Meromorphic Extendibility and Rigidity of Interpolation

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In 2000, Edgar Stout obtained a characterization of continuous functions on boundaries of certain domains D in $\mathbb{C}^n$, $n \geq 1$, which extend holomorphically through D, in terms of a generalized argument principle. In the special case of the complex plane his result is

Theorem (Stout, 2000)

A continuous function f on a smooth Jordan curve T extends holomorphically throughout the interior domain D if and only if

$$\mathrm{w}_T\big(Q(z, f(z))\big) \geq 0$$

for any polynomial of two complex variables $Q(z, w)$ such that $Q(z, f(z)) \neq 0$ on T, where w_T stands for the **winding number** on T.

Josip Globevnik realized that the conditions in the previous theorem can be relaxed (first for the unit disk and then in the general case).

Theorem (Globevnik, 04 & 04)

Let $D \subset \mathbb{C}$ be a bounded domain whose boundary, say T, consists of finitely many pairwise disjoint simple closed curves. Then a continuous function f on T extends holomorphically throughout D if and only if

$$\mathrm{w}_T(f + h) \geq 0$$

for any function h holomorphic in D and continuous in $\overline{D}$ such that $f + h \neq 0$ on T.

Dmitry Khavinson pointed out that in the case of the unit disk the proof can be significantly shortened.

Theorem (Khavinson, 05)

A continuous function f on $\mathbb{T}$ extends holomorphically throughout $\mathbb{D}$ if and only if

$$\mathrm{w}_{\mathbb{T}}\big(f+h\big) \geq 0$$

for any function h in the disk algebra such that $f+h \neq 0$ on $\mathbb{T}$.

Proof

Let $\{f_n\}$ be a sequence of rational approximants to f. Denote by h_n the best H^∞ approximant to f_n. In fact, h_n belongs to the disk algebra. If f is not in the disk algebra, the function $f_n - h_n$ is non-zero and is **badly approximable**. Then

$$\mathrm{w}_{\mathbb{T}}(f-h_n) = \mathrm{w}_{\mathbb{T}}\left((f_n-h_n)\left(1-\frac{f_n-f}{f_n-h_n}\right)\right) = \mathrm{w}_{\mathbb{T}}(f_n-h_n) < 0.$$

Theorem (Globevnik, 08 & 08)

Let D be an open set in $\mathbb{C}$ whose boundary T consists of a finite number of pairwise disjoint simple closed curves. A continuous function f on T extends meromorphically through D with at most N poles there if and only if

$$\mathrm{w}_T(gf + h) \geq -N$$

for all g, h holomorphic in D and continuous in $\overline{D}$ such that $gf + h \neq 0$ on T.

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Proof in the case of $\mathbb{D}$ (not original)

Let $\{f_n\}$ be a sequence of rational approximants to f. Denote by m_n the best H^∞_N approximant to f_n. Again, $g_n m_n$ belongs to the disk algebra for some polynomial $\deg(g_n) \leq N$. Then

$$\mathrm{w}_\mathbb{T}(g_n f - g_n m_n) = \mathrm{w}_\mathbb{T}\left(g_n(f_n - m_n)\left(1 - \frac{f_n - f}{f_n - m_n}\right)\right) = \mathrm{w}_\mathbb{T}(g_n(f_n - g_n)) < -N$$

since according to Adamyan-Arov-Krein Theory $|f_n - m_n|$ is constant on $\mathbb{T}$ and $\mathrm{w}_T(f_n - m_n) < -2N$.

Question (Globevnik, 08)

Can the condition

$$\mathrm{w}_T(gf + h) \geq -N$$

be replaced by

$$\mathrm{w}_T(f + h) \geq -N?$$

Let f be Dini-continuous. Then $f = f_+ + f_-$, where f_+ is in the disk algebra and f_- is holomorphic outside of the unit disk. Assume that

$$Z_{|z|>1}(f_- + q) \leq \deg(q) + N \tag{1}$$

for any polynomial q. Then for any h in the disk algebra such that $f + h \neq 0$, there exists q satisfying

$$\mathrm{w}_{\mathbb{T}}(f + h) = \mathrm{w}_{\mathbb{T}}(f_- + q) = \deg(q) - Z_{|z|>1}(f_- + q) \geq -N.$$

If $f_- + q \neq 0$, then $\mathrm{w}_{\mathbb{T}}(f_- + q) \geq -N$ implies (1). Assume $f_- + q = 0$ somewhere on $\mathbb{T}$. If it is true that $(f_- + q)(\mathbb{T})$ has no interior, then there exists δ arbitrarily small satisfying $f_- + p \neq 0$ on $\mathbb{T}$ for $p = q + \delta$. Then

$$\begin{aligned} -N &\leq \mathrm{w}_{\mathbb{T}}(f + (p - f_+)) = \mathrm{w}_{\mathbb{T}}(f_- + p) = \deg(p) - Z_{|z|>1}(f_- + p) \\ &= \deg(q) - Z_{|z|>1}(f_- + q) \end{aligned}$$

by Rouche's theorem. Thus, again, $\mathrm{w}_{\mathbb{T}}(f_- + h) \geq -N$ implies (1).

Proposition (Raghupathi-Y)

Let f be an α-Hölder continuous function on $\mathbb{T}$, $\alpha > 1/2$. Let $N \in \mathbb{N}$. Then

$$\mathrm{w}_{\mathbb{T}}(f + h) \geq -N$$

for any function h in the disk algebra such that $f + h \neq 0$ on $\mathbb{T}$ if and only if

$$Z_{\mathbb{D}}(f_n + p) \leq N + n$$

holds for any $n \in \mathbb{Z}_+$ and any $\deg(p) \leq n$, where $f_n(z) = z^n f_-(1/z)$, $z \in \mathbb{D}$.

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Theorem (Raghupathi-Y)

Let g be a holomorphic function in $\mathbb{D}$ such that

$$Z_{\mathbb{D}}\big(z^n g(z) + p(z)\big) \leq N + n \quad \text{for any} \quad \deg(p) \leq n,$$

for any $n \in \mathbb{Z}_+$. Then g is a rational function of type (N, N) holomorphic in $\mathbb{D}$.

Theorem (Raghupathi-Y)

Let f be an α-Hölder continuous function on $\mathbb{T}$, $\alpha > 1/2$. Let $N \in \mathbb{Z}_+$. Then f extends to a meromorphic function with at most N poles in $\mathbb{D}$ if and only if

$$\mathrm{w}_{\mathbb{T}}(f+h) \geq -N$$

for every h in the disk algebra such that $f + h \neq 0$ on $\mathbb{T}$.