D.

$$h(t) = e^t \quad \text{and} \quad E_{2n} = \{\overbrace{0,\ldots,0}^{n \text{ times}}, \overbrace{-4i/3,\ldots,-4i/3}^{n \text{ times}}\}$$



Zeros of q_8 (disks) and q_{24} (diamonds).

The contour F is generated by

$$e_1 := (i-3)/4, \;\; e_2 := (87+6i)/104, \;\text{ and }\; e_3 := -i/10,$$

in the sense that

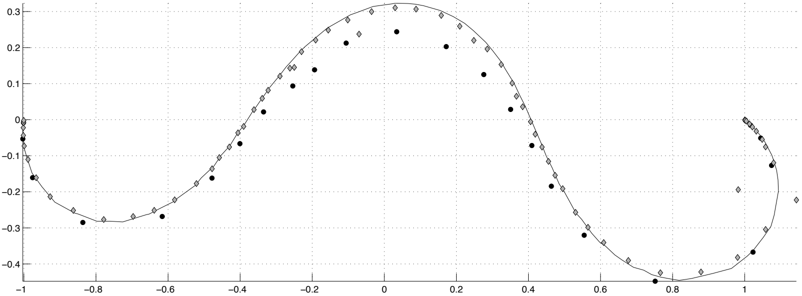
$$\left|(r(e_1;t)r(e_2;t)r(e_3;t))^{\pm}\right| \equiv 1,$$

i.e.,

$$E_{3n} := \{\overbrace{e_1,\ldots,e_1}^{n \text{ times}}, \overbrace{e_2,\ldots,e_2}^{n \text{ times}}, \overbrace{e_3,\ldots,e_3}^{n \text{ times}}\},$$

and is computed numerically.

$$h(t) = \begin{cases} t, & \text{if } \mathrm{Im}(t) \geq 0, \\ \bar{t}, & \text{otherwise.} \end{cases}$$



Zeros of q_{24} (disks) and q_{66} (diamonds).