Weighted Extremal Domains and H^2-best Rational Approximants to Algebraic Functions

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Let T be a rectifiable Jordan curve with interior domain G and exterior domain O and $E^p(G)$ be the *Smirnov class* of holomorphic functions in G. The *space of meromorphic functions* of the degree n is defined as

$$E^p_n(G) := E^p(G) + R_n(G)$$

where $R_n(G)$ is the set of rational functions of type $(n-1, n)$ with all their poles in G.

Meromorphic approximation problem consists in the following: given a continuous function f on T, find

$$||f - g_n||_{p,T} = \inf_{g \in E^p_n(G)} ||f - g||_{p,T}.$$

This problem always admits a solution:

- Adamjan, Arov, and Krein[1], $p=\infty$ and $T=\mathbb{T}$;
- Baratchart and Seyfert[2], $p\in[1,\infty)$ and $T=\mathbb{T}$;
- Prokhorov[3] & Baratchart, Mandrèa, Saff, and Wielonsky[4].

The error of approximation is given by the n-th singular number of a certain Hankel operator and the best approximants are described in terms of the corresponding singular vectors.

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[1] Analytic properties of Schmidt pairs for a Hankel operator on the generalized Schur-Takagi problem. *Math. USSR Sb.*, 15: 31-73, 1971

[2] An L^p analog of AAK theory for $p\geq 2$. *J. Funct. Anal.*, 191(1): 52-122, 2002

[3] On L^p-generalization of a theorem of Adamyan, Arov, and Krein. *Comput. Methods Funct. Theory*, 1(2): 501-520, 2001

[4] 2-D inverse problems for the Laplacian: a meromorphic approximation approach. J. Math. Pures Appl., 86:1–41, 2006.

When $T = \mathbb{T}$, write $f = f_+ + f_-$, where f_+ is the *analytic projection* of f and f_- is the *anti-analytic projection* of f. Let $g_n = g_{n+} + r_n$, where $r_n \in R_n(\mathbb{D})$, be a best approximant for f in MAP with $p = 2$. Then

$$||f - g_n||_2^2 = ||f_+ - g_{n+}||_2^2 + ||f_- - r_n||_2^2.$$

Therefore, we arrive at the *rational approximation problem*: given f holomorphic outside of $\mathbb{D}$ and vanishing at infinity, find

$$||f - r_n||_2 = \inf_{r \in R_n(\mathbb{D})} ||f - r||_2.$$

$r \in R_n(\mathbb{D})$ is called a *critical point* in RAP$_2$ for f if $D\Theta_{f,n}(r) = 0$, where $\Theta_{f,n}(r) := \|f - r\|_2^2$. A critical point r_n is called *irreducible* if r_n has exactly n poles. (It is known that all best and locally best rational approximants are always irreducible critical points.)

Let $r_n = p_{n-1}/q_n$ be an irreducible critical point. Then r_n interpolates f at the reflections of the zeros of q_n with order 2 in the Hermite sense[5]. In other words, r_n is a *multipoint Padé approximant* with the implicitly defined interpolation set.

Irreducible critical points converge to f in the complement of $\overline{\mathbb{D}}$. Can we *extend* the domain of convergence knowing analytic continuation properties of f in $\mathbb{D}$?

[5] A.L. Levin. The distribution of poles of rational functions of best approximation and related questions. Math. USSR Sbornik, 9(2):267–274, 1969.

We say that $f \in \mathscr{A}(G)$ if

- f admits holomorphic and single-valued continuation from infinity to an open neighborhood of $\overline{O}$;
- f admits meromorphic continuation along any arc in $\overline{G} \setminus E_f$ starting from T, where E_f is a finite set of points in G;
- E_f is non-empty, the meromorphic continuation of f from infinity has a branch point at each element of E_f.

We say that $f \in \mathscr{A}$ if it belongs to $f \in \mathscr{A}(G)$ for some G.

Given $f \in \mathscr{A}$ and a triangular scheme $\{E_n\}$ with $|E_n| = 2n$, the n-th diagonal Padé approximant to f associated with $\{E_n\}$ is the unique rational function $\Pi_n = p_n/q_n$ such that $\deg p_n \leq n$, $\deg q_n \leq n$, $q_n \not\equiv 0$, and the ratio

$$\frac{q_n(z)f(z) - p_n(z)}{v_n(z)}$$

has an analytic extension to $\overline{\mathbb{C}} \setminus E_f$ and behaves like $O\left(1/z^{n+1}\right)$ as $z \to \infty$, where

$$v_n(z) := \prod_{e \in E_n} \begin{cases} (z-e), & |e| \leq 1, \\ (1-z/e), & |e| > 1. \end{cases}$$

Padé approximants are called *classical* if $v_n \equiv 1$ for all n.

We say that a compact K is *admissible* for $f \in \mathscr{A}$ if $\overline{\mathbb{C}} \setminus K$ is connected and f has meromorphic and single-valued extension there.

An admissible set K is a *smooth cut* for f if $K = E_0 \cup E_1 \cup \bigcup \gamma_j$, where

- $\bigcup \gamma_j$ is a finite union of open analytic arcs such that the jump of f is not identically zero across any of them;
- $E_0 \subseteq E_f$ and each point in E_0 is the endpoint of exactly one γ_j;
- E_1 is a finite set of points each element of which is the endpoint of at least three arcs γ_j.

Theorem (Stahl[6,7])

Given $f \in \mathscr{A}$, there exists a unique admissible compact Γ_* such that $\mathrm{cp}(\Gamma_*) \leq \mathrm{cp}(K)$ for any admissible K and $\Gamma_* \subset \Gamma$ for any admissible Γ satisfying $\mathrm{cp}(\Gamma) = \mathrm{cp}(\Gamma_*)$. The set Γ_* is a smooth cut for f and

$$\frac{\partial g_{D_*}}{\partial \mathbf{n}^+} = \frac{\partial g_{D_*}}{\partial \mathbf{n}^-}$$

where g_{D_*} is the Green's function for $D_* := \overline{\mathbb{C}} \setminus \Gamma_*$ with pole at infinity and $\partial / \partial \mathbf{n}^\pm$ are the partial derivatives with respect to the one-sided normals on each γ_j.

[6] Extremal domains associated with an analytic function. I, II. Complex Variables Theory Appl., 4:311–324, 325–338, 1985.

[7] Structure of extremal domains associated with an analytic function. Complex Variables Theory Appl.,4:339–356, 1985.

Theorem (Stahl[8])

Let $f \in \mathscr{A}$ and $\{\Pi_n\}$ be the sequence of classical Padé approximants to f. Then

$$|f - \Pi_n|^{1/2n} \xrightarrow{\text{cp}} \exp\left\{-g_{D_*}\right\}$$

in D_*, and the counting measures of poles of Π_n converge weak to the* **logarithmic equilibrium distribution** *on Γ_*.*

[8] The convergence of Padé approximants to functions with branch points. J. Approx. Theory, 91:139–204, 1997.

Let ν be a probability Borel measure supported in $\overline{\mathbb{D}}$. Set

$$U^\nu(z) := -\int \log|1 - z\bar{u}| \mathrm{d}\nu(u).$$

U^ν is, in fact, a spherically normalized logarithmic potential of ν^*, where

$$\nu^*(B) = \nu(\{z : 1/z \in B\}).$$

Thus, U^ν is harmonic outside of $\mathrm{supp}(\nu^*)$, in particular, $\mathbb{D}$. When $\nu = \delta_0$ is the Dirac delta at the origin, $U^\nu \equiv 0$ and $\nu^* = \delta_\infty$.

Let $K \subset \mathbb{D}$ be non-polar. For a Borel measure ω, set

$$I_\nu[\omega] := \int \log \frac{1}{|x-y|} \mathrm{d}\omega(x)\mathrm{d}\omega(y) - 2\int U^\nu \mathrm{d}\omega.$$

The *ν-capacity* of K is defined by

$$\mathrm{cp}_\nu(K) := \exp\left\{ -\inf I_\nu[\omega] \right\}$$

where the infimum is taken over all probability Borel measures supported on K.

Theorem

Given $f \in \mathscr{A}(\mathbb{D})$*, there exists a unique admissible compact* Γ_ν*, minimal set for Problem* (f,ν)*, such that* $\mathrm{cp}_\nu(\Gamma_\nu) \leq \mathrm{cp}_\nu(K)$ *for any admissible* K *and* $\Gamma_\nu \subset \Gamma$ *for any admissible* Γ *satisfying* $\mathrm{cp}_\nu(\Gamma) = \mathrm{cp}_\nu(\Gamma_\nu)$*. The set* Γ_ν *is a smooth cut for* f *and*

$$\frac{\partial V_{D_\nu}^{\nu^*}}{\partial \mathbf{n}^+} = \frac{\partial V_{D_\nu}^{\nu^*}}{\partial \mathbf{n}^-}$$

where $\partial/\partial \mathbf{n}^\pm$ *are the partial derivatives with respect to the one-sided normals on each* γ_j^ν*,* $D_\nu := \overline{\mathbb{C}} \setminus \Gamma_\nu$*, and*

$$V_{D_\nu}^{\nu^*}(z) = \int g_{D_\nu}(z,u)\mathrm{d}\nu^*(u)$$

is the ***Green's potential*** *of* ν^* *in* D_**.*

[9] Weighted extremal domains and best rational approximation. Adv. Math. 229, 357–407, 2012

It is enough to consider only admissible sets that are unions of a finite number of disjoint continua each of which contains at least two point of E_f.

The weighted energy functional I_ν is finite and continuous on the Hausdorff closure of the above sets contained in $\overline{\mathbb{D}}_\rho$, $\rho := \max_{z\in E_f} |z|$.

Radial projection onto $\overline{\mathbb{D}}_\rho$ decreases ν-capacity. As the Hausdorff closure is compact, Γ_ν exists.

Using the connection between the weighted energy and the Green's energy of $\tilde{\nu}^*$ over the corresponding domain and the connection between the later and the Dirichlet integral of the Green's potential of $\tilde{\nu}^*$, one shows that the *symmetry property* uniquely characterizes Γ_ν.

Theorem (adjustment of the proof of Gonchar and Rakhmanov[10])

Let $f \in \mathscr{A}(\mathbb{D})$ and $\{\Pi_n\}$ be a sequence of rational interpolants to f whose interpolation points are distributed asymptotically as ν^* for a probability Borel measure ν supported in $\overline{\mathbb{D}}$. Then

$$|f - \Pi_n|^{1/2n} \stackrel{\text{cp}}{\to} \exp\left\{-V_{D_\nu}^{\nu^*}\right\}$$

in $D_\nu \setminus \operatorname{supp}(\nu^*)$, and the counting measures of poles of Π_n converge weak* to $\widehat{\nu^*}$, the balayage of ν^* onto Γ_ν.

[10] Equilibrium distributions and the degree of rational approximation of analytic functions. Mat. Sb., 134(176)(3):306–352, 1987

Let $K \subset G$ be compact and non-polar. There exists the unique measure $\omega_{(K,T)}$, the *Green's equilibrium distribution on K relative to G*, such that

$$\int g_G(x,y)\mathrm{d}\omega_{(K,T)}(x)\mathrm{d}\omega_{(K,T)}(y) \leq \int g_G(x,y)\mathrm{d}\omega(x)\mathrm{d}\omega(y)$$

for any probability Borel measure ω supported on K.

The quantity

$$\mathrm{cp}(K,T) := \left(\int g_G(x,y)\mathrm{d}\omega_{(K,T)}(x)\mathrm{d}\omega_{(K,T)}(y) \right)^{-1}$$

is called the *condenser capacity of K relative to G*. It is known that

$$\mathrm{cp}(K,T) = \mathrm{cp}(T,K).$$

Theorem (Stahl[5,6,8])

Given $f \in \mathscr{A}(G)$, there exists a unique admissible compact $K_\circ$ such that $\mathrm{cp}(K_\circ, T) \leq \mathrm{cp}(K, T)$ for any admissible K and $K_\circ \subset K$ for any admissible K satisfying $\mathrm{cp}(K, T) = \mathrm{cp}(K_\circ, T)$. The set $K_\circ$ is a smooth cut for f and

$$\frac{\partial}{\partial \mathbf{n}^+} V_{\overline{\mathbb{C}}\setminus K_\circ}^{\omega_{(T,K_\circ)}} = \frac{\partial}{\partial \mathbf{n}^-} V_{\overline{\mathbb{C}}\setminus K_\circ}^{\omega_{(T,K_\circ)}}$$

where $\omega_{(T,K_\circ)}$ is the Green's equilibrium distribution on T relative to $\overline{\mathbb{C}} \setminus K_\circ$. The above symmetry property uniquely characterizes $K_\circ$.

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[5] Extremal domains associated with an analytic function. I, II. Complex Variables Theory Appl., 4:311–324, 325–338, 1985.

[6] Structure of extremal domains associated with an analytic function. Complex Variables Theory Appl.,4:339–356, 1985.

[8] Weighted extremal domains and best rational approximation. Adv. Math. 229, 357–407, 2012

Theorem

Let $f \in \mathscr{A}(\mathbb{D})$ *and* $\{r_n\}$ *be a sequence of irreducible critical points in RAP for* f*. Then*

$$|f - r_n|^{1/2n} \xrightarrow{\text{cp}} \exp\left\{-V_{\overline{\mathbb{C}}\setminus K_\circ}^{\omega^*_{(K_\circ,\mathbb{T})}}\right\}$$

in $\overline{\mathbb{C}} \setminus (K_\circ \cup K_\circ^*)$*, and the counting measures of poles of* r_n *converge weak** *to* $\omega_{(K_\circ,\mathbb{T})}$*. Moreover, it holds that*

$$\lim_{n\to\infty} \|f - r_n\|_2^{1/2n} = \lim_{n\to\infty} \|f - r_n\|_{\mathbb{T}}^{1/2n} = \exp\left\{-\frac{1}{\text{cp}(K_\circ,\mathbb{T})}\right\}.$$

[9] Weighted extremal domains and best rational approximation. Adv. Math. 229, 357–407, 2012

Take a weak* limit point of counting measures of poles of r_n, say ν.

r_n are multipoint Padé approximants corresponding to an interpolation scheme which is asymptotically distributed as ν^*.

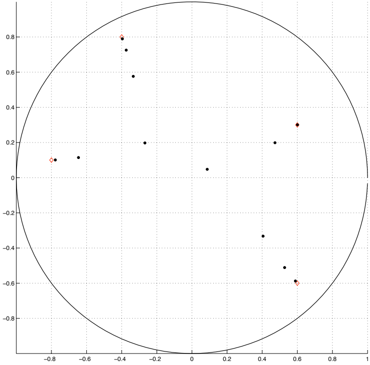
The counting measures of the poles of r_n converge weak* to $\widehat{\nu^*}$ on Γ_ν.

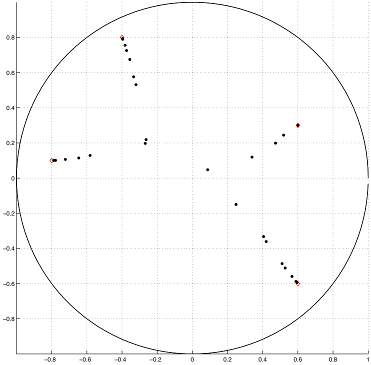
Equality $\nu = \widehat{\nu^*}$ implies that $\nu = \omega_{(\Gamma_\nu, \mathbb{T})}$.

$V_{D_\nu}^{\widetilde{\nu}}$ enjoys the same symmetry property as $V_{\overline{\mathbb{C}}\setminus K_\circ}^{\omega_{(\mathbb{T}, K_\circ)}}$ that uniquely characterizes $K_\circ$, where $\widetilde{\nu}$ is the balayage of ν onto $\mathbb{T}$.

$$f_1(z) = \frac{1}{\sqrt[4]{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}} + \frac{1}{z-z_1},$$

where $z_1 = 0.6 + 0.3i$, $z_2 = -0.8 + 0.1i$, $z_3 = -0.4 + 0.8i$, $z_4 = 0.6 - 0.6i$, and $z_5 = -0.6 - 0.6i$.

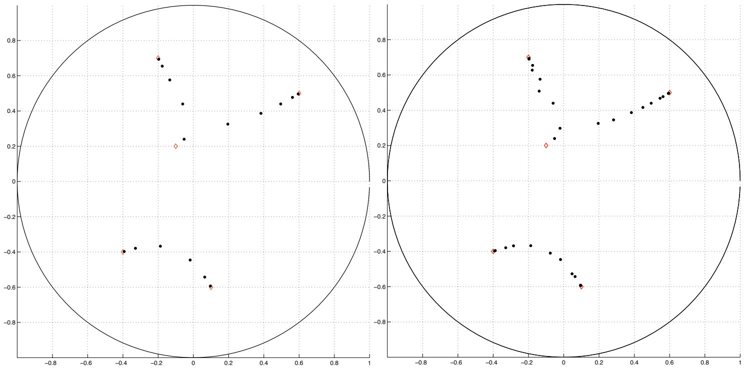




The poles of rational approximants to f_1 of degree 12 and both 12 and 16.

$$f_2(z) = \frac{1}{\sqrt[3]{(z-z_1)(z-z_2)(z-z_3)}} + \frac{1}{\sqrt{(z-z_4)(z-z_5)}},$$

where $z_1 = 0.6 + 0.5i$, $z_2 = -0.1 + 0.2i$, $z_3 = -0.2 + 0.7i$, $z_4 = -0.4 - 0.4i$, and $z_5 = 0.1 - 0.6i$.



The poles of rational approximants to f_2 of degree 16 and both 13 and 16.

Theorem

Let $f \in \mathscr{A}(G)$ and $\{g_n\}$ be a sequence of best approximants in MAP_2 for f. Then

$$|f - g_n|^{1/2n} \xrightarrow{\text{cp}} \exp\left\{ V_G^{\omega_{(K_\circ,T)}} - \frac{1}{\text{cp}(K_\circ, T)} \right\}$$

in $G \setminus K_\circ$ and the counting measures of poles of g_n converge weak to $\omega_{(K_\circ,T)}$.*

[9] Weighted extremal domains and best rational approximation. Adv. Math. 229, 357–407, 2012

Theorem

Let $f \in \mathscr{A}(G)$. *Then*

$$\lim_{n\to\infty} \rho_{n,2}^{1/2n}(f,T) = \lim_{n\to\infty} \rho_{n,\infty}^{1/2n}(f,T) = \exp\left\{-\frac{1}{\mathrm{cp}(K_\circ, T)}\right\}$$

where

$$\rho_{n,p}(f,T) := \inf\left\{\|f-r\|_{p,T} :\ r \in R_n(G)\right\}.$$

[9] Weighted extremal domains and best rational approximation. Adv. Math. 229, 357–407, 2012