# Asymptotics of Padé approximants to a certain class of elliptic-type functions

Maxim Yattselev
University of Oregon, Eugene, OR
joint work with
Laurent Baratchart
INRIA, Sophia Antipolis, France

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Let

$$f(z) = \sum_{j=1}^{\infty} \frac{f_j}{z^j}$$

be holomorphic at infinity. A rational function $\pi_n = \frac{p_n}{q_n}$ of type $(n, n)$ is called the diagonal Padé approximant to $f$ of order $n$ if

$$(q_n f - p_n)(z) = O\left(\frac{1}{z^{n+1}}\right) \quad \text{as} \quad z \to \infty.$$

Polynomials $q_n$ and $p_n$ may not be unique, but $\pi_n$ is. It is characterized by the property

$$(f - \pi_n)(z) = O\left(\frac{1}{z^{2n+1}}\right) \quad \text{as} \quad z \to \infty.$$

That is, $\pi_n$ has the highest order of tangency with $f$ at infinity.

A tremendous step forward in the investigation of the behavior of Padé approximants was done by Stahl[1].

**Theorem (Stahl)**

Let $f$ be holomorphic at infinity, multiple-valued, and with all its singularities contained in a compact set $F$, $\mathrm{cap}(F) = 0$.

[1] The convergence of Padé approximants to functions with branch points, J. Approx. Theory, 91, 139–204, 1997

A tremendous step forward in the investigation of the behavior of Padé approximants was done by Stahl[1].

**Theorem (Stahl)**

Let $f$ be holomorphic at infinity, multiple-valued, and with all its singularities contained in a compact set $F$, $\mathrm{cap}(F) = 0$. Then

- there exists a domain $D$, unique up to a polar set, such that the sequence $\{\pi_n\}$ converges in capacity to $f$ in $D$;
- $\Delta := \overline{\mathbb{C}} \setminus D$ has empty interior and consists ''essentially'' of analytic arcs.

$\Delta$ is said to be the set of minimal capacity for $f$ as it has the smallest logarithmic capacity among all compacts that make $f$ single-valued in their complement.

[1] The convergence of Padé approximants to functions with branch points, J. Approx. Theory, 91, 139–204, 1997

Let $h$ be an integrable function with compact support. Set

$$f_h(z) := \frac{1}{\pi i} \int \frac{h(t)\mathrm{d}t}{t - z}.$$

Such a function is called the Cauchy integral of $h$.

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**Theorem (Stahl)**

Let $\Delta$ be a set of minimal capacity and $h$ be a q.e. non-vanishing function on $\Delta$. Then the sequence $\{\pi_n\}$ converges in capacity to $f_h$ in $D$.

Recall that

$$q_n(z)f_h(z) - p_n(z) = \mathcal{O}\left(1/z^{n+1}\right) \quad \text{as} \quad z \to \infty.$$

Hence,

$$\begin{aligned} 0 &= \oint_\Gamma z^k (q_n f_h - p_n)(z)\mathrm{d}z, \\ &= \oint_\Gamma z^k q_n(z) f_h(z)\mathrm{d}z, \\ &= 2\int_\Delta t^k q_n(t)h(t)\mathrm{d}t \quad k \in \{0, \dots, n-1\}, \end{aligned}$$

where $\Gamma$ is any positively oriented Jordan curve encompassing $\Delta$.

Let $\Delta = [-1,1]$ and $D := \overline{\mathbb{C}} \setminus \Delta$. Set

$$w(z) := \sqrt{z^2-1}, \quad w(z)/z \to 1 \quad \text{as} \quad z \to \infty,$$

where holomorphic in $D$ branch is selected. Define

$$\varphi(z) := z + w(z), \quad z \in D.$$

Then $\varphi$ is the conformal map of $D$ onto $\{|z|>1\}$, $\varphi(\infty)=\infty$, and $\varphi'(\infty) > 0$.

Let $h$ be Dini-continuous non-vanishing complex-valued function on $[-1,1]$. Then there exists a function $S$, called the Szegő function of $h$, such that $S$ is analytic and non-vanishing in $D$, $S(\infty)=1$, and

$$h = GS^{+}S^{-},$$

where $G$ is the geometric mean of $h$, i.e.,

$$G := \exp\left\{\int_{[-1,1]} \log h(t) \frac{i\mathrm{d}t}{\pi w^{+}(t)}\right\}.$$

Observe that $\frac{i\mathrm{d}t}{\pi w^{+}(t)}$ is the equilibrium measure on $[-1,1]$.

Using Nuttall's method of singular integral equations[2] one can prove:

**Theorem (is the error rate known?)**

Let $h$ be Dini-continuous non-vanishing complex-valued function on $[-1,1]$ and

$$f_h(z) := \frac{1}{\pi i}\int \frac{h(t)}{t-z}\frac{\mathrm{d}t}{w^+(t)}, \quad z \in D.$$

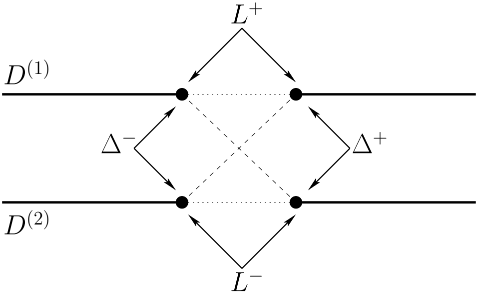
Then it holds locally uniformly in $D$ that

$$(f_h - \pi_n) = \frac{2}{w}\frac{S_n^*}{S_n}\left(1 + \mathcal{O}(\omega_n)\right),$$

where $\omega_n := \min \|1/h - l_n\|_{[-1,1]}$, $\deg(l_n) \leq n$, $S_n := \left(\frac{\varphi}{2}\right)^n \frac{1}{S}$, and $S_n^* = GS\left(\frac{1}{2\varphi}\right)^n$.

[2] Padé polynomial asymptotic from a singular integral equation. Constr. Approx., 6(2):157–166, 1990

Let $\mathfrak{R} = D^{(1)} \cup D^{(2)} \cup L$ be the Riemann surface of $w$ $(g = 0)$.




Domains $D^{(1)}$ and $D^{(2)}$ are represented as upper and lower layers, (two thick horizontal lines each). Each pair of disks joint by a dotted line represents the same point on $\boldsymbol{\Delta}$ as approached from the left ($\boldsymbol{\Delta}^-$) and from the right ($\boldsymbol{\Delta}^+$). Each pair of disk joint by a dashed line represents the same point on $L$ as approached from the left ($L^-$) and from the right ($L^+$). The left and right sides are chosen according to the orientation of each contour in question.

Denote by $\pi$ the canonical projection and set

$$S_n(z^{(1)}) := S_n(z) \quad \text{and} \quad S_n(z^{(2)}) = S_n^*(z).$$

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## Proposition

Let $h$ be a Dini-continuous non-vanishing function on $\Delta$. Then the function $S_n$ has continuous traces on both sides of $L$ that satisfy

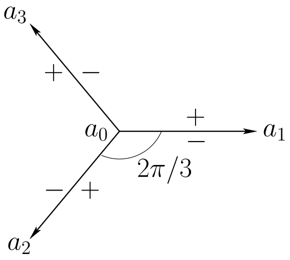
$$S_n^- = S_n^+ \cdot (h \circ \pi). \tag{1}$$

Moreover, under the normalization

$$S_n(z^{(1)})z^{-n} \to 1 \quad \text{as} \quad z^{(1)} \to \infty^{(1)},$$

$S_n$ is the unique function meromorphic in $\mathfrak{R} \setminus L$ with the principle divisor $n\infty^{(2)} - n\infty^{(1)}$ and continuous traces on $L$ that satisfy (1).

Let $a_1$, $a_2$, and $a_3$ be three non-collinear points in the complex plane $\mathbb{C}$. There exists a unique connected compact $\Delta$, called Chebotarëv continuum, containing these points that has minimal logarithmic capacity among all continua joining $a_1$, $a_2$, and $a_3$.




It consists of three analytic arcs $\Delta_k$, $k \in \{1, 2, 3\}$, that emanate from a common endpoint, say $a_0$, and end at each of the given points $a_k$, respectively. It is also known that the tangents at $a_0$ of two adjacent arcs form an angle of magnitude $2\pi/3$.

Set

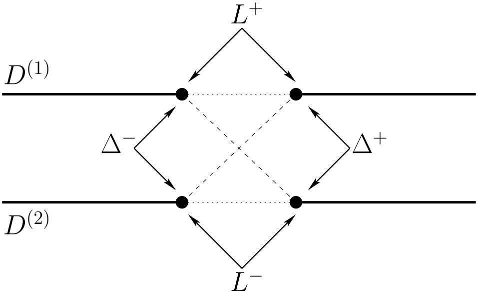
$$w(z) := \sqrt{\prod_{k=0}^{3}(z-a_k)}, \quad \frac{w(z)}{z^2} \to 1 \quad \text{as} \quad z \to \infty,$$

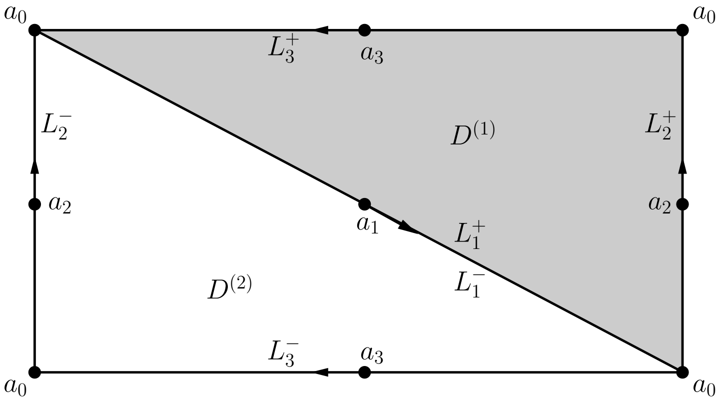
to be a holomorphic function in $D \setminus \{\infty\}$.

Define $\varphi$ to be the conformal map of $D$ onto $\{|z| > 1\}$ such that

$$\varphi(z) = \frac{z}{\mathrm{cap}(\Delta)} + \cdots .$$

Let $\mathfrak{R} = D^{(1)} \cup D^{(2)} \cup L$ be the Riemann surface of $w$ $(g = 1)$, $L := L_1 \cup L_2 \cup L_3$, $\pi(L_k) = \Delta_k$.






Elliptic Riemann surface $\mathfrak{R}$ has genus 1 and therefore is homeomorphic to a torus. We represent $\mathfrak{R}$ as a torus cut along curves $L_2$ and $L_3$. In this case domains $D^{(1)}$ and $D^{(2)}$ can be represented as the upper and lower triangles, respectively.

## Proposition (BY)

Let $h$ be a Dini-continuous non-vanishing function on $\Delta$. Then there exists $\mathbf{z}_n \in \mathfrak{R}$ such that $\mathbf{z}_n + (n-1)\infty^{(2)} - n\infty^{(1)}$ is the principle divisor of a function $S_n$ which is meromorphic in $\mathfrak{R} \setminus L$ and has continuous traces on both sides of $L$ that satisfy

$$S_n^- = S_n^+ \cdot (h \circ \pi). \tag{2}$$

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Moreover, under the normalization

$$S_n(z^{(1)}) z^{-k_n} \to 1 \quad \text{as} \quad z^{(1)} \to \infty^{(1)},$$

where $k_n = n-1$ if $\mathbf{z}_n = \infty^{(1)}$ and $k_n = n$ otherwise, $S_n$ is the unique function meromorphic in $\mathfrak{R} \setminus L$ with the principle divisor of the form $\mathbf{w} + (n-1)\infty^{(2)} - n\infty^{(1)}$, $\mathbf{w} \in \mathfrak{R}$, and continuous traces on $L$ that satisfy (2).

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Furthermore, if $\mathbf{z}_n = \infty^{(1)}$ then $\mathbf{z}_{n-1} = \infty^{(2)}$ and $S_n = S_{n-1}$.

## Recall

$$S_n S_n^* = G\,\mathrm{cap}^{2n}([-1,1]).$$

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## Proposition (BY)

It holds that

$$\frac{(S_nS_n^*)(z)}{(\mathrm{cap}(\Delta))^{2n-1}} = \xi_n G \begin{cases} (z-z_n)/|\varphi(z_n)|, & \mathbf{z}_n \in D^{(2)} \setminus \{\infty^{(2)}\}, \\ \mathrm{cap}(\Delta), & \mathbf{z}_n = \infty^{(2)}, \\ (z-z_n)|\varphi(z_n)|, & \mathbf{z}_n \in L \cup D^{(1)} \setminus \{\infty^{(1)}\}, \end{cases}$$

where $|\xi_n| = 1$, $z_n = \pi(\mathbf{z}_n)$, and

$$G := \exp\left\{ \int_\Delta \log h(t) \frac{i(t-a_0)\mathrm{d}t}{\pi w^+(t)} \right\}.$$

Observe that $\frac{i(t-a_0)\mathrm{d}t}{\pi w^+(t)}$ is the equilibrium measure on $\Delta$.

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## Proposition (BY)

Moreover, it holds that

$$\frac{S_n^*(z)}{S_n(z)} = \frac{\xi_n G \Upsilon(\mathbf{z}_n; z)}{\varphi^{2n-1}(z)} \begin{cases} \dfrac{z - z_n}{\varphi(z)|\varphi(z_n)|}, & \mathbf{z}_n \in D^{(2)} \setminus \{\infty^{(2)}\}, \\ 1/\varphi(z), & \mathbf{z}_n = \infty^{(2)}, \\ \dfrac{\varphi(z)|\varphi(z_n)|}{z - z_n}, & \mathbf{z}_n \in L \cup D^{(1)} \setminus \{\infty^{(1)}\}, \end{cases}$$

where $\{\Upsilon(\mathbf{a}; \cdot)\}$, $\mathbf{a} \in \mathfrak{R}$, is a normal family of non-vanishing functions in $D$.

Denote by $\mathbf{Z}$ the derived set of $\{\mathbf{z}_n\}$. The following proposition is essentially due to Suetin[3].

## Proposition

It holds that

- $\mathbf{Z} = \mathfrak{R}$ when the numbers $\omega_\Delta(\Delta_k)$ are rationally independent;
- $\mathbf{Z}$ is a finite set of points when $\omega_\Delta(\Delta_k)$ are rational;
- $\mathbf{Z}$ is the union of a finite number of pairwise disjoint arcs when $\omega_\Delta(\Delta_k)$ are rationally dependent but at least one of them is irrational.

[3] Convergence of Chebyshëv continued fractions for elliptic functions. Mat. Sb., 194(12):63–92, 2003. English transl. in Math. Sb. 194(12):1807–1835, 2003

## Theorem

Let $h$ be a complex-valued Dini-continuous non-vanishing function on $\Delta$ and

$$f_h(z) := \frac{1}{\pi i}\int_\Delta \frac{h(t)}{t-z}\frac{dt}{w^+(t)}, \quad z \in D.$$

Then it holds locally uniformly in $D$ that

$$(f_h - \pi_n) = \frac{2}{w}\frac{S_n^*}{S_n}\frac{1+E_n^*}{1+\mathcal{O}(\delta^n)+E_n},$$

where $E_n$ is a sectionally meromorphic function on $\mathfrak{R}\setminus L$ with at most one pole at $\mathbf{z}_n$,

$$\oint_{\partial D}\left(|E_n|^2 + |E_n^*|^2\right)\left|\frac{dt}{w}\right| \leq \text{const.}\omega_n^2, \quad \mathbf{z}_n \notin L,$$

and $\omega_n := \min\|1/h - l_n\|_\Delta$, $\deg(l_n) \leq n$.