On Convergence of AAK Approximants for Cauchy Transforms with Polar Singularities

M. Yattselev joint work with L. Baratchart
Project APICS, INRIA, Sophia Antipolis, France

Universidad de Almería, Almería, ESPAÑA
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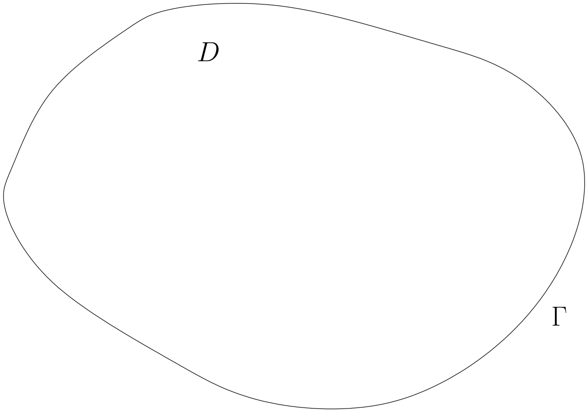
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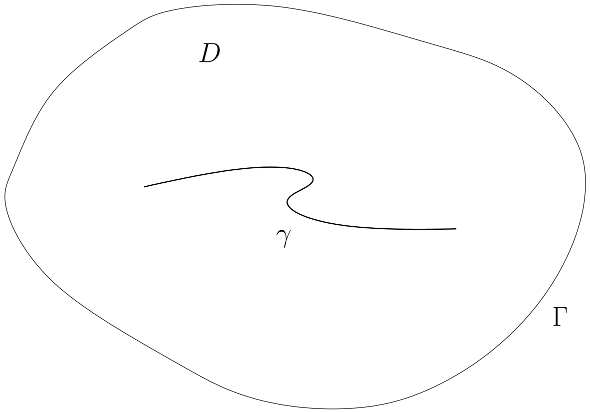
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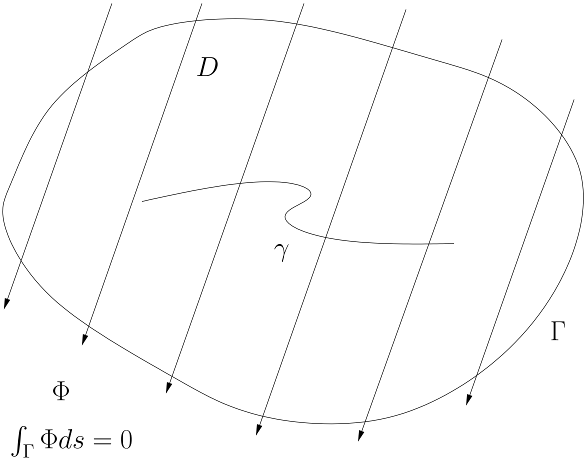
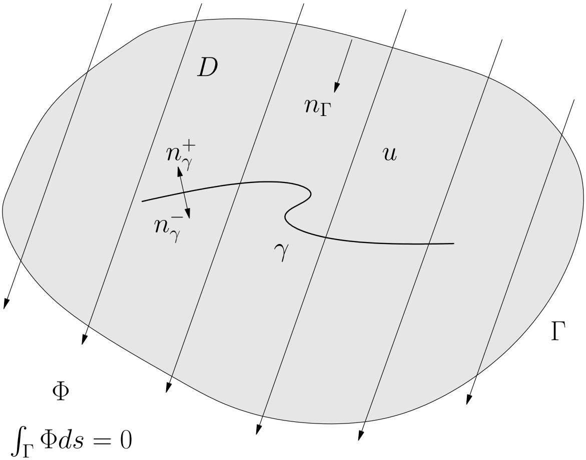
“Crack” Problem




D
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$\int_\Gamma \Phi ds = 0$

Let u be the equilibrium distribution of heat or current. Then

$$\begin{cases} \Delta u = 0 & \text{in } D \setminus \gamma \\[2ex] \dfrac{\partial u}{\partial n_\Gamma} = \Phi & \text{on } \Gamma := \partial D \\[2ex] \dfrac{\partial u^\pm}{\partial n_\gamma^\pm} = 0 & \text{on } \gamma \setminus \{\gamma_0, \gamma_1\} \end{cases},$$

where Δu is the Laplacian of u.

u has well-defined conjugate in $D \setminus \gamma$ and

$$\mathcal{F}(\xi) = u(\xi) - i \int_{\xi_0}^{\xi} \Phi ds, \quad \xi \in \partial D.$$

Further,

$$\mathcal{F}(z) = h(z) + \frac{1}{2\pi i} \int_{\gamma} \frac{(\mathcal{F}^- - \mathcal{F}^+)(t)}{z - t} dt, \quad z \in D \setminus \gamma,$$

where h is analytic in D and continuous in $\overline{D}$.

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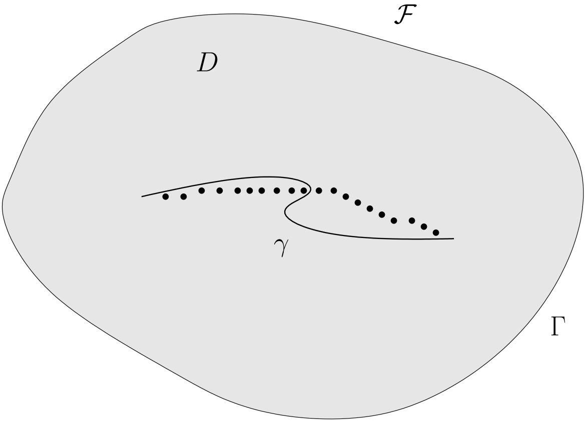
One approximates $\mathcal{F}$ on Γ by meromorphic in D functions and observes the asymptotic behavior of their poles as the number of poles grows large.


$\mathcal{F}$
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ElectroEncephaloGraphy problem consists in detecting epileptic foci located in the brain from the measurements of electric potential, U, on the scalp.

The brain, the skull, and the scalp are modeled by three nested spheres with the same center[1].

From measurements of U on the outer sphere, one needs to recover U on the inner sphere, inside of which it satisfies Neumann boundary value problem[2].

[1] L. Baratchart, J. Leblond, and J-P. Marmorat. Inverse source problem in a 3D ball from best meromorphic approximation on 2D slices. *Electron. Trans. Numer. Anal.*, 25:41–53, 2006.

[2] B. Atfen, L. Baratchart, J. Leblond, and J. R. Partington. Bounded extremal and Cauchy-Laplace problems on 3D spherecal domains. *In preparation.*

The inner ball is sliced into parallel disks. For each disk, d, there exists a function, f_d, analytic in d except branch points and poles such that

$$U^2\Big|_{\partial d} = f_d|_{\partial d}\,.$$

The epileptic foci are recovered from the knowledge of the branch points and poles of f_d for each disk d. The latter are localized using the meromorphic approximation approach.

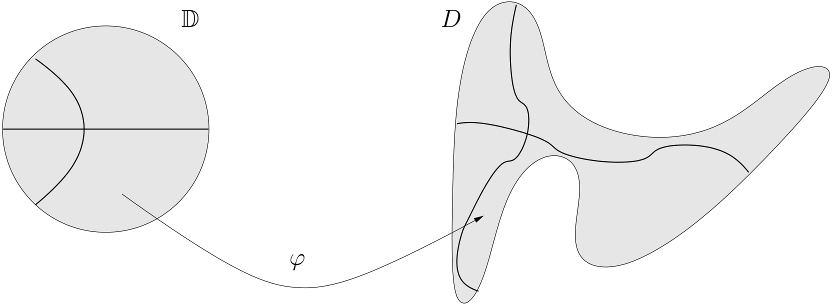
We want to answer the following questions:

1. What is asymptotic distribution of poles of best meromorphic approximants to $\mathcal{F}$?
2. Do some of these poles converge to the polar singularities of $\mathcal{F}$?
3. What can be said about the convergence of such approximants to $\mathcal{F}$?

Note (Baratchart, Mandrèa, Saff, and Wielonsky[3])

In the following we set D to be the unit disk, $\mathbb{D}$, and γ to be a subset of $(-1, 1)$. It was shown by Baratchart et al. that all these considerations translate to domains with piecewise $C^{1,\alpha}$ boundary without outward-pointing cusps, where γ is supposed to be a subset of a hyperbolic geodesic of the corresponding domain.

[3] 2-D inverse problems for the Laplacian: a meromorphic approximation approach. *J. Math. Pures Appl.*, 86:1–41, 2006


$\mathbb{D}$
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Let

- μ be a complex Borel measure, $S_\mu := \mathrm{supp}(\mu) \subset (-1, 1)$;
- R be rational function whose set of poles $S' \subset \mathbb{D}$;
- $\mathcal{F}(\mu; R; z) = \displaystyle\int \frac{d\mu(t)}{z-t} + R(z)$;
- $D_\mathcal{F} := \overline{\mathbb{C}} \setminus (S_\mu \cup S')$ stand for the domain of analyticity of $\mathcal{F}$.

Let h be a complex-valued function on the unit circle, $\mathbb{T}$. Then

$$\begin{aligned} h \in L^2 &\quad \text{iff} \quad \|h\|_2^2 := \sum |h_j|^2 < \infty, \;\; h_j := \frac{1}{2\pi}\int_{\mathbb{T}} \xi^{-j} h(\xi)|d\xi|, \\ h \in L^\infty &\quad \text{iff} \quad \|h\|_\infty := \operatorname{ess.sup}_{\mathbb{T}} |h| < \infty. \end{aligned}$$

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\end{array}
$$

Let $p = 2, \infty$. The Hardy spaces are defined by

$$
\begin{array}{lll}
H^p & := & \left\{ h \in L^p : \; h_j = 0, \; j < 0 \right\}, \\
\bar{H}_0^p & := & \left\{ h \in L^p : \; h_j = 0, \; j > -1 \right\}.
\end{array}
$$

It is clear that

$$
L^2 = H^2 \oplus \bar{H}_0^2.
$$

Orthogonal projections:

$$\begin{aligned} \mathcal{P}_- : L^2 &\rightarrow \bar{H}^2_0 \\ \mathcal{P}_+ : L^2 &\rightarrow H^2. \end{aligned}$$

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Let $n \in \mathbb{Z}_+$. The n-th singular number of $\mathcal{H}_f$:

$$\sigma_n(\mathcal{H}_f) := \inf\left\{ \|\mathcal{H}_f - \mathcal{O}\| : \ \mathcal{O} : H^2 \to \bar{H}_0^2, \quad \operatorname{rank}(\mathcal{O}) \leq n \right\},$$

$$\sigma_\infty(\mathcal{H}_f) := \lim_{n\to\infty} \sigma_n(\mathcal{H}_f);$$

The set of Blaschke products of degree at most n:

$$B_n := \left\{ b(z) : \; b(z) = e^{ic} \prod_{j=1}^{m} \frac{z - z_j}{1 - \bar{z}_j z}, \; m \leq n, \; z_j \in \mathbb{D}, \; c \in \mathbb{R} \right\}.$$

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The set of meromorphic functions of degree n:

$$H_n^\infty := H^\infty B_n^{-1}.$$

Inner functions:

- Blaschke products;
- singular inner functions

$$\exp\left\{-\int \frac{\xi+z}{\xi-z}d\nu(\xi)\right\},$$

where ν is a positive measure on $\mathbb{T}$ which is singular with respect to the Lebesgue measure.

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Outer functions:

- $w \in H^2$ such that

$$w(z) = \exp\left\{\frac{1}{2\pi}\int\frac{\xi+z}{\xi-z}\log|w(\xi)||d\xi|\right\}.$$

Theorem (Adamyan, Arov, and Krein[4])

Let $f \in L^\infty$ and $n \in \mathbb{Z}_+$. Then

$$\inf_{g \in H_n^\infty} \|f - g\|_\infty = \sigma_n(\mathcal{H}_f).$$

[4] Analytic properties of Schmidt pairs for a Hankel operator on the generalized Schur-Takagi problem. *Math. USSR Sb.*, 15:31-73, 1971.

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Further, if $\sigma_n(\mathcal{H}_f) > \sigma_\infty(\mathcal{H}_f)$ then there exists a function of the unit norm $v_n \in H^2$ such that

$$f - g_n = \frac{\mathcal{H}_f(v_n)}{v_n}.$$

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- v_n is called a singular vector associated to g_n, $\|v_n\|_2 = 1$;
- v_n is not necessarily unique;
- there always exists a v_n with the inner-outer factorization

$$v_n(z) = b_n(z)w_n(z), \quad z \in \mathbb{D},$$

  where b_n is a Blaschke product of exact degree n and w_n is an outer function.

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Then

$$\int t^j q_n(t) Q(t) \frac{w_n(t)}{\widetilde{q}_n^2(t)} d\mu(t) = 0, \quad j = 0, \dots, n-m-1.$$

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- $d\mu(t) = e^{i\Theta(t)} d|\mu|(t)$, where $|\mu|$ is the total variation and Θ is real-valued argument function of bounded variation, i.e.

$$\sup\left\{\sum_{j=1}^{N} |\Theta(x_j) - \Theta(x_{j-1})|\right\} < \infty,$$

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  $x_0 < x_1 < \ldots < x_N \subset S_\mu$;
- $|\mu|([x-\delta,x+\delta]) \geq c\delta^L$, where c and L are some constants, $x \in S_\mu$, and $\delta \in (0,1)$.

Lemma (Baratchart et al.[5])

The family $\mathcal{W} := \{w_n\}$ is normal in $D^*_{\mathcal{F}}$, where $D^*_{\mathcal{F}}$ is the reflection of $D_{\mathcal{F}}$ across $\mathbb{T}$. Moreover, any limit point of $\mathcal{W}$ is zero free in $\mathbb{D}$.

[5] L. Baratchart and F. Seyfert. An L^p analog of AAK theory for $p \geq 2$. *J. Func. Anal.*, 191(1):52–122, 2002;
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Remark

This lemma, in fact, does not require the hypothesis $\mu \in$ **BVT**. It is sufficient for the lemma to hold to have a measure with an argument of bounded variation and infinitely many points in the support.

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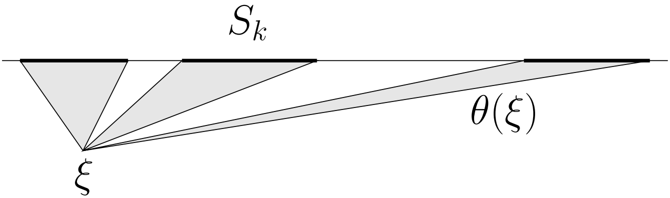
Lemma (Baratchart, Küstner, and Totik[6])

Let S_k be a covering of S_μ by k disjoint closed intervals. Then

$$\sum(\pi-\theta(\xi_{j,n}))\leq V(\Theta,\mathcal{W},Q,k).$$



[6] Zero distribution via orthogonality, *Ann. Inst. Fourier*, 55(5):1455–1499, 2005.

Denote by $V_{\mathbb{D}}^{\omega}$ the Green potential of a probability measure ω, $\operatorname{supp}(\omega) \subset \mathbb{D}$, relative to $\mathbb{D}$, i.e.

$$V_{\mathbb{D}}^{\omega}(z) := \int \log\left|\frac{1-\bar{t}z}{z-t}\right| d\omega(t), \quad z \in \mathbb{D} \setminus \operatorname{supp}(\omega).$$

It is known that there exists the unique measure $\omega^* = \omega_{(S_\mu, \mathbb{T})}$ that minimizes the Green energy functional

$$\int\int \log\left|\frac{1-\bar{t}z}{z-t}\right| d\omega(t)d\omega(z) = \int V_{\mathbb{D}}^{\omega}(z)d\omega(z),$$

among all probability Borel measures supported on S_μ.

It holds that

$$V_{\mathbb{D}}^{\omega^*} \equiv 0 \quad \text{on} \quad \mathbb{T}$$

by the definition of the Green potential and

$$V_{\mathbb{D}}^{\omega^*} \equiv \frac{1}{\operatorname{cap}(S_\mu, \mathbb{T})} \quad \text{on} \quad S_\mu$$

by the properties of the Green equilibrium measure, where $\operatorname{cap}(S_\mu, \mathbb{T})$ is the Green capacity of S_μ relative to $\mathbb{D}$.

Theorem (Baratchart and Y.[7])

Let $\{g_n\}$ be a sequence of irreducible best approximants to $\mathcal{F}(\mu; R; \cdot)$ with $\mu \in$ **BVT**. Then

- the counting measures of the poles of g_n converge to ω^* in the weak* sense;

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- in particular, if z is not a limit point of poles of g_n then $\lim_{n\to\infty} |b_n(z)|^{1/n} = \exp\left\{-V_{\mathbb{D}}^{\omega^*}(z)\right\}$;

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- $|(\mathcal{F} - g_n)(z)|^{1/2n} \stackrel{\text{cap}}{\to} \exp\left\{V_{\mathbb{D}}^{\omega^*}(z) - \dfrac{1}{\text{cap}(S_\mu, \mathbb{T})}\right\}$ on compact subsets of $\mathbb{D} \setminus S_\mu$;

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- $|(\mathcal{F} - g_n)(z)|^{1/2n} \overset{\text{cap}}{\to} \exp\left\{ V_{\mathbb{D}}^{\omega^*}(z) - \dfrac{1}{\text{cap}(S_\mu, \mathbb{T})} \right\}$ on compact subsets of $\mathbb{D} \setminus S_\mu$;
- for each n large enough there exists $q_{n,m}$, divisor of q_n, such that $q_{n,m} = Q + o(1)$.

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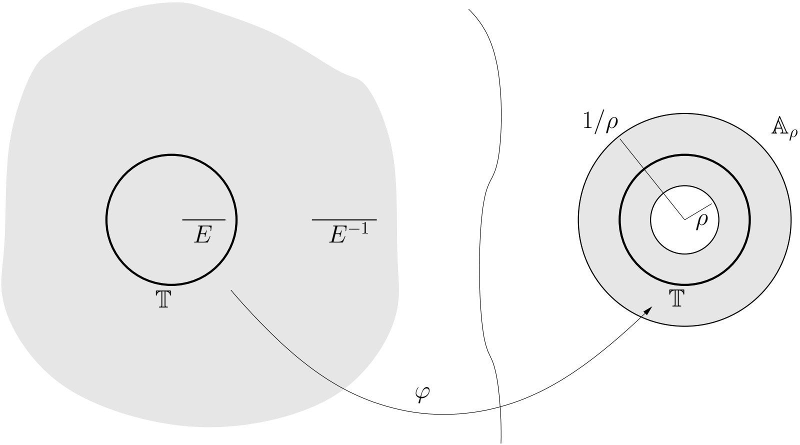
Conformal Map



Let $S_\mu = E := [a, b]$. Then

$$\exp\left\{-V_{\mathbb{D}}^{\omega^*}(z)\right\} = |\varphi(z)|$$

and

$$\exp\left\{\frac{-1}{\operatorname{cap}(E,\mathbb{T})}\right\} = \varphi(b) = -\varphi(a) =: \rho,$$

where

$$\varphi(z) := \exp\left\{2\pi\tau^2 \int_1^z \frac{dt}{\sqrt{(t-a)(b-t)(1-at)(1-bt)}}\right\}$$

is the conformal map of $\overline{\mathbb{C}} \setminus (E \cup E^{-1})$ onto annulus $\mathbb{A}_\rho$.

Definition (Class of measures BND)

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- $d\mu(t) = (t-a)^\alpha(b-t)^\beta s(t)d\mu_E(t)$, where $\alpha,\beta \in [0,1/2)$ and μ_E is the arcsine distribution on $E=[a,b]$;

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- s is a non-vanishing Dini-continuous function on E;
- μ has an argument of bounded variation on E.

Theorem (Y.[8])

Let $\{g_n\}$ be a sequence of irreducible best approximants to $\mathcal{F}(\mu;R;\cdot)$ with $\mu\in\mathbf{BND}$ and R analytic on E. Then the outer factors w_n are such that

$$w_n=\frac{\tau+o(1)}{\sqrt{(1-az)(1-bz)}}+\frac{l_n}{\widetilde{Q}},\quad \widetilde{Q}(z)=z^m\overline{Q(1/\bar{z})},$$

where $o(1)$ holds locally uniformly in $\overline{\mathbb{C}}\setminus E^{-1}$ and the polynomials l_n, $\deg(l_n)<m$, converge to zero and are coprime with $\widetilde{Q}$.

[8] On Approximation of Complex Cauchy Transforms with Polar Singularities. *To be submitted*

Theorem

Further,

$$\frac{b_n(z)}{\varphi^n(z)} = \frac{1+o(1)}{\mathcal{D}_n(z)} \frac{b(z)}{\varphi^m(z)}$$

locally uniformly in $D_{\mathcal{F}} \cap D^*_{\mathcal{F}}$, where $b = Q/\widetilde{Q}$.

Each $\mathcal{D}_n$ is such that

- it is an outer function in $\overline{\mathbb{C}} \setminus (E \cup E^{-1})$;
- there exist constants m and M independent of n such that $0 < m < \mathcal{D}_n(z) < M < \infty$ in $\overline{\mathbb{C}}$;
- it holds that $\mathcal{D}_n(z)\overline{\mathcal{D}_n(1/\bar{z})} = 1$;
- it has winding number zero on any curve separating E from E^{-1}.

Theorem

Moreover,

$$(\mathcal{F} - g_n)(z) =$$

$$\left(\frac{2\mathcal{D}}{\tau} + o(1)\right) \sqrt{\frac{(1-az)(1-bz)}{(z-a)(z-b)}} \left(\frac{\rho}{\varphi(z)}\right)^{2(n-m)} \frac{\mathcal{D}_n^2(z)}{b^2(z)}$$

locally uniformly in $D_{\mathcal{F}} \cap \mathbb{D}$.

Theorem

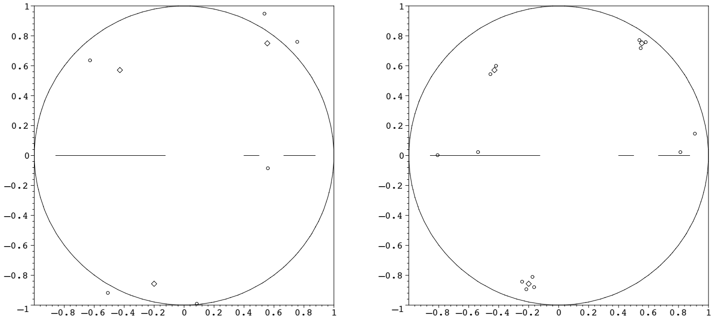
Finally, for each η and all n large enough, there exists an arrangement of $\eta_{1,n}, \ldots \eta_{m(\eta),n}$, the zeros of b_n approaching η, such that

$$\eta_{k,n} = \eta + A^{\eta}_{k,n} \left(\frac{\rho}{\varphi(\eta)} \right)^{2(n-m)/m(\eta)} \exp\left\{ \frac{2\pi ki}{m(\eta)} \right\},$$

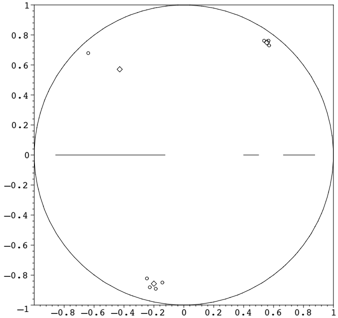
$k = 1, \ldots, m(\eta)$, where the sequences $\{A^{\eta}_{k,n}\}$ are convergent with finite nonzero limit independent of k.

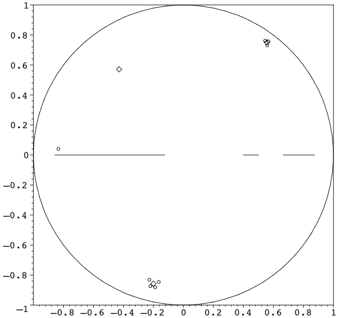
$$\begin{aligned}\mathcal{F}(z) &= 7\int_{[-6/7,-1/8]}\frac{e^{it}dt}{z-t} - (3+i)\int_{[2/5,1/2]}\frac{1}{t-2i}\frac{dt}{z-t} \\ &+ (2-4i)\int_{[2/3,7/8]}\frac{\ln(t)dt}{z-t} + \frac{2}{(z+3/7-4i/7)^2} \\ &+ \frac{6}{(z-5/9-3i/4)^3} + \frac{24}{(z+1/5+6i/7)^4}.\end{aligned}$$

On the figures the solid lines stand for the support of the measure, diamonds depict the polar singularities of $\mathcal{F}$, and circles denote the poles of the correspondent approximants. Note that the poles of $\mathcal{F}$ seem to attract the singularities first.

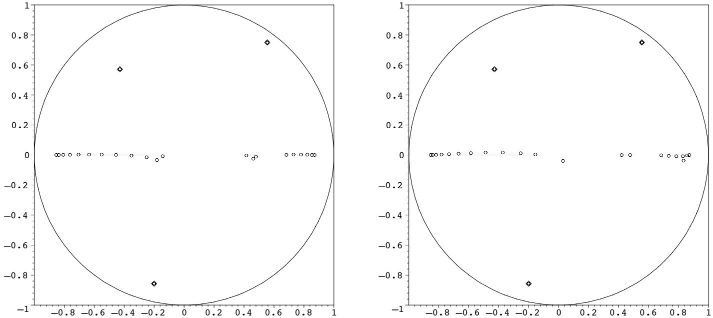


Padé approximants to $\mathcal{F}$ of degree 8 and 13





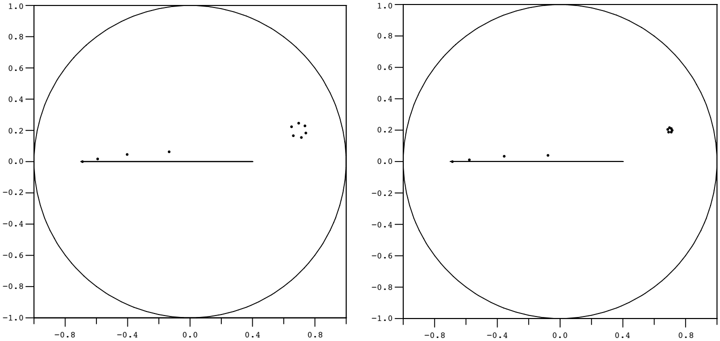
AAK (left) and rational (right) approximants to $\mathcal{F}$ of degree 8



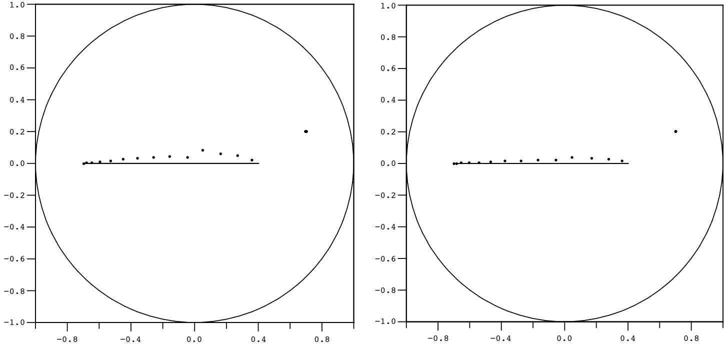
Padé (left) and AAK (right) approximants to $\mathcal{F}$ of degree 30

$$\begin{aligned}\mathcal{F}(z) &= 7\int_{[-0.7,0]}\frac{e^{it}}{z-t}\frac{dt}{\sqrt{(t+0.7)(0.4-t)}}\\ &+ \int_{[0,0.4]}\frac{it+1}{z-t}\frac{dt}{\sqrt{(t+0.7)(0.4-t)}}.\\ &+ \frac{1}{5!(z-0.7-0.2i)^6}\end{aligned}$$

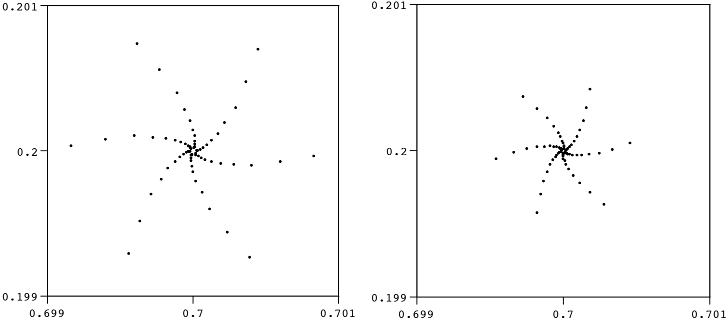
On the figures the solid line stands for the support of the measure and circles denote the poles of the correspondent approximants.



Poles of Padé (left) and AAK (right) approximants of degree 10 to $\mathcal{F}$.



Poles of Padé (left) and AAK (right) approximants of degree 20 to $\mathcal{F}$.



Poles of Padé (left) and AAK (right) approximants of degrees 21-33 to $\mathcal{F}$ lying in an neighborhood of the polar singularity.