On Strong Asymptotics of MOPs for Angelesco Systems

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Multiple Orthogonal Polynomials

Let μ_1 and μ_2 be compactly supported Borel measures on the real line such that

$$[\alpha_1, \beta_1] < [\alpha_2, \beta_2],$$

where $[\alpha_i, \beta_i]$ is the convex hull of the support of μ_i. This is an Angelesco system of two measures.

Type II multiple orthogonal polynomial corresponding to a multi-index $\vec{n} = (n_1, n_2)$ is defined as the unique monic polynomial of degree $|\vec{n}| := n_1 + n_2$ such that

$$\int x^k P_{\vec{n}}(x) d\mu_i(x) = 0, \quad k = \overline{0, n_i - 1}.$$

This polynomial has n_i zeros on Δ_i. We always assume that $\operatorname{supp} \mu_i = \Delta_i$.

Weak Asymptotics

Tools developed to understand OPs yield that the asymptotic behavior of MOPs for Angelesco systems is governed by the following potential theoretic extremal problem:

if $c(\vec{n}) := n_1/|\vec{n}| \to c \in (0,1)$*, one needs to find measures* ω_1 *and* ω_2 *such that*

$$\begin{cases} \operatorname{supp}\omega_i \subseteq \Delta_i, \quad |\omega_1| = c, \quad |\omega_2| = 1-c, \\ V^{2\omega_i+\omega_{3-i}}(x) = \ell_i, \quad x \in \operatorname{supp}\omega_i, \\ V^{2\omega_i+\omega_{3-i}}(x) < \ell_i, \quad x \in \Delta_i \setminus \operatorname{supp}\omega_i, \end{cases}$$

for some constants ℓ_i*, where* $V^{\nu}(z) = -\int \log|z-x|d\nu(x)$ *is the logarithmic potential of* ν.

Weak Asymptotics

Theorem (Gonchar-Rakhmanov 1981)

For any $c \in (0,1)$ the pair of measures (ω_1, ω_2) exists, is unique, and it holds that

$$\operatorname{supp} \omega_i = [\alpha_{c,i}, \beta_{c,i}], \quad \alpha_{c,1} = \alpha_1, \ \ \beta_{c,2} = \beta_2.$$

If μ_i is absolutely continuous w.r.t. Lebesgue measure and $\mu_i'(x) > 0$ a.e. on $[\alpha_i, \beta_i]$, then

$$\frac{1}{|\vec{n}|} \log |P_{\vec{n}}(z)| \sim V^{\omega_1+\omega_2}(z)$$

and zero counting measures of $P_{\vec{n}}$ converge weak* to $\omega_1 + \omega_2$ as $c(\vec{n}) \to c$.

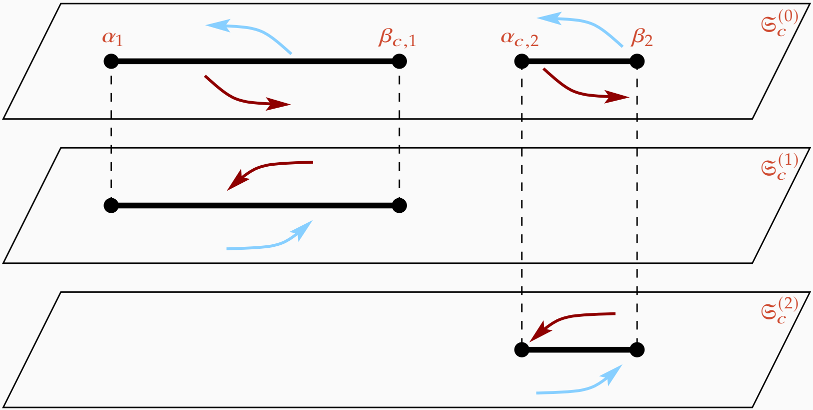
- This theorem was proven for Angelesco systems of any number of measures.

Riemann Surface $\mathfrak{S}_c$

There exists an increasing continuous function $z(c) : [0, 1] \to [\alpha_1, \beta_2]$ such that

$$\beta_{c,1} = \min\{\beta_1, z(c)\} \quad \text{and} \quad \alpha_{c,2} = \max\{\alpha_2, z(c)\}.$$

Thus, there exists $0 < c^* < c^{**} < 1$ for which $[\alpha_{c,i}, \beta_{c,i}] = [\alpha_i, \beta_i]$, $c \in [c^*, c^{**}]$.



Strong Asymptotics

Theorem (Ya. 2016)

Let $\mu_i'(x)$ be Fisher-Hartwig perturbations of analytic and non-vanishing densities. Then

$$P_{\vec{n}}(z) \sim \left(S_{\vec{n}}\Phi_{\vec{n}}\right)^{(0)}(z),$$

as $c(\vec{n}) \to c \in (0,1)$, where $\Phi_{\vec{n}}(z)$ is a rational function on $\mathfrak{S}_{c(\vec{n})}$ with the zero/pole divisor

$$|\vec{n}|\infty^{(0)} - n_1\infty^{(1)} - n_2\infty^{(2)}$$

and $S_{\vec{n}}(z)$ is the solution of a certain boundary value problem on $\mathfrak{S}_{c(\vec{n})}$ (Szegő function for the densities μ_1', μ_2').

- This theorem was proven for Angelesco systems of any number of measures.
- Strong asymptotics for Szegő densities along $n_1 = n_2$ was proven by Aptekarev 1988.

Nearest-Neighbor Recurrence Relations

Theorem (Van Assche 2011)

It holds that

$$xP_{\vec{n}}(x) = P_{\vec{n}+\vec{e}_i}(x) + b_{\vec{n},i}P_{\vec{n}}(x) + a_{\vec{n},1}P_{\vec{n}-\vec{e}_1}(x) + a_{\vec{n},2}P_{\vec{n}-\vec{e}_2}(x)$$

for each $i \in \{1, 2\}$, where $\vec{e}_1 = (1, 0)$ and $\vec{e}_2 = (0, 1)$.

- This theorem holds for any number of measures and any system of MOPs.

Asymptotics of the Recurrence Coefficients

Theorem (Aptekarev-Denisov-Ya. 2020)

Let $\mu_i'(x)$ be Fisher-Hartwig perturbations of analytic and non-vanishing densities. Then

$$a_{\vec{n},i} \to A_{c,i} \quad \text{and} \quad b_{\vec{n},i} \to B_{c,i}$$

as $c(\vec{n}) \to c \in (0,1)$, where $\chi_c : \mathfrak{S}_c \to \overline{\mathbb{C}}$ is the conformal map such that

$$\chi^{(0)}(z) = z + O\left(\frac{1}{z}\right) \quad \text{and} \quad \chi^{(i)}(z) =: B_{c,i} + \frac{A_{c,i}}{z} + O\left(\frac{1}{z^2}\right).$$

- This theorem was proven for Angelesco systems of any number of measures.
- The main problem with $c = 0, 1$ is that $\beta_{c,1} \to \alpha_1$ as $c \to 0$ and $\alpha_{c,2} \to \beta_2$ as $c \to 1$.

Strong Asymptotics II

Theorem (Aptekarev-Denisov-Ya. 2021)

Let $\mu_i'(x)$ be analytic and positive. Then

$$a_{\vec{n},i} \to A_{c,i} \quad \text{and} \quad b_{\vec{n},i} \to B_{c,i}$$

as $c(\vec{n}) \to c \in [0,1]$, where

$$A_{0,2} = \left(\frac{\beta_2 - \alpha_2}{4}\right)^2, \quad B_{0,2} = \frac{\beta_2 + \alpha_2}{2}, \quad A_{0,1} = 0, \quad B_{0,1} = B_{0,2} + \varphi_2(\alpha_1),$$

and $\varphi_2(z) = z + O(1)$ is the conformal map of the complement of $[\alpha_2, \beta_2]$ to the complement of a disk. It is also true that

$$P_{\vec{n}}(z) \sim (S_{\vec{n}}\Phi_{\vec{n}})^{(0)}(z)$$

as long as $c(\vec{n}) \to c \in [0,1]$ and $\min\{n_1, n_2\} \to \infty$.

Strong Asymptotics III

Theorem (Ya. to be submitted)

Let $\mu_i'(x)$ be analytic and positive. All formulae of strong asymptotics hold uniformly in $|\vec{n}|$ as long as

$$\varepsilon_{\vec{n}} := 1/\min\{n_1, n_2\} \to 0.$$

The error terms are (uniform in $\vec{n}$ and) of order $\varepsilon_{\vec{n}}^{1/3}$. In particular,

$$a_{\vec{n},i} = A_{c(\vec{n}),i} + O\left(\varepsilon_{\vec{n}}^{1/3}\right) \quad \text{and} \quad b_{\vec{n},i} = B_{c(\vec{n}),i} + O\left(\varepsilon_{\vec{n}}^{1/3}\right).$$

- The results are achieved through finer analysis of local parametrices + an extra step to handle $b_{\vec{n},i}$ when $c(\vec{n})$ is close to 0 or 1.

Differential Equations

Theorem (Aptekarev-Kozhan 2020)

Assuming that the recurrence coefficients converge with rate $o(1/|\vec{n}|)$, it holds that

$$\begin{cases} cB'_{1,c} + (1-c)B'_{2,c} = 0, \\ c(1-c)B_c B'_c + A'_{1,c} + A'_{2,c} = 0, \quad B_c = B_{2,c} - B_{1,c}, \\ c^2 \dfrac{A'_{1,c}}{A_{1,c}} = (1-c)^2 \dfrac{A'_{2,c}}{A_{2,c}} = c(1-c)\dfrac{B'_c}{B_c}. \end{cases}$$

- This theorem was proven for Angelesco systems of any number of measures.

- The results in Ya. 2016 allow to prove this theorem without the rate of convergence assumption.

- The first equation allows to prove uniformity of the asymptotics of $b_{\vec{n},i}$.

Strong Asymptotics IV

Theorem (Ya. in preparation)

Uniformity of the asymptotics holds for densities $\mu_i'(x)$ that belong to certain fractional Sobolev spaces only (no analyticity necessary).

Strong Asymptotics V

Theorem-ish (Denisov-Ya.)

High degree of confidence: Strong asymptotics for Szegő densities holds along ray sequences $c(|\vec{n}|) \to c \in (c^*, c^{**})$, i.e., when $\operatorname{supp} \omega_i = [\alpha_i, \beta_i]$.

Moderate degree of confidence: Strong asymptotics along ray sequences $c(\vec{n}) \to c \in (0, c^*] \cup [c^{**}, 1)$ holds for uniformly Szegő densities:

$$\lim_{\epsilon,\delta \to 0^+} \int_{\alpha^*+\epsilon}^{\beta^*-\delta} \frac{\log \mu'(x)\,dx}{\sqrt{(\beta^* - \delta - x)(x - \alpha^* - \epsilon)}} = \int_{\alpha^*}^{\beta^*} \frac{\log \mu'(x)\,dx}{\sqrt{(\beta^* - x)(x - \alpha^*)}}$$

for any $[\alpha^*, \beta^*] \subseteq [\alpha, \beta] = \operatorname{supp} \mu$.