On $L^2_{\mathbb{R}}$-best rational approximants to Markov functions on several intervals

Maxim L. Yattselev
Indiana University-Purdue University Indianapolis

IUPUI SCHOOL OF SCIENCE
Department of Mathematical Sciences

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Padé Approximation

Let $F(z)$ be a holomorphic at ∞:

$$F(z) = f_0 + \frac{f_1}{z} + \cdots + \frac{f_{2n}}{z^{2n}} + \frac{f_{2n+1}}{z^{2n+1}} + \cdots$$

We would like to find a rational function of type (n, n) such that

$$\frac{P_n(z)}{Q_n(z)} = f_0 + \frac{f_1}{z} + \cdots + \frac{f_{2n}}{z^{2n}} + \mathcal{O}\left(\frac{1}{z^{2n+1}}\right).$$

The right way to approach this problem is to solve the linear system

$$R_n(z) := (Q_n F - P_n)(z) = \mathcal{O}\left(\frac{1}{z^{n+1}}\right).$$

This system always has a non-trivial solution and $[n/n]_F(z) := (P_n/Q_n)(z)$ is unique. It is called the n-th diagonal Padé approximant. The denominator $Q_n(z)$ is normalized to be monic and of minimal possible degree.

Markov Functions

Let μ be a positive Borel measure with infinite compact support in $\mathbb{R}$. Set

$$F_\mu(z) := \int \frac{d\mu(x)}{z-x},$$

which is known as a Markov function. The Cauchy and Fubini-Tonelli theorems yield that

$$\int x^m Q_n(x) d\mu(x) = 0, \quad m \in \{0, \ldots, n-1\}.$$

The denominator of the n-th diagonal Padé approximant is the n-th monic orthogonal polynomial with respect to μ.

Theorem (Markov)

$$F_\mu(z) - [n/n]_{F_\mu}(z) \rightrightarrows 0$$

on closed subsets of the complement of the convex hull of μ.

Szegő Theorem

Let $w(z) := \sqrt{(z-a)(z-b)}$ be holomorphic away from $[a,b]$ and $w(z) \sim z$ around infinity. Measure μ belongs to the Szegő class on $[a,b]$ if

$$\boxed{\int \log \dot{\mu}(x)dx > -\infty, \quad d\mu(x) = \frac{\dot{\mu}(x)dx}{|w(x)|} + d\mu_s(x),}$$

where μ_s is singular to the Lebesgue measure. The Szegő function is given by

$$\boxed{S_{\dot{\mu}}(z) := \exp\left\{\frac{w(z)}{2\pi \mathrm{i}}\int_a^b \frac{\log \dot{\mu}(x)}{x-z}\frac{dx}{w_+(x)}\right\}.}$$

It is a non-vanishing analytic function off $[a,b]$ such that $|S_{\dot{\mu}\pm}(x)|^2 = \dot{\mu}(x)$.

Theorem (Szegő)

$$F_\mu(z) - [n/n]_{F_\mu}(z) = (2+o(1))\frac{S^2_{\dot{\mu}}(z)}{w(z)}\psi^{2n}(z)$$

locally uniformly in $\overline{\mathbb{C}} \setminus [a,b]$, where $\psi(z) = \frac{2}{b-a}\left(z - \frac{b+a}{2} + w(z)\right)$ is the conformal map from $\overline{\mathbb{C}} \setminus [a,b]$ to $\mathbb{D}$ with $\psi(\infty) = 0$ and $\psi'(\infty) > 0$.

Multipoint Padé Approximants

It might be interested to distribute interpolation point according to some rule rather than putting all of them at infinity. Let

$$\boxed{E_n := \{e_1, \dots, e_{2n}\} \subset \overline{\mathbb{C}} \setminus [a,b]}$$

be an interpolation multi-set (interpolation points can coincide). Set

$$\boxed{V_n(z) := \prod_{e \in E_n, e \neq \infty} (z - e).}$$

We would like to find $P_n(z), Q_n(z)$ so that $R_n(z)$ is analytic off $[a,b]$ and

$$\boxed{R_n(z) = \frac{(Q_n F_\mu - P_n)(z)}{V_n(z)} = \mathcal{O}\left(\frac{1}{z^{n+1}}\right).}$$

Again, the solution corresponding to the monic denominator of the smallest degree is unique. We write $[n/n; E_n]_{F_\mu}(z) := (P_n/Q_n)(z)$.

Szegő Asymptotics

As in the case of classical Padé approximants it holds that

$$\boxed{\int x^m Q_n(x)\frac{d\mu(x)}{V_n(x)} = 0, \quad m \in \{0,\dots,n-1\}.}$$

Theorem (Totik / Calle Ysern-López Lagomasino / Stahl)

Let μ be a Szegő measure on an interval $[a,b]$ and $\{E_n\}$ be a sequence of conjugate-symmetric interpolation multi-sets such that

$$\lim_{n\to\infty}\sum_{e\in E_n}\big(1-|\psi(e)|\big) = \infty.$$

Then it holds locally uniformly in $\overline{\mathbb{C}}\setminus[a,b]$ that

$$F_\mu(z) - [n/n;E_n]_{F_\mu}(z) = (2+o(1))\frac{S_\mu^2(z)}{w(z)}\prod_{e\in E_n}\frac{\psi(z)-\psi(e)}{1-\psi(z)\overline{\psi(e)}}.$$

$L^2_{\mathbb{R}}$-best Rational Approximants

Let $\mathcal{R}_n$ be the set of rational functions with real coefficients of type $(n-1,n)$ with all the poles in $\mathbb{D}$ and $f(z)$ be a conjugate-symmetric function analytic off $K \subset \mathbb{D}$, $f(\infty)=0$. A function $r_n(z) \in \mathcal{R}_n$ is $L^2_{\mathbb{R}}$-best rational approximant

$$\boxed{\|f - r_n\|_2 = \inf_{r \in \mathcal{R}_n} \|f - r\|_2,}$$

where $\|h\|_2 = \int_{\mathbb{T}} |h(\tau)|^2 |d\tau|$. The best approximant always exists and has exactly n poles, however, it might not be unique.

The results that follow equally apply to locally best approximants or even critical points in rational approximation problem.

Theorem (Levin)

Let $r_n(z)$ be a critical point and $\{z_1, \dots, z_n\}$ be the poles of $r_n(z)$. Set

$$E_n := \left\{1/\overline{z_1}, 1/\overline{z_1}, 1/\overline{z_2}, 1/\overline{z_2}, \dots, 1/\overline{z_n}, 1/\overline{z_n}\right\}.$$

Then $r_n(z) = [n/n; E_n]_f(z)$.

The Condenser Szegő Function

Let $[a,b]\subset(-1,1)$ and $\tilde{w}(z)=zw(1/z)$. Let μ be a Szegő measure on $[a,b]$.

$$G_{\dot\mu}=\exp\left\{\int\log\dot\mu(x)\frac{\Lambda_{[a,b]}dx}{|(w\tilde{w})(x)|}\right\},$$

where the measure $\Lambda_{[a,b]}dx/|(w\tilde{w})(x)|$ has mass 1 (equilibrium distribution of the condenser $([a,b],\mathbb{T})$). The condenser Szegő function is given by

$$D_{\dot\mu}(z):=\exp\left\{\frac{(w\tilde{w})(z)}{2\pi\mathrm{i}}\int_a^b\frac{1-2xz+x^2}{(x-z)(1-xz)}\log\left(\frac{\dot\mu(x)}{G_{\dot\mu}}\right)\frac{dx}{(w_+\tilde{w})(x)}\right\}.$$

The function $D_{\dot\mu}(z)$ is non-vanishing and analytic in $\overline{\mathbb{C}}\setminus\left([a,b]\cup[a,b]^{-1}\right)$, its argument has zero increment along $\mathbb{T}$ and $|D_{\dot\mu}(\tau)|\equiv1$ for $\tau\in\mathbb{T}$. Moreover, its traces exist almost everywhere on $[a,b]\cup[a,b]^{-1}$ and satisfy

$$\begin{aligned}G_{\dot\mu}|D_{\dot\mu\pm}(x)|^2 &= \dot\mu(x), \quad x\in[a,b],\\ G_{\dot\mu}/|D_{\dot\mu\pm}(x)|^2 &= \dot\mu(1/x), \quad x\in[a,b]^{-1}.\end{aligned}$$

Define

$$\varphi(z) := \exp\left\{\pi\Lambda_{[a,b]}\int_1^z \frac{ds}{(w\tilde{w})(s)}\right\}.$$

This is the conformal map of $\overline{\mathbb{C}} \setminus \left([a,b] \cup [a,b]^{-1}\right)$ onto the annulus $\{z : \varrho < |z| < 1/\varrho\}$, where $\varrho := \varphi(b)$.

Theorem (Baratchart-Stahl-Wielonsky)

Let $\{r_n(z)\}$ be a sequence of critical points in $L^2_{\mathbb{R}}$ rational approximation of $F_\mu(z)$, where μ is a Szegő measure supported in $(-1,1)$. Then

$$F_\mu(z) - r_n(z) = (2G_{\dot{\mu}} + o(1))\frac{D^2_{\dot{\mu}}(z)}{w(z)}\left(\frac{\varrho}{\varphi(z)}\right)^{2n}.$$

holds locally uniformly in $\overline{\mathbb{C}} \setminus \left([a,b] \cup [a,b]^{-1}\right)$.

Measures

Let now $\mathrm{supp}(\mu) = \cup_{i=1}^{g+1}[a_i, b_i] =: \Delta$ and

$$w(z) := \sqrt{(z-a_1)(z-b_1)\cdots(z-a_{g+1})(z-b_{g+1})}$$

be such that $w(z) \sim z^{g+1}$ near infinity. Consider measure of the form

$$d\mu(x) = -\frac{1}{\pi \mathrm{i}}\frac{\rho(x)dx}{w_+(x)},$$

where $\rho(x)$ is real-valued and non-vanishing. Further, let $m(x)$ be a monic polynomial of degree g with exactly one zero in each gap of Δ. Set

$$\dot{\mu}(x) := \rho(x)/m(x).$$

Assume that $\dot{\mu}(x)$ is a positive function and there exists $p > 4$ such that

$$\iint_{\Delta\times\Delta}\left|\frac{\log\dot{\mu}(x) - \log\dot{\mu}(y)}{x-y}\right|^p dxdy < \infty.$$

Given $e \in \mathbb{C} \setminus [a_1, b_{g+1}]$, let $m_e(z)$, be such that

$$\frac{1}{2\pi\mathrm{i}} \int_{|s-e|=r} \frac{m_e(s)}{s-e} \frac{ds}{w(s)} = 1, \quad \int_{b_i}^{a_{i+1}} \frac{m_e(x)}{x-e} \frac{dx}{w(x)} = 0,$$

where $|s-e| = r$ is positively oriented and is exterior to Δ. $m_\infty(z)$ is defined similarly with $(s-e)$ replaced by 1 and $|s| = r$ negatively oriented. Define

$$\boxed{\psi_n(z) := \exp\left\{ \sum_{e \in E_n} \int_{b_{g+1}}^{z} \frac{m_e(s)}{s-e} \frac{ds}{w(s)} \right\},}$$

where again $s - e$ is replaced by 1 if $e = \infty$ and E_n is conjugate-symmetric.

The functions $\psi_n(z)$ is analytic in $\overline{\mathbb{C}} \setminus [a_1, b_{g+1}]$ and a has a zero at each $e \in E_n$ of order equal to the multiplicity of e in E_n. It holds that $|\psi_n(z)| < 1$, $z \notin \Delta$, $|\psi_{n\pm}(x)| \equiv 1$, $x \in \Delta$, and

$$\psi_{n+}(x) = \psi_{n-}(x) e^{-4\pi\mathrm{i}\omega_{n,k}}, \quad x \in (b_k, a_{k+1}),$$

for some constants $\omega_{n,k} \in [0, 1)$ with an explicit integral expressions.

Multi-Interval Szegő Function

Given a measure μ as described, let

$$S_{\dot{\mu}}(z) := \exp\left\{ \frac{w(z)}{2\pi\mathrm{i}} \left[\int_{\Delta} \frac{\log \dot{\mu}(x)}{x-z} \frac{dx}{w_+(x)} - \sum_{i=1}^{g} \int_{b_i}^{a_{i+1}} \frac{2\pi\mathrm{i} c_{\dot{\mu},i}}{y-z} \frac{dy}{w(y)} \right] \right\}$$

for some constants $c_{\dot{\mu},i}$ with an explicit integral expressions.

The function $S_{\dot{\mu}}(z)$ is analytic in $\overline{\mathbb{C}} \setminus [a_1, b_{g+1}]$. It holds that

$$|S_{\dot{\mu}\pm}(x)|^2 = \dot{\mu}(x), \quad x \in \Delta,$$

and

$$S_{\dot{\mu}+}(x) = S_{\dot{\mu}-}(x) e^{-2\pi\mathrm{i} c_{\dot{\mu},k}}, \quad x \in (b_k, a_{k+1}).$$

Adjustment Functions

There exists a family $\{T_n(z)\}$ of analytic and non-vanishing functions that is normal in $\overline{\mathbb{C}} \setminus [a_1, b_{g+1}]$, $|T_{n\pm}(x)| \equiv 1$, $x \in \Delta$, and

$$T_{n+}(x) = T_{n-}(x) e^{4\pi \mathrm{i}(c_{\dot{\mu},k} + \omega_{n,k})}, \quad x \in (b_k, a_{k+1}).$$

Each $T_n(z)$ can be meromorphically continued through each gap (b_k, a_{k+1}) and any continuation will have a simple pole at each zero of $m(x)$ and also an additional simple pole/zero.

The functions $T_n(z)$ are constructed as ratios of Riemann theta functions. The locations of unspecified poles/zeros are determined by a certain Jacobi inversion problem.

Szegő Asymptotics

Theorem (Ya.)

Let μ be as described and $\{E_n\}$ be a sequence of conjugate-symmetric interpolation multi-sets separated from Δ. Then it holds locally uniformly in $\overline{\mathbb{C}} \setminus [a_1, b_{g+1}]$ that

$$F_\mu(z) - [n/n; E_n]_{F_\mu}(z) = (2 + o(1))\big(T_n\psi_n\big)(z)\,\frac{\big(mS_{\dot\mu}^2\big)(z)}{w(z)}.$$

Multi-Interval Condenser Map

Let $\tilde{w}(z) = z^{g+1}w(1/z)$. Define

$$\boxed{\varphi(z) := \exp\left\{\pi \int_1^z \frac{u(s)ds}{(w\tilde{w})(s)}\right\},}$$

where $u(x)$, $\deg(u) = 2g$, is a symmetric polynomial such that

$$\int_\Delta \frac{u(x)dx}{(w_+\tilde{w})(x)} = \mathrm{i}, \quad \int_{b_i}^{a_{i+1}} \frac{u(x)dx}{(w\tilde{w})(x)} = 0.$$

This function is holomorphic in $\overline{\mathbb{C}} \setminus \left([a_1, b_{g+1}] \cup [a_1, b_{g+1}]^{-1}\right)$. The increment of its argument along the unit circle is equal to 2π and $|\varphi(\tau)| \equiv 1$, $\tau \in \mathbb{T}$. Moreover,

$$|\varphi(x)| \equiv \varrho^{\pm 1}, \quad x \in \Delta^{\pm 1}, \quad \varrho := \varphi(b_{g+1}) < 1.$$

Finally, there exist constants ω_k, with explicit integral expressions, such that

$$\varphi_+(x) = \varphi_-(x)e^{-2\pi \mathrm{i}\omega_k}, \quad x \in (b_k, a_{k+1}).$$

Multi-Interval Condenser Szegő Function

Given a sufficiently smooth function $\lambda(x)$, let

$$D_\lambda(z) := \exp\left\{ \frac{(w\tilde{w})(z)}{2\pi\mathrm{i}u(z)} \left[\int_\Delta K(z;x) \log\left(\frac{\lambda(x)}{G_\lambda}\right) \frac{u(x)dx}{(w_+\tilde{w})(x)} - H(z) \right] \right\},$$

where $K(z;x) := (1 - 2xz + x^2)/(x-z)(1-xz)$ and

$$H(z) := \sum_{i=1}^{g} \int_{b_i}^{a_{i+1}} 2\pi\mathrm{i}\kappa_{\lambda,i} K(z;y) \frac{u(y)dy}{(w\tilde{w})(y)}$$

for some constants G_λ and $\kappa_{\dot{\mu},i}$ with an explicit integral expressions.

$D_\lambda(z)$ is holomorphic in $\overline{\mathbb{C}} \setminus \left([a_1, b_{g+1}] \cup [a_1, b_{g+1}]^{-1}\right)$. It has zero increment of its argument along the unit circle and $|D_\lambda(\tau)| \equiv 1, \tau \in \mathbb{T}$. Moreover,

$$G_\lambda |D_{\lambda\pm}(x)|^2 = \lambda(x), \quad x \in \Delta,$$

and $D_\lambda(1/z) = 1/D_\lambda(z)$. Finally, it holds that

$$D_{\lambda+}(x) = D_{\lambda+}(x) e^{-2\pi\mathrm{i}\kappa_{\lambda,k}}, \quad x \in (b_k, a_{k+1}).$$

Szegő Asymptotics

Let $\{r_n(z)\}$ be a sequence of critical points in $L^2_{\mathbb{R}}$ rational approximation of $F_\mu(z)$. Write $r_n(z) = [n/n; E_n](z)$. Let $\{x_{n,1}, \dots, x_{n,g}\}$ be points coming from Jacobi inversion problem associated with μ and E_n. Define

$$m_n(z) = \prod_{k=1}^{g}(z - x_{n,k}) \quad \text{and} \quad B_n(z) := \prod_{\text{poles of } T_n} \frac{z - x_{n,i}}{1 - x_{n,i}z}$$

Theorem (Ya.)

Let $d\mu(x) = -\frac{1}{\pi \mathrm{i}}\frac{\rho(x)dx}{w_+(x)}$, where $|\rho(x)| \in W_1^p, p > 4$. Set

$$\lambda_n(x) := \rho(x)B_n^2(x)/m_n(x)$$

and denote by d_n the number of factors in $B_n(z)$. Then it holds that

$$F_\mu(z) - r_n(z) = (2G_{\lambda_n} + o(1))\frac{m_n(z)}{B_n^2(z)}\frac{D^2_{\lambda_n}(z)}{w(z)}\left(\frac{\varrho}{\varphi(z)}\right)^{2(n-d_n)}$$

locally uniformly in $\overline{\mathbb{C}} \setminus \left([a_1, b_{g+1}] \cup [a_1, b_{g+1}]^{-1}\right)$.