# Strong Asymptotics of Hermite-Padé Approximants for Angelesco Systems with Complex Weights

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Let $\mu$ be a positive Borel measure $\mathrm{supp}(\mu) \subseteq [a, b]$. There exists a monic polynomial $Q_n$, $\deg(Q_n) = n$, such that

$$\int x^k Q_n(x)\mathrm{d}\mu(x) = 0,$$

$k \in \{0, \ldots, n-1\}$. It holds that

$$\int |Q_n(x)|^2 \mathrm{d}\mu(x) = \min_Q \int |Q(x)|^2 \mathrm{d}\mu(x)$$

for any monic polynomial of degree $n$.

To guess the behavior of $Q_n$, lets look at

$$\min_{Q} \sup_{x\in[a,b]} |Q(x)|.$$

Write

$$V^{\sigma_Q}(z) = -\frac{1}{n}\log|Q(z)| = -\frac{1}{n}\int \log|z-x|\mathrm{d}\sigma_Q(x)$$

Then the problem becomes

$$\max_{\sigma_Q} \min_{x\in[a,b]} V^{\sigma_Q}(x).$$

Look at all the probability measures $\sigma$ on $[a, b]$. It is known that there exists the unique measure $\omega$ such that

$$\ell := \min_{x \in [a,b]} V^{\omega}(x) = \max_{\sigma} \min_{x \in [a,b]} V^{\sigma}(x).$$

The measure $\omega$ is called the **logarithmic equilibrium distribution** on $[a, b]$ and the constant $\ell$ is called **Robin constant**. It is known that

$$\begin{cases} \ell - V^{\omega} \equiv 0 & \text{on} \quad [a, b], \\ \ell - V^{\omega} > 0 & \text{in} \quad \overline{\mathbb{C}} \setminus [a, b]. \end{cases}$$

Let $f$ be a function holomorphic at infinity. The **diagonal Padé approximant** $[n/n]_f = P_n/Q_n$ is the unique rational function such that

$$\begin{cases} \deg\left(Q_n\right) \leq n, \\ \left(Q_n f - P_n\right)(z) = \mathcal{O}\left(z^{-(n+1)}\right). \end{cases}$$

Let $\mu$ be such that $\mu' > 0$ almost everywhere on $[a, b]$. If

$$f(z) = \int \frac{\mathrm{d}\mu(x)}{x - z},$$

then $Q_n$ is the $n$-th orthogonal polynomials w.r.t. to $\mu$ and it holds locally uniformly in $\overline{\mathbb{C}} \setminus [a, b]$ that

$$\begin{cases} \lim\limits_{n\to\infty} n^{-1} \log\left|f - [n/n]_f\right| \leq -2\left(\ell - V^{\omega}\right) \\ \lim\limits_{n\to\infty} n^{-1} \log\left|Q_n\right| = -V^{\omega}. \end{cases}$$

If the measure $\mu$ satisfies Szegő condition $\int \log \mu' \mathrm{d}\omega > -\infty$, then there exists a non-vanishing holomorphic function $S$ such that

$$S_+ S_- = \mu' w_+,$$

where $w(z) = \sqrt{(z-a)(z-b)}$. In this case it holds that

$$\begin{cases} Q_n = C_n \big[ 1 + o(1) \big] S \Phi^n \\ Q_n f - P_n = C_n \big[ 1 + o(1) \big] / (w S \Phi^n), \end{cases}$$

where $\Phi$ is the conformal map of $\overline{\mathbb{C}} \setminus [a,b]$ to the complement of the unit disk such that $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$.

## Remarks

When $\mathrm{d}\mu(x) = (\rho/w_+)(x)\mathrm{d}x$ and $\rho$ is Hölder continuous, complex-valued, and non-vanishing on $[a,b]$, this theorem is due to **Nuttall** and when $\mathrm{d}\mu(x)/\mathrm{d}x$ is a Jacobi-type weight with $\rho$ **"smooth"**, it is due **Baratchart-Y**.

Denote by $\mathfrak{R}$ the Riemann surface of $w$ (two copies of the complex plane cut along $[a, b]$ and glued crosswise). Set

$$\begin{cases} \Phi^{(0)} & = & \Phi \\ \Phi^{(1)} & = & 1/\Phi \end{cases} \quad \text{and} \quad \begin{cases} S^{(0)} & = & S \\ S^{(1)} & = & 1/S. \end{cases}$$

Then $\Phi$ is a rational function on $\mathfrak{R}$ with a simple pole at $\infty^{(0)}$ and a simple zero at $\infty^{(1)}$.

The asymptotic formula can be written as

$$\begin{cases} Q_n & = & C_n\left[1+o(1)\right]\left(S\Phi^n\right)^{(0)} \\ Q_n f - P_n & = & C_n\left[1+o(1)\right]\left(S\Phi^n\right)^{(1)}/w. \end{cases}$$

Let $f_i$, $i \in \{1, \dots, p\}$, be functions holomorphic at infinity, $p \in \mathbb{N}$. Given a multi-index $\vec{n} \in \mathbb{N}^p$, **Hermite-Padé** approximant to the vector

$$\vec{f} = (f_1, \dots, f_p)$$

associated with $\vec{n}$, is a vector of rational functions

$$[\,\vec{n}\,]_{\vec{f}} := \left(P^{(1)}_{\vec{n}}/Q_{\vec{n}}, \dots, P^{(p)}_{\vec{n}}/Q_{\vec{n}}\right)$$

such that

$$\begin{cases} \deg\left(Q_{\vec{n}}\right) \leq |\,\vec{n}\,| := n_1 + \cdots + n_p, \\ \left(Q_{\vec{n}} f_i - P^{(i)}_{\vec{n}}\right)(z) = \mathcal{O}\left(z^{-(n_i+1)}\right), \quad i \in \{1, \dots, p\}. \end{cases}$$

The vector $\vec{f}$ is called an **Angelesco system** if

$$f_i(z) = \int \frac{\mathrm{d}\mu_i(x)}{x - z}, \qquad i \in \{1, \ldots, p\},$$

where $\mu_i$'s are positive measures on the real line with mutually disjoint convex hulls of their supports, i.e.,

$$\operatorname{supp}(\mu_i) \subseteq [a_i, b_i] \quad \text{and} \quad [a_j, b_j] \cap [a_k, b_k] = \varnothing.$$

For such systems it holds that

$$\int x^k Q_{\vec{n}}(x) \mathrm{d}\mu_i(x) = 0, \qquad k \in \{0, \ldots, n_i - 1\}, \quad i \in \{1, \ldots, p\}.$$

Assume now that

$$n_i = c_i|\,\vec{n}\,| + o\left(|\,\vec{n}\,|\right), \quad \vec{c} = (c_1, \ldots, c_p) \in (0,1)^p, \quad |\,\vec{c}\,| = 1.$$

There exists the unique vector of positive Borel measures

$$(\omega_1, \ldots, \omega_p), \quad |\omega_i| = c_i, \quad \operatorname{supp}(\omega_i) = [a_{\vec{c},i}, b_{\vec{c},i}] \subseteq [a_i, b_i],$$

such that $(\sigma_1, \ldots, \sigma_p) = (\omega_1, \ldots, \omega_p)$ if

$$\ell_i := \min_{x\in[a_i,b_i]} V^{\omega+\omega_i}(x) \leq \min_{x\in[a_i,b_i]} V^{\sigma+\sigma_i}(x)$$

for each $i \in \{1, \ldots, p\}$, where $\sigma := \sum_{i=1}^p \sigma_i$. It holds that

$$\begin{cases} \ell_i - V^{\omega_i+\omega} \equiv 0 & \text{on} \quad [a_{\vec{c},i}, b_{\vec{c},i}], \\ \ell_i - V^{\omega_i+\omega} < 0 & \text{on} \quad [a_i, b_i] \setminus [a_{\vec{c},i}, b_{\vec{c},i}]. \end{cases}$$

## Theorem (Gonchar-Rakhmanov)

Let $\mu_i$ be such that $\mu_i' > 0$ almost everywhere on $[a_i, b_i]$. Then

$$\begin{cases} \lim\limits_{|\vec{n}|\to\infty} |\vec{n}|^{-1} \log \left| f_i - P_{\vec{n}}^{(i)}/Q_{\vec{n}} \right| = -\left(\ell_i - V^{\omega_i+\omega}\right), \\ \lim\limits_{|\vec{n}|\to\infty} |\vec{n}|^{-1} \log |Q_{\vec{n}}| = -V^{\omega}. \end{cases}$$

New feature of the Hermite-Padé approximation is the appearance of divergence domains. Set

$$\begin{cases} D_i^+ & := & \left\{z : \ \ell_i - V^{\omega_i+\omega}(z) > 0\right\}, \\ D_i^- & := & \left\{z : \ \ell_i - V^{\omega_i+\omega}(z) < 0\right\}. \end{cases}$$

The domain $D_i^+$ is unbounded, this is precisely the domain in which the approximants $P_{\vec{n}}^{(i)}/Q_{\vec{n}}$ converge to $f_i$. The open set $D_i^-$ is bounded and possibly empty, within this set the approximants diverge to infinity.

Let $\mathfrak{R}$ be a Riemann surface obtained by

- taking $p+1$ copies of the extended complex plane
- cutting one of them, say $\mathfrak{R}^{(0)}$, along the union $\bigcup_{i=1}^{p}[a_{\vec{c},i}, b_{\vec{c},i}]$
- cutting each of the remaining copies $\mathfrak{R}^{(i)}$ along only one interval so that no two copies have the same cut
- gluing $\mathfrak{R}^{(0)}$ to $\mathfrak{R}^{(i)}$ crosswise.

Denote by $\mathfrak{R}_{\vec{n}}$ the Riemann surface constructed as above corresponding to the vector equilibrium problem for

$$\left(\frac{n_1}{|\,\vec{n}\,|}, \dots, \frac{n_p}{|\,\vec{n}\,|}\right).$$

All surfaces have genus zero.

Denote $\Phi_{\vec{n}}$ the rational function on $\mathfrak{R}_{\vec{n}}$ which is non-zero and finite except for a pole of order $|\,\vec{n}\,|$ at $\infty^{(0)}$ and a zero of multiplicity $n_i$ at each $\infty^{(i)}$; and satisfies $\prod_{k=0}^{p}\Phi_{\vec{n}}\big(z^{(k)}\big)\equiv 1$. It holds that

$$\frac{1}{|\,\vec{n}\,|}\log\left|\Phi_{\vec{n}}(\mathbf{z})\right| = \begin{cases} -V^{\omega_{\vec{n}}}(z)+\frac{1}{p+1}\sum_{k=1}^{p}\ell_{\vec{n},k}, & \mathbf{z}\in\mathfrak{R}_{\vec{n}}^{(0)}, \\ V^{\omega_{\vec{n},i}}(z)-\ell_{\vec{n},i}+\frac{1}{p+1}\sum_{k=1}^{p}\ell_{\vec{n},k}, & \mathbf{z}\in\mathfrak{R}_{\vec{n}}^{(i)}. \end{cases}$$

It is true that

$$\frac{1}{|\,\vec{n}\,|}\log\left|\frac{\Phi_{\vec{n}}^{(i)}(z)}{\Phi_{\vec{n}}^{(0)}(z)}\right| = V^{\omega_{\vec{n},i}+\omega_{\vec{n}}}(z)-\ell_{\vec{n},i} = V^{\omega_i+\omega}(z)-\ell_i+o(1)$$

locally uniformly in $\overline{\mathbb{C}}\setminus\bigcup_{i=1}^{p}[a_{\vec{c},i},b_{\vec{c},i}]$ as $|\,\vec{n}\,|\to\infty$, $i\in\{1,\dots,p\}$.

### Theorem

Given $\vec{c} \in (0,1)^p$ such that $|\,\vec{c}\,| = 1$ and a sequence of multi-indices $\{\,\vec{n}\,\}$,

$$n_i = c_i|\,\vec{n}\,| + o\left(|\,\vec{n}\,|\right),$$

let $[\,\vec{n}\,]_{\vec{f}}$ be the Hermite-Padé approximant to $\vec{f} = (f_1, \dots, f_p)$, where

$$\mu_i'(x) = \rho_i(x) \prod_{j=0}^{J_i} |x - x_{ij}|^{\alpha_{ij}} \prod_{j=1}^{J_i} \left\{ \begin{array}{ll} 1, & x < x_{ij} \\ \beta_{ij}, & x > x_{ij} \end{array} \right\},$$

$\alpha_{ij} > -1$, $\Re(\beta_{ij}) > 0$, and $\rho_i$ is a holomorphic function on $[a_i, b_i]$. Then

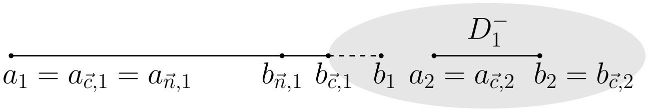
$$\left\{ \begin{array}{rcl} Q_{\vec{n}} & = & C_{\vec{n}}\big[1 + o(1)\big]\big(S\Phi_{\vec{n}}\big)^{(0)} \\ Q_{\vec{n}} f_i - P_{\vec{n}}^{(i)} & = & C_{\vec{n}}\big[1 + o(1)\big]\big(S\Phi_{\vec{n}}\big)^{(i)}/w_i, \end{array} \right.$$

where $S$ is a non-vanishing function on $\mathfrak{R}$ satisfying $S_{\pm}^{(i)} = S_{\mp}^{(0)}\big(\rho_i w_{i+}\big)$ on $(a_{\vec{c},i}, b_{\vec{c},i})$ and $w_i(z) := \sqrt{(z - a_{\vec{c},i})(z - b_{\vec{c},i})}$.

Recall that $\omega_{\vec{n},i}$ is the weighted equilibrium measure in the field $(\omega_{\vec{n}} - \omega_{\vec{n},i})/2$:

$$\min_{x\in[a_i,b_i]} V^{\omega_{\vec{n}}+\omega_{\vec{n},i}}(x) = \max \min_{x\in[a_i,b_i]} V^{\sigma+\sigma_i}(x)$$

In general, $\left[a_{\vec{n},i}, b_{\vec{n},i}\right]$, the support of $\omega_{\vec{n},i}$, is a proper subset of $[a_i, b_i]$.




## Local Riemann-Hilbert analysis

- Hard Edge: $b_{\vec{n},1} = b_{\vec{c},1} = b_1 \notin \partial D_1^-$ (Bessel)
- Soft Edge: $b_{\vec{n},1} = b_{\vec{c},1} < b_1$ (Airy)
- Soft-Type Edge I: $b_{\vec{n},1} \in \partial D_{\vec{n},1}^-$ (includes soft edge)
- Soft-Type Edge II: $b_{\vec{n},1} \notin \partial D_{\vec{n},1}^-$ but $b_{\vec{c},1} \in \partial D_1^-$

The following Riemann-Hilbert problem is needed for the local analysis around soft-type edges. It also corresponds to a certain family of solutions to Painlevé XXXIV equation.

(a,c) $\mathbf{\Psi}^i_{\alpha,\beta}$ is analytic off the rays with properly specified behavior at the origin;
(b)

$$\mathbf{\Psi}^i_{\alpha,\beta+} = \mathbf{\Psi}^i_{\alpha,\beta-} \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{on} \quad (-\infty, 0), \\ \begin{pmatrix} 1 & 0 \\ e^{\pm \mathrm{i}\pi\alpha} & 1 \end{pmatrix} & \text{on} \quad \left\{ \arg(\zeta) = \pm 2\pi/3 \right\}, \\ \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} & \text{on} \quad (0, \infty); \end{cases}$$

($d_1$,$d_2$)

$$\mathbf{\Psi}^i_{\alpha,\beta}(\zeta; s) = \left(\mathbf{I} + \mathcal{O}\left(\zeta^{-1}\right)\right) \frac{\zeta^{-\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} \exp\left\{\theta^i(\zeta; s)\sigma_3\right\}$$

where $\theta^1(\zeta; s) = -\frac{2}{3}(\zeta + s)^{3/2}$ and $\theta^2(\zeta; s) = -\left(\frac{2}{3}\zeta^{3/2} + s\zeta^{1/2}\right)$.

**Theorem**

Given $\alpha \in \mathbb{R}$ and $\Re(\beta) \geq 0$, $\mathbf{\Psi}^i_{\alpha,\beta}$ exists for all $s \in \mathbb{R}$. Assuming $\beta \neq 0$, it holds that

$$\mathbf{\Psi}^1_{\alpha,\beta}(\zeta; s) = \frac{\zeta^{-\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} \left( \mathbf{I} + \mathcal{O}\left( \sqrt{\frac{|s|+1}{|\zeta|+1}} \right) \right) \exp\left\{ \theta^1(\zeta; s)\sigma_3 \right\}$$

uniformly for all $\zeta$ and $s \in (-\infty, \infty)$; moreover, we have that

$$\mathbf{\Psi}^2_{\alpha,0}(\zeta; s) = \frac{\zeta^{-\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} \left( \mathbf{I} + \mathcal{O}\left( \sqrt{\frac{|s|+1}{|\zeta|+1}} \right) \right) \exp\left\{ \theta^2(\zeta; s)\sigma_3 \right\}$$

uniformly for all $\zeta$ and $s \in (-\infty, 0]$.

The case $\beta = 1$ was worked out by Its, Kuijlaars, and Östensson. The case $\alpha = 0$ is the current Master Thesis project of Bogadskiy under supervision of Its.