Szegő-type Asymptotics of Frobenius–Padé Approximants

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Let F be a function holomorphic at the origin:

$$F(z) = f_0 + f_1 z + f_2 z^2 + \cdots + f_m z^m + \cdots$$

and P_n be its n-th Taylor polynomial:

$$P_n(z) = f_0 + f_1 z + f_2 z^2 + \cdots + f_n z^n.$$

Then

$$P_n(z) \rightrightarrows F(z) \quad \text{as} \quad n \to \infty$$

in the largest disk of holomorphy of F.

Let P_n and Q_m be polynomials of respective degrees at most n and m such that

$$\big(Q_m F - P_n\big)(z) = \mathcal{O}\big(z^{m+n+1}\big) \quad \text{as} \quad z \to 0.$$

The rational function P_n/Q_m is unique and is called (n,m)-th Padé approximant at the origin.

Theorem (de Montessus de Ballore)

If D is the largest disk centered at the origin where F has exactly m poles counting multiplicities, then

$$(P_n/Q_m)(z) \rightrightarrows F(z) \quad \text{as} \quad n \to \infty$$

in the spherical metric.

Let f be a function holomorphic and vanishing at infinity:

$$f(z) = \frac{f_1}{z} + \frac{f_2}{z^2} + \cdots + \frac{f_n}{z^n} + \cdots .$$

Further, let p_n, q_n be a pair of polynomials of degree at most n such that

$$R_n(z) := \big(q_n f - p_n\big)(z) = \frac{1}{m_n z^{n+1}} + \mathcal{O}\big(z^{-n-2}\big) \quad \text{as} \quad z \to \infty.$$

The rational function p_n/q_n is always unique and is called the n-th diagonal Padé approximant to f at infinity.

Theorem (Markov)

If σ is a compactly supported positive Borel measure on the real line, then

$$(p_n/q_n)(z) \rightrightarrows f(z) = \int \frac{\mathrm{d}\sigma(t)}{t-z} \quad \text{as} \quad n \to \infty$$

locally uniformly in $\overline{\mathbb{C}} \setminus I$, where I is the smallest interval containing $\operatorname{supp}(\sigma)$.

It is easy to show that

$$\int x^i q_n(x)\mathrm{d}\sigma(x) = 0, \quad i \in \{0,\dots,n-1\}.$$

Let $w(z) := \sqrt{z^2-1}$ and $\rho(z)$ be holomorphic and non-vanishing in a neighborhood of $[-1,1]$. By assuming

$$\mathrm{d}\sigma(x) = \frac{\rho(x)}{2\pi\mathrm{i}}\frac{\mathrm{d}x}{w^+(x)}$$

and studying (thanks to Fokas, Its, and Kitaev) the matrices

$$\boldsymbol{Y}_n := \begin{pmatrix} q_n & R_n \\ m_{n-1}q_{n-1} & m_{n-1}R_{n-1} \end{pmatrix}$$

via the steepest descent method of Deift and Zhou, one can get very precise asymptotics of q_n and R_n.

In particular, one deduces Szegő's asymptotics:

$$\begin{cases} q_n(z) & = & (1+o(1))\left(\Phi^n S_\rho\right)(z), \\ (wR_n)(z) & = & (1+o(1))\left(\Phi^n S_\rho\right)^{-1}(z), \end{cases}$$

where $\Phi(z) := z + w(z)$ and

$$S_\rho(z) = \exp\left\{\frac{w(z)}{2\pi\mathrm{i}}\int_{[-1,1]}\frac{\log\rho(x)}{x-z}\frac{\mathrm{d}x}{w^+(x)}\right\},$$

which satisfies $S_\rho^+ S_\rho^- = \rho$ on $(-1,1)$. We can rewrite the asymptotic formulae as

$$\begin{cases} q_n(z) & = & (1+o(1))\left(\Phi^n S\right)^{(0)}(z), \\ (wR_n)(z) & = & (1+o(1))\left(\Phi^n S\right)^{(1)}(z). \end{cases}$$

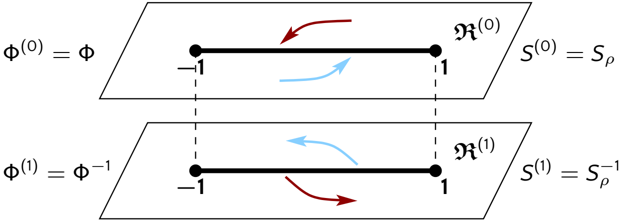
Orthogonality $\int x^i q_n(x)\mathrm{d}\sigma(x) = 0$ tell us that the logarithmic potential

$$V^\tau(z) := -\int \log|z-x|\mathrm{d}\tau(x)$$

of a weak* limit point of the normalized counting measures of the zeros of q_n is such that

$$2V^\tau = \min_{\mathrm{supp}(\mu)} 2V^\tau \quad \text{on} \quad \mathrm{supp}(\tau) \subseteq \mathrm{supp}(\mu).$$

When $\mathrm{supp}(\mu) = [-1,1]$, the measure τ is necessarily the arcsine distribution on $[-1,1]$. Then we construct a Riemann surface $\mathfrak{R}$ using $\mathrm{supp}(\tau) = [-1,1]$ and look for a rational function on $\mathfrak{R}$ with a simple pole at $\infty^{(0)}$ and a simple zero at $\infty^{(1)}$ and a function S that solves a certain boundary value problem.



Let μ be a positive Borel measure on $[a,b]$ and p_n be orthonormal polynomials w.r.t μ, i.e., $\int p_n p_m \mathrm{d}\mu = \delta_{mn}$. Given $f \in L^2(\mu)$, associate

$$f \sim \sum_{i=0}^{\infty} c_i(f) p_i, \quad c_i(f) := \int f p_i \mathrm{d}\mu.$$

Theorem (Freud + Mastroianni & Totik)

If $f \in \mathrm{Lip}^{\frac{1}{2}+\epsilon}$ and the Christoffel functions satisfy

$$n\lambda_n(x,\mu) \lesssim 1$$

uniformly on a set $S \subseteq [a,b]$, then

$$\sum_{i=0}^{n-1} c_i(f) p_i \rightrightarrows f \quad \text{on} \quad S.$$

The condition on the Christoffel functions is satisfied if μ is doubling on $[a,b]$.

A Frobenius-Padé approximant of type (m,n) to f is a rational function

$$P_{m,n}/Q_{m,n}, \quad \deg(P_{m,n}) \leq m, \quad \deg(Q_{m,n}) \leq n,$$

such that

$$c_i\big(Q_{m,n}f - P_{m,n}\big) = 0, \quad i \in \{0,\dots,m+n\}.$$

A Frobenius-Padé approximant always exists as its construction boils down to solving a linear system with one more unknown than equations.
A Frobenius-Padé approximant corresponding to $Q_{m,n}$ of the smallest degree is unique.

$$\deg(Q_{m,n}) = n \quad \Rightarrow \quad \text{Uniqueness.}$$

Let

$$f(z) = \int \frac{\mathrm{d}\sigma(x)}{x - z}.$$

The measures μ and σ are such that

$$\Delta_\mu := \mathrm{supp}(\mu) = [b_\mu, a_\mu], \quad \Delta_\sigma := \mathrm{supp}(\sigma) = [a_\sigma, b_\sigma]$$

and

$$\Delta_\mu \cap \Delta_\sigma = \varnothing.$$

We shall also assume that

$$n - 1 \leq m, \quad \frac{n}{n+m} \to c \in (0, 1/2] \quad \text{as} \quad n \to \infty.$$

Assume for now that μ and σ are positive measures. Recall that

$$c_i\big(Q_{m,n}f - P_{m,n}\big) = 0, \quad i \in \{0,\dots,m+n\}.$$

Write $R_{m,n} := Q_{m,n}f - P_{m,n}$. Then

$$\int x^i R_{m,n}(x)\mathrm{d}\mu(x) = 0, \quad i \in \{0,\dots,m+n\}.$$

Let $V_{m,n}$ be the polynomial vanishing at the zeros of $R_{m,n}$ on Δ_μ . Cauchy tells us that

$$\int \frac{x^i Q_{m,n}(x)}{V_{m,n}(x)}\mathrm{d}\sigma(x) = 0, \quad i \leq \min\{n-1, m\} = n-1.$$

Using Cauchy's work again, we get that

$$\int \frac{x^i V_{m,n}(x)}{Q_{m,n}(x)} \left(\int \frac{Q^2_{m,n}(t)}{V_{m,n}(t)} \frac{\mathrm{d}\sigma(t)}{t-x} \right) \mathrm{d}\mu(x) = 0, \quad i \in \{0,\dots,m+n\}.$$

We have that

$$\int \frac{x^i Q_{m,n}(x)}{V_{m,n}(x)} \mathrm{d}\sigma(x) = 0, \quad i \in \{0, \dots, n-1\},$$

and

$$\int \frac{x^i V_{m,n}(x)}{Q_{m,n}(x)} \Big(\cdot \Big) \mathrm{d}\mu(x) = 0, \quad i \in \{0, \dots, m+n\}.$$

Weak* limits of the counting measures of zeros then give us

$$\begin{cases} Q_{m,n} & \Rightarrow \quad \tau_{\sigma,c}, \ |\tau_{\sigma,c}| = c, \ \operatorname{supp}(\tau_{\sigma,c}) \subseteq \Delta_\sigma, \\ V_{m,n} & \Rightarrow \quad \tau_{\mu,c}, \ |\tau_{\mu,c}| = 1, \ \operatorname{supp}(\tau_{\mu,c}) \subseteq \Delta_\mu, \end{cases}$$

and we expect their logarithmic potentials to satisfy

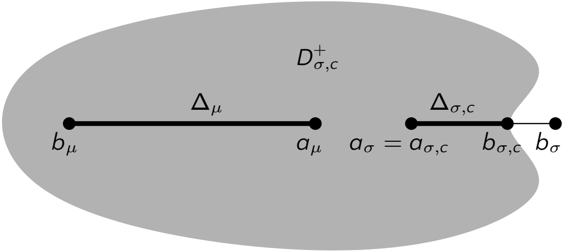
$$\begin{cases} 2V^{\tau_{\sigma,c}} - V^{\tau_{\mu,c}} = \min_{\Delta_\sigma}\left(2V^{\tau_{\sigma,c}} - V^{\tau_{\mu,c}}\right) =: \ell_{\sigma,c} \quad \text{on} \quad \operatorname{supp}(\tau_{\sigma,c}), \\ 2V^{\tau_{\mu,c}} - V^{\tau_{\sigma,c}} = \min_{\Delta_\mu}\left(2V^{\tau_{\mu,c}} - V^{\tau_{\sigma,c}}\right) =: \ell_{\mu,c} \quad \text{on} \quad \operatorname{supp}(\tau_{\mu,c}). \end{cases}$$

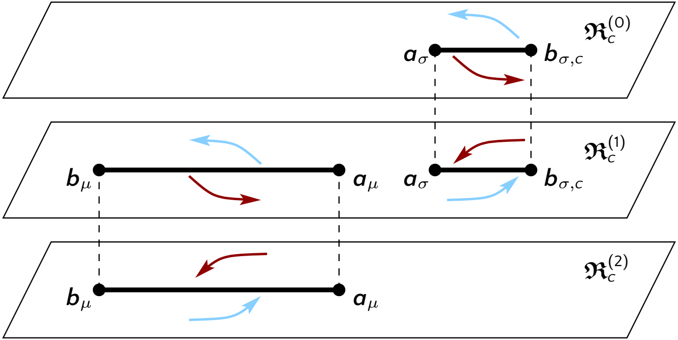
We are looking for measures such that

$$\begin{cases} 2V^{\tau_{\sigma,c}} - V^{\tau_{\mu,c}} = \ell_{\sigma,c} & \text{on} \quad \operatorname{supp}(\tau_{\sigma,c}) \subseteq \Delta_\sigma, \\ 2V^{\tau_{\mu,c}} - V^{\tau_{\sigma,c}} = \ell_{\mu,c} & \text{on} \quad \operatorname{supp}(\tau_{\mu,c}) \subseteq \Delta_\mu. \end{cases}$$

Proposition (Gonchar, Rakhmanov, & Sorokin)

Such a pair of measures exists and is unique, $\operatorname{supp}(\tau_{\mu,c}) = \Delta_\mu$, and $\operatorname{supp}(\tau_{\sigma,c}) =: \Delta_{\sigma,c}$ is an interval. Set $D^+_{\sigma,c} := \{2V^{\tau_{\sigma,c}} - V^{\tau_{\mu,c}} - \ell_{\sigma,c} < 0\}$. Then it is non-empty, contains $\Delta_{\sigma,c}$ in its boundary, is bounded when $c < 1/2$, and is equal to $\overline{\mathbb{C}} \setminus \Delta_\sigma$ when $c = 1/2$.





Define $\Phi_{m,n}$ on $\mathfrak{R}_{\frac{n}{n+m}}$ as having a divisor

$$(n+m)\infty^{(2)} - n\infty^{(0)} - m\infty^{(1)}$$

and normalized so that

$$\Phi_{m,n}^{(0)}(z)\Phi_{m,n}^{(1)}(z)\Phi_{m,n}^{(2)}(z) \equiv 1.$$

Theorem (Aptekarev, Bogolubsky, & Y.)

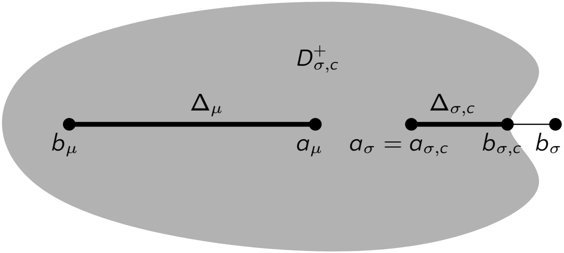
Let

$$d\nu(x) = \frac{\rho_\nu(x)}{2\pi i} \frac{dx}{w_\nu^+(x)}, \quad w_\nu(z) = \sqrt{(z-a_\nu)(z-b_\nu)}, \quad \nu \in \{\mu, \sigma\},$$

where ρ_ν is holomorphic and non-vanishing around Δ_ν. Assume μ possesses the full system of orthonormal polynomials. Then

$$\begin{cases} Q_{m,n}(z) & = & (1+o(1))\left(\Phi_{m+1,n}S_c\right)^{(0)}(z), \\ \left(w_{\sigma,c}R_{m,n}\right)(z) & = & (1+o(1))\left(\Phi_{m+1,n}S_c\right)^{(1)}(z). \end{cases}$$

locally uniformly in $\overline{\mathbb{C}} \setminus \Delta_\sigma$. It holds that $\left|\Phi^{(1)}_{m+1,n}/\Phi^{(0)}_{m+1,n}\right| < 1$ in $D^+_{\sigma,c}$.



Steepest descent is performed on

$$\boldsymbol{Y}_{m,n} := \boldsymbol{C}_{m,n}\begin{pmatrix} Q_{m,n} & R_{m,n} & H_{m,n} \\ Q_{m+1,n-1} & R_{m+1,n-1} & H_{m+1,n-1} \\ Q_{m,n-1} & R_{m,n-1} & H_{m,n-1} \end{pmatrix}$$

where $\boldsymbol{C}_{m,n}$ is a diagonal matrix of constants,

$$R_{m+1,n-1}(z) = \big(Q_{m+1,n-1}f - P_{m+1,n-1}\big)(z) = \mathcal{O}\big(z^{m+1}\big) \quad \text{as} \quad z \to \infty,$$

and

$$H_{m,n-1}(z) := \int \frac{R_{m,n-1}(x)}{x-z}\mathrm{d}\mu(x) = \mathcal{O}\big(z^{-(m+n+1)}\big) \quad \text{as} \quad z \to \infty.$$