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THE GONCHAR–CHUDNOVSKIES CONJECTURE AND A FUNCTIONAL ANALOGUE OF THE THUE–SIEGEL–ROTH THEOREM

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Dedicated to the 90th anniversary of Andrei Aleksandrovich Gonchar

Abstract. This article examines the Gonchar–Chudnovskies conjecture about the limited size of blocks of diagonal Padé approximants of algebraic functions. The statement of this conjecture is a functional analogue of the famous Thue–Siegel–Roth theorem. For algebraic functions with branch points in general position, we will show the validity of this conjecture as a consequence of recent results on the uniform convergence of the continued fraction for an analytic function with branch points. We will also discuss related problems on estimating the number of "spurious" ("wandering") poles for rational approximations (Stahl's conjecture), and on the appearance and disappearance of defects (Froissart doublets).

1. Introduction

The article is devoted to functional analogues of Diophantine approximations of algebraic numbers, i.e., we will talk about the approximation of algebraic functions by rational ones. More specifically, we will consider the functional analogue of the famous theorem of Thue–Siegel–Roth on the rate of approximation of algebraic numbers by rational ones (see [18, 11]), which is the so-called Gonchar–Chudnovskies conjecture (see [14, 8]). For algebraic functions with branch points in general position, the validity of this conjecture will be deduced from recent results on the uniform convergence of the continued fraction for an analytic function with branch points (see [4]). We begin the introduction with basic definitions and concepts.

1.1. **Continued fractions and diagonal Padé approximants.** *Euclid's algorithm* (separating the integer part, inverting the fractional part, again separating the integer part, and so on) assigns to a real number $\alpha \in \mathbb{R}$ the continued fraction

$$\alpha = a_0 + \frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+\dots}, \tag{1.1}$$

whose coefficients (*partial quotients*) are the natural numbers $\{a_l \in \mathbb{N}\}$. Finite truncations of a continued fraction (*convergents*)

$$a_0 + \frac{1}{a_1+\dots}\,\frac{1}{a_n} =: \frac{p_n}{q_n} \in \mathbb{Q} \tag{1.2}$$

are rational (*Diophantine*) approximations of the real number α.

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Similarly, Euclid's algorithm (inverting, separating the polynomial part, again inverting the regular part, and so on) assigns to the power series (*germ* of the function) in the neighbourhood of the point at ∞

$$f(z) = \sum_{k=1}^{\infty} \frac{f_k}{z^k}, \quad f_1 = 1, \tag{1.3}$$

the *continued fraction* with polynomial coefficients $\{t_l(z)\}_{l=1}^{\infty}$:

$$f(z) = \frac{1}{t_1(z)+}\,\frac{1}{t_2(z)+}\,\frac{1}{t_3(z)+\dots}. \tag{1.4}$$

If $f(z)$ is not rational, then the continued fraction (1.4) does not terminate. A finite part of a continued fraction is a rational function

$$\frac{1}{t_1(z)+\dots}\,\frac{1}{t_n(z)} =: \frac{p_n(z)}{q_n(z)} = \pi_n(z), \tag{1.5}$$

which is called a *convergent*. Another name for the rational function (1.5) is a *diagonal Padé approximant* of the power series (1.3), which is constructively determined by the system of linear equations

$$R_n(z) := q_n(z)f(z) - p_n(z) = \mathcal{O}(1/z^{n+1}) \quad \text{as } z \to \infty \tag{1.6}$$

for the coefficients of the polynomials q_n and p_n of degree not higher than n. Generally speaking, this system does not have a unique solution, but the relation between the polynomials in (1.6) determines the rational function (1.5) uniquely (after cancellation). The index n and the rational function (1.5) are called *normal* if $\deg q_n(z) = n$ (after cancellation). In this case, we normalize the denominator

$$q_n(z) = z^n + \dots$$

1.2. **Functions with branch points and poles of rational approximations.** We consider a class of analytic functions whose germ (1.3) has an analytic continuation in $\bar{\mathbb{C}}$ along any path that does not pass through a finite number of points A:

$$f \in \mathcal{A}(\bar{\mathbb{C}} \setminus A), \quad \#A < \infty, \tag{1.7}$$

which we consider, for concreteness, to be branch points of the function f.

The rational approximants (1.5), being single-valued functions, approximate (in a certain sense) a holomorphic (i.e., single-valued) branch of the germ (1.3), (1.7) in some domain D (where this holomorphic branch can be selected):

$$\mathcal{D}_{f,\infty} := \{D\}\colon\ f \in \mathcal{H}(D), \quad \infty \in D \in \mathcal{D}_{f,\infty}. \tag{1.8}$$

Moreover, their poles (the zeros $q_n(z) := \prod_{k=1}^{n}(z - z_{k,n})$) approach (in a certain sense) the border ∂D of this area of holomorphy $D \in \mathcal{D}_{f,\infty}$.

1.3. **Nuttall's conjecture and Stahl's theorem.** In [19], J. Nuttall put forward a hypothesis about the convergence in capacity of convergents for (1.7)

$$\pi_n \xrightarrow[D^*]{\text{cap}} f, \quad D^* \in \mathcal{D}_{f,\infty}, \tag{1.9}$$

in the domain of holomorphy with a minimal (in the sense of logarithmic capacity) boundary

$$D^* \in \mathcal{D}_{f,\infty}\colon \quad \partial D^* = \min_{\partial D,\, D \in \mathcal{D}_{f,\infty}} \operatorname{cap}(\partial D). \tag{1.10}$$

Nuttall's conjecture was proven by G. Stahl [23, 24, 25] in an even wider class than (1.7), namely

$$\operatorname{cap}(A) = 0. \tag{1.11}$$

Moreover, Stahl found a weak limit of the poles π_n

$$\nu_{q_n}(z) := \frac{1}{n}\sum_{k=1}^{n}\delta(z - z_{k,n}) \xrightarrow{*} \omega(z), \quad \operatorname{supp}\omega = \partial D^*, \tag{1.12}$$

where ω is the equilibrium measure of the compactum $\bar{\mathbb{C}}\backslash D^*$, and proved the logarithmic (in capacity) asymptotic behavior of the polynomial $q(z)$:

$$\lim_{n\to\infty}\frac{1}{n}\log|q_n(z)| = -V^{\omega}(z) := \int \log|z-t|\,d\omega(t). \tag{1.13}$$

1.4. **"Wandering" poles and uniform convergence.** Stahl's remarkable result describes the behavior of the poles π_n "in general" (see (1.12)), but does not allow control of the dynamics (over n) of individual zeros q_n, called wandering or spurious, which unlike most zeros q_n, do not approach ∂D^*. To account for these subtle effects, more accurate asymptotics π_n are needed than (1.13).

In [4], we investigated the so called *strong asymptotics* for q_n (and for R_n)

$$\lim_{n\to\infty}\frac{q_n}{\Phi^n} = \,? \quad \text{in} \quad D^*, \tag{1.14}$$

where Φ is a suitably defined (see below) normalizing function. The strong asymptotics (1.14) allow control of wandering poles and answer questions about uniform convergence of the convergents (1.5) of the continued fraction (1.4) for analytic germs (1.3) from the class (1.7):

$$f - \pi_n = \frac{R_n}{q_n} \rightrightarrows \,?. \tag{1.15}$$

1.5. **Aim and structure of the work.** We would like to discuss here some applications of our result in [4] about the asymptotics of convergents of continued fractions for an analytic function with branch points. We will talk about well-known problems: on the normality of Padé approximants for algebraic functions (a functional analogue of the Thue–Siegel–Roth theorem and the Gonchar–Chudnovskies '$\varepsilon = 0$' conjecture), on estimating the number of "spurious" ("wandering") poles for rational approximations (Stahl's conjecture), and on appearance and disappearance of defects (Froissart doublets). In §2, we will introduce the necessary concepts to formulate the theorem on strong asymptotics from [4]. In §3, we present the formulation of this theorem and discuss the formulation itself and the immediate consequences of the theorem. §4 is devoted to the connection between the normality of the Padé approximants for the functions under consideration and the emergence of special divisors in the Jacobi problem of inversion of Abelian integrals. It is this connection that will make it possible to clarify the well-known problems mentioned above in §§5 and 6. For a preview of this article, see [3].

2. Necessary concepts

2.1. **Geometry of the extremal domain of holomorphy D^*.** We note that the original prooof of Stahl's theorem is quite difficult (it takes up several articles), and over the past 40 years no significant simplification has been obtained (even when the class under consideration has been narrowed from (1.11) to (1.7)).

However, for functions of the class (1.7), the existence proof of the extremal domain of holomorphy D^* (see (1.10)) and the description of the structure of the extremal sets D^* and ∂D^* thanks to E. A. Rahmanov (see [21], and also [17, 4]) currently look short and transparent. It is known that, in the class (1.7), the compactum of minimal capacity (1.10)

$$\Delta := \partial D^* = \bar{\mathbb{C}} \setminus D^* = E \cup \cup\Delta_k \tag{2.1}$$

consists of a finite number of open analytical arcs $\{\Delta_k\}$ and a finite set of points E, each of which is the end point of at least one of the arcs. For Green's function $g_\Delta(z)$ of the compactum Δ (with a singularity at ∞), there is a known representation that has the form for its derivative

$$h(z) := (2\partial_z g_\Delta)(z) = \frac{1}{z} + \ldots = \sqrt{\frac{B(z)}{A(z)}}, \tag{2.2}$$

where $2\partial_z := \partial_x - i\partial_y$, $\deg B = \deg A - 2$ and

$$A(z) := \prod_{k=1}^{p}(z - a_k), \quad \{a_1, \ldots, a_p\} := A \cap E, \tag{2.3}$$

and the polynomial B is determined from the condition of imaginary periods of the Abelian integral $\int h\,dz$ and the nature of monodromy of the approximated function f.

2.2. **Riemann surface and its standard characteristics**

2.2.1. *Definition and structure of sheets.* Let $\mathfrak{R}$ be the Riemann surface for the function h (see (2.2)):

$$\mathfrak{R} := \mathfrak{R}_h \quad \Leftrightarrow \quad h^2 - \frac{B(z)}{A(z)} = 0. \tag{2.4}$$

This is a hyperelliptic Riemann surface, whose two-sheeted cover $\bar{\mathbb{C}}$ has the form: $\mathfrak{R} = \mathfrak{R}^{(0)} \cup \mathfrak{R}^{(1)}$, $\mathfrak{R}^{(l)} := \pi^{-1}(\bar{\mathbb{C}} \setminus \bar{\Delta}) =: D^{(l)}$, $l = 0, 1$, with a cross-wise re-gluing of the sheets along the analytical arcs $\{\Delta_k\}$ and identification of the branch points E (see (2.1)). Thus, to each arc $\overline{\Delta_k}$ corresponds a cycle $L_k := \pi^{-1}(\overline{\Delta_k})$ on $\mathfrak{R} = \mathfrak{R}^{(0)} \cup L \cup \mathfrak{R}^{(1)}$, $L := \cup_k L_k$, oriented so that the domain $D^{(0)}$ remains on the left when its border L_k is traversed in a positive direction.

2.2.2. *Genus and homological basis.* The number and multiplicity of zeros of the polynomials $A(z)$ and $B(z)$ in (2.2), (2.3) uniquely define the genus $\mathfrak{R}$

$$g := \mathrm{gen}(\mathfrak{R}). \tag{2.5}$$

Let us fix a homological basis of cycles[1] on $\mathfrak{R}$

$$\{\mathbf{a}_k, \mathbf{b}_k\}_{k=1}^{g}: \quad \{\mathbf{b}_k\}_{k=1}^{g} \subset \{L_k\},$$

and $\mathbf{a}_k = a_k^{(0)} \cup a_k^{(1)}$, $a_k^{(j)} \subset \mathfrak{R}^{(j)}$, $j = 0, 1$, $\pi(a_k^{(0)}) = \pi(a_k^{(1)}) = \Delta_k^a$. We denote the canonical cuts of $\mathfrak{R}$ by

$$\widetilde{\boldsymbol{\mathfrak{R}}} := \boldsymbol{\mathfrak{R}} \setminus \cup_{k=1}^{g}(\mathbf{a}_k \cup \mathbf{b}_k) \quad \text{and} \quad \widehat{\boldsymbol{\mathfrak{R}}} := \boldsymbol{\mathfrak{R}} \setminus \cup_{k=1}^{g}\mathbf{a}_k.$$

2.2.3. *The Abelian integral and the leading-order term of the asymptotics.* We define on $\mathfrak{R}$

$$G = \int h\,dz \quad \text{and} \quad \Phi = e^G. \tag{2.6}$$

Fixing the unitary factor, we assume

$$\Phi(z) := \Phi^{(0)}(z) = \frac{z}{\mathrm{cap}(\Delta)} + O(1), \quad z \to \infty. \tag{2.7}$$

[1]To denote sets on a Riemann surface we use bold font or a superscript denoting a sheet. In the future (when this does not lead to confusion), we will omit the superscript (0) to denote the main (i.e., taken from $\mathfrak{R}^{(0)}$) branch of the analytic function on $\mathfrak{R}$.

From (2.2), we have $\Phi^{(0)}\Phi^{(1)} \equiv 1$ on $\bar{\mathbb{C}} \setminus \cup_k \Delta_k^a$. We denote the jumps on the cycles by

$$\frac{\Phi^+}{\Phi^-} = \begin{cases} \exp\{2\pi i\omega_k\} & \text{on} \quad \mathbf{a}_k, \\ \exp\{2\pi i\tau_k\} & \text{on} \quad \mathbf{b}_k, \end{cases} \tag{2.8}$$

where the real constants ω_k and τ_k are

$$\omega_k := -\frac{1}{2\pi i}\oint_{\mathbf{b}_k} h(t)dt \quad \text{and} \quad \tau_k := \frac{1}{2\pi i}\oint_{\mathbf{a}_k} h(t)dt. \tag{2.9}$$

2.2.4. *Basis of holomorphic differentials.* We fix

$$d\vec{\Omega} := (d\Omega_1, \dots, d\Omega_g)^T$$

to be a column vector of the normalized basis of holomorphic differentials

$$\oint_{\mathbf{a}_k} d\vec{\Omega} = \delta_{k,l}, \quad k, l = 1, \dots, g. \tag{2.10}$$

We denote the (symmetric with positive definite imaginary part) Riemann matrix by

$$\mathcal{B}_\Omega := \left[\oint_{\mathbf{b}_j} d\Omega_k\right]_{j,k=1}^g. \tag{2.11}$$

2.3. **Jacobi problem of inversion of Abelian integrals.** Assuming that

$$\rho(z) := (f^+ - f^-)(z)|_{\Delta\setminus E} \neq 0, \tag{2.12}$$

we fix a continuous branch of $\log(\rho/h^+)$ on $\Delta \setminus E$ and define the vector

$$\vec{c}_\rho := \frac{1}{2\pi i}\oint_L \log(\rho/h^+)\, d\vec{\Omega}, \tag{2.13}$$

which, together with the vectors (see (2.8), (2.9))

$$\vec{\omega} := (\omega_1, \dots, \omega_g)^T, \quad \vec{\tau} := (\tau_1, \dots, \tau_g)^T, \tag{2.14}$$

define the right-hand side of the following Jacobi problem for the inversion of Abelian integrals:

$$\sum_{j=1}^{g} \int_{b_j^{(1)}}^{\mathbf{t}_{n,j}} d\vec{\Omega} \equiv \vec{c}_\rho + n(\vec{\omega} + \mathcal{B}_\Omega\vec{\tau}) \qquad (\text{mod periods } d\vec{\Omega}). \tag{2.15}$$

Here, $\{b_j^{(1)}\}_{j=1}^g$ are the zeros of the polynomial $B(z)$. Simple zeros correspond to branch points, and zeros with even multiplicity are located on the $\mathfrak{R}^{(1)}$ sheet of the Riemann surface, and the number of times they are counted is half of their multiplicity.

We also recall that

$$\vec{c} \equiv \vec{e} \quad (\text{mod periods } d\vec{\Omega}) \quad \Leftrightarrow \quad \vec{c} - \vec{e} = \vec{n} + \mathcal{B}_\Omega\vec{m}, \quad \vec{n}, \vec{m} \in \mathbb{Z}^g.$$

It is known that the solution $\{\mathbf{t}_{n,j}\} =: \mathbf{t}_n \subset \mathfrak{R}$ of the Jacobi problem (2.15) always exists but may not be the only one. Any solution to the Jacobi problem has the form

$$\{\mathbf{t}_{n,j}\}_{j=1}^{g-2k} \cup \{z_j^{(0)}\}_{j=1}^k \cup \{z_j^{(1)}\}_{j=1}^k, \tag{2.16}$$

where, in the first group, there are no involution points (i.e., $\mathbf{a} = \widetilde{\mathbf{b}} \Leftrightarrow \mathbf{a} \neq \mathbf{b}$, $\pi(\mathbf{a}) = \pi(\mathbf{b})$), and involution points can be placed anywhere on $\mathfrak{R}$, and moreover, (2.16) remains the solution of (2.15). The agreement is to place all involution points of the solution on $\infty^{(0)}$ and $\infty^{(1)}$.

3. Formulas for strong asymptotics of rational approximations

3.1. **Limiting conditions.** First, we fix some restrictions on the class (1.7) and the index n, which we will use in the proof of the strong asymptotics theorem.

Nature of branching. We assume that all branches are algebro-logarithmic in nature (**AL**-condition), i.e.,

$$(3.1)\qquad (\mathbf{AL})\quad f\colon\quad f(z)=h_1(z)\psi(z)+h_2(z),\quad \psi(z)=\begin{cases}(z-a_k)^{\alpha(a_k)},\\ \log(z-a_k),\end{cases}$$

where $h_l(z)\in H(D^{\varepsilon}_{a_k})$, $l=1,2$. Condition (**AL**) is local.

Location of branch points. The following condition excludes some "degenerate" geometries of a compactum with minimal capacity Δ (see (1.10), (2.1)). We assume that the points of the set A (branch points of f) are arranged in such a way (**GP**-condition) that, in the expression (2.2) for the function h of the compactum Δ, the zeros of odd multiplicity of the polynomial $B(z)$ are simple.

$$(3.2)\qquad (\mathbf{GP})\quad A\colon\quad h(z)=\sqrt{\frac{B(z)}{A(z)}},\quad B(z)=\prod_{j=1}^{p-2m}(z-b_j)\prod_{j=p-2m+1}^{g}(z-b_j)^2.$$

Here, we count zeros of even multiplicity as before (see (2.15)), and the polynomial $A(z)$ is specified in (2.3).

It is clear that the points of the set A, subject to the **GP**-condition, are in general position. The equivalent form of the **GP**-condition fixes the geometry of the compactum (2.1).

$$(3.3)\qquad (\mathbf{GP})\quad A\colon\quad \begin{array}{rl} i) & e\in E\cap A \text{ — end point of exactly one arc from } \cup\Delta_k;\\ ii) & e\in E\setminus A \text{ — end point of exactly three arcs from } \cup\Delta_k.\end{array}$$

Jump of f on Δ. Condition for the non-degeneracy of the jump of f on Δ (**$f\Delta$**-condition):

$$(3.4)\qquad (\boldsymbol{f\Delta})\quad f\colon\quad f^+-f^-\neq 0\quad \text{on}\quad \Delta\setminus(A\cap E).$$

ε-normal indices. The sequence $\mathbb{N}_\varepsilon$ of (ε-normal) indices is determined (for a fixed $\varepsilon>0$) using the solutions $\mathbf{t}_n$ of the Jacobi problem (2.15)):

$$(3.5)\qquad (\mathbb{N}_{\boldsymbol{\varepsilon}})\quad n\in\mathbb{N}_\varepsilon\colon\quad \begin{cases}\forall\mathbf{t}_n & \Rightarrow\quad |\pi(\mathbf{t}_{n,j})|\le 1/\varepsilon\ \forall\mathbf{t}_{n,j}\in\mathfrak{R}^{(0)};\\ \forall\mathbf{t}_{n-1} & \Rightarrow\quad |\pi(\mathbf{t}_{n-1,j})|\le 1/\varepsilon\ \forall\mathbf{t}_{n-1,j}\in\mathfrak{R}^{(1)}.\end{cases}$$

We note that ε-normal indices, for sufficiently large n, are normal (for Padé approximants of functions from the class (1.7) with conditions (3.1) and (3.2)) in the sense of the definition from §1.1. (This is one of the corollaries of the proven theorem.) We also note that there is a unique solution to the problem (2.15) for $n\in\mathbb{N}_\varepsilon$.

3.2. **Nuttall–Szegő function.** The function $S_n(z)$ is defined as the solution to the following homogeneous Riemann boundary-value problem on $\mathfrak{R}$ – (3.2) (for proof of existence, uniqueness, and useful properties of $S_n(z)$, see [4]):

$$\textbf{(I)}\qquad (S_n\Phi^n)\in\mathfrak{M}(\mathfrak{R}\setminus L),\quad \exists S_n^{\pm}\in C\Big(L\cup\cup_{k=1}^{p}a_k\Big);$$

$$\textbf{(II)}\qquad (S_n\Phi^n)^-=(\rho/h^+)(S_n\Phi^n)^+\quad \text{on}\quad L\setminus E;$$

$$\textbf{(III)}\qquad S(t)=0,\quad t\in\{\mathbf{t}_{n,j}\}_{j=1}^{g}\setminus\Big(\{a_j\}_{j=1}^{p}\cup\{b_j^{(1)}\}_{j=1}^{g}\Big);$$

$$\textbf{(IV)}\qquad \begin{cases} |S_n(z^{(k)})| \sim |z-a|^{m(a)/2-(-1)^k(1+2\alpha_a)/4} & \text{as}\quad z^{(k)} \to a \in \{a_j\}_{j=1}^p, \\ |S_n(z^{(k)})| \sim |z-b|^{m(b)/2-1/2+(-1)^k/4} & \text{as}\quad z^{(k)} \to b \in \{b_j^{(1)}\}_{j=1}^{p-2m}, \\ |S_n(z^{(1)})| \sim |z-b|^{m(b)-1} & \text{as}\quad z^{(1)} \to b \in \{b_j^{(1)}\}_{j=p-2m+1}^{g}. \end{cases}$$

At the remaining (with the exception of **(III)** and **(IV)**) points of $\mathfrak{R}$ the function S_n is finite and does not vanish. Here, $m(t)$ denotes the number of times t appears among the elements of the set of solutions to the Jacobi problem (2.15) – $\mathbf{t}_n$.

3.3. **Formulation of the result.** The following holds true (see [4])

Theorem 3.1. *Let the germ of f – (1.3) belong to the class (1.7) with the additional conditions (**AL**), (**GP**), and (**$f\Delta$**) (see (3.1)–(3.4)). Then, for the denominators q_n of the rational approximants (1.5) and the remainder functions R_n – (1.6) for $n \in \mathbb{N}_\varepsilon$ for a fixed $\varepsilon, n \to \infty$, we have*

$$\begin{cases} q_n = (1+\nu_{n1})\gamma_n S_n \Phi^n + \nu_{n2}\,\gamma_n^* S_{n-1}\Phi^{n-1}, \\ R_n = (1+\nu_{n1})\gamma_n \frac{hS_n^{(1)}}{\Phi^n} + \nu_{n2}\;\gamma_n^* \frac{h\,S_{n-1}^{(1)}}{\Phi^{n-1}} \end{cases} \tag{3.6}$$

locally uniformly in D^* and

$$\begin{cases} q_n = (1+\nu_{n1})\gamma_n \Psi_n + \nu_{n2}\gamma_n^* \Psi_{n-1}, \quad \Psi_n := (S_n\Phi^n)^+ + (S_n\Phi^n)^-, \\ R_n^\pm = (1+\nu_{n1})\gamma_n \Big(\frac{hS_n^{(1)}}{\Phi^n}\Big)^\pm + \nu_{n2}\gamma_n^* \Big(\frac{hS_{n-1}^{(1)}}{\Phi^{n-1}}\Big)^\pm \end{cases}$$

locally uniformly in $\Delta \setminus E$. Here,

$$\gamma_n := \frac{\operatorname{cap}(\Delta)^n}{S_n(\infty)}, \quad \gamma_n^* := \frac{\operatorname{cap}(\Delta)^{n+1}}{S_{n-1}^{(1)}(\infty)},$$

and

$$|\nu_{n,j}| \le \frac{c(\varepsilon)}{n} \quad \textit{in} \quad \bar{\mathbb{C}} \quad \textit{and} \quad \nu_{n,j}(z) = O\Big(\frac{1}{z}\Big), \quad z \to \infty, \quad j = 1, 2.$$

3.4. **Discussion of the formulation of the result.** We will make a few remarks in relation to the statement of Theorem 3.1.

Remark 3.1. If the set A consists of two points, then Δ is the segment connecting them. In this case, Theorem 3.1 gives well-known formulas for the asymptotics of polynomials orthogonal with respect to a complex-valued weight (if $\rho > 0$ in (2.12), then it turns into the classical theorem of Bernstein–Szegő). Moreover, the Riemann surface $\mathfrak{R}$ has zero genus, therefore Φ is a simple conformal mapping of D^* onto $\{|z| > 1\}$ with a fixed point at infinity and a positive derivative there, and $S_n = S_\rho$ is the classical Szegő function.

Remark 3.2. The conditions (**GP**), (**AL**), and (**$f\Delta$**) (see (3.1), (3.2), and (3.4)) additional to (1.7) are technical in nature. In their absence, the method of the Riemann–Hilbert matrix problem used to prove Theorem 3.1 requires the solution of a series of special local boundary-value problems (for some advances in this direction, see [6]). Evidently, this obstacle can be circumvented by proving the existence of a solution to these problems without explicitly finding it (see similar examples in [10, 1]). Apart from that, there are other ways to potentially remove these restrictions. For example, in the absence of the condition (**$f\Delta$**) (see (3.4)), the asymptotic formulas can be obtained from the formulas of Theorem 3.1 using Christoffel's transformation (see [28]).

Remark 3.3. We note that the projections onto $\mathbb{C}$ of both functions S_n and hS_n^* are holomorphic in D^*. Moreover, S_n has exactly g zeros on $\mathfrak{R}$, which *depend* on n. We can also conclude that q_n has a zero in the neighborhood of every zero of S_n located on $D^{(0)}$. These zeros are called *spurious* or *wandering*, since their location is determined by the geometry of $\mathfrak{R}$ and by the weight function ρ, and, generally speaking, they do not approach Δ as n grows, whereas the remaining zeros q_n do so. On the other hand, those zeros of S_n, which are located on $D^{(1)}$, are the zeros of the projection onto $\mathbb{C}$ of the function S_n^* and, therefore, determine the location of the zeros of R_n (additional interpolation points).

Remark 3.4. Although Theorem 3.1 applies only to those normal indices which are also asymptotically normal, formula (3.6) explains what happens in degenerate cases. If, for some index n, the solution (2.15) is unique and contains l instances of the point $\infty^{(1)}$, then the function $S_n^{(1)}$ has a zero at infinity of order l, which, in combination with the second line in (3.6), shows that π_n "almost interpolated" the function f by additional l orders. Thus, there is a small perturbation of f that does not change the vector $\vec{c}_\rho$ in (2.13), which turns the index n into the last normal index before the apperance of a block of non-normal indices of size l. This corresponds to the fact that the solutions (2.15) will be special for the subsequent $l-1$ indices, and the solution for the index $n+l$ contains l instances of the point $\infty^{(0)}$. In the next section, we will dwell on the correspondence between the theorem on blocks of non-normal indices for Padé approximants and the structure and dynamics with respect to n of special (non-unique) solutions to the Jacobi problem (2.15). It is this correspondence that underlies the applications of Theorem 3.1 to the functional analogues of the fundamental theorems on the rate of Diophantine approximations.

We note that $\widehat{\rho} - [n/n]_{\widehat{\rho}} = R_n/q_n$ applied to $f := \widehat{\rho}$ gives Corllary 3.1 to Theorem 3.1 on the uniform convergence of Padé approximants.

Corollary 3.1. *Under the conditions of Theorem* 3.1, *for $n \in \mathcal{N}_\varepsilon$, it is true that*

$$f - [n/n]_{\widehat{\rho}} = [1 + \mathcal{O}(1/n)]\frac{S_n^*}{S_n}\frac{h}{\Phi^{2n}} \quad in \quad D_{n,\varepsilon}^*, \tag{3.7}$$

where $D_{n,\varepsilon}^* := D^* \setminus \cup_{j=1}^g \{|z - \mathbf{t}_{n,j}| < \varepsilon\}$, and furthermore, the set $\{|z - \mathbf{t}_{n,j}| < \varepsilon\}$ is replaced by $\{|z| > 1/\varepsilon\}$ if $\mathbf{t}_{n,j} = \infty$, and $\mathcal{O}(1/n)$ uniformly for a fixed $\varepsilon > 0$.

4. Block structure of the Padé table and special divisors

In this section, we will establish a connection between the Jacobi problem of inversion of Abelian integrals (2.15) and the normality of the Padé approximants of analytic germs with a finite number of branch points.

4.1. **Normality and the block theorem.**

We recall the main features of the theory of normal indices of diagonal Padé approximants (1.4) (see [18]). From (1.4), we have a formula for the denominators of the Padé fractions

$$q_n(z) := \frac{1}{H_n}\begin{vmatrix} f_0 & \dots & f_n \\ \multicolumn{3}{c}{\dotfill} \\ f_{n-1} & \dots & f_{2n-1} \\ 1 & \dots & z^n \end{vmatrix}, \qquad H_n := \begin{vmatrix} f_0 & \dots & f_{n-1} \\ \multicolumn{3}{c}{\dotfill} \\ f_{n-1} & \dots & f_{2n-2} \end{vmatrix},$$

where H_n are the Hankel determinants. Then, for the remainder function (1.6), we have

$$2\pi i R_n(z) = \oint_\infty \frac{q_n(t)f(t)}{z-t}\,dt = \frac{1}{z}\sum_{j=0}^\infty \oint_\infty \left(\frac{t}{z}\right)^j q_n(t)f(t)\,dt = \frac{m_n}{z^{n+1}} + \dots, \tag{4.1}$$

where $m_n := \frac{H_{n+1}}{H_n}$.

By definition (see §1.1), the rational approximant π_n from (1.3) and the index $n \in \mathcal{N}$ are called normal ($\mathcal{N}(f)$ is the set of normal indices) if

$$\exists(p_n\colon \deg p_n = n-1), \quad (q_n\colon \deg q_n = n)\colon \qquad \begin{cases} \pi_n := \frac{p_n}{q_n} & \text{– irreducible,} \\ f - \pi_n = O\Big(\frac{1}{z^{2n+1}}\Big). \end{cases}$$

Then, from (4.1) follows the normality criterion:

$$H_n \neq 0 \quad \Leftrightarrow \quad n \in \mathcal{N}.$$

Since the block theorem states that all Padé approximants with non-normal indices are combined into blocks, i.e., if n and $n+k$ are two consecutive normal indices, then (after reduction) it is true that

$$\pi_m = \pi_n \quad \forall m \in [n, n+k).$$

Thus, if $\mathcal{N} \equiv \mathbb{Z}_+$, then the size of all blocks is equal to 1 (trivial blocks). Scenario for the occurrence of a non-trivial block: let $n \in \mathcal{N}$, but

$$(f - \pi_n)(z) = \frac{A}{z^{2n+k}} + \dots, \quad A \neq 0, \tag{4.2}$$

then k is the size of the block and $A = \frac{m_n}{2\pi i}$.

The following statements are consequences of the block theorem:

$$\text{a)} \quad \begin{cases} n \in \mathcal{N}, \\ m_n = 0 \end{cases} \quad \Rightarrow \quad n \text{ is the start of the block of size} \geqslant 2;$$

and, more generally, for $k > 2$

$$\text{b)} \quad \begin{cases} n \in \mathcal{N}, \\ m_n = 0, \\ m_n m_{n+1} = 0, \\ \dots\dots\dots\dots\dots \\ m_n \dots m_{n+k-2} = 0 \end{cases} \quad \Rightarrow \quad n \text{ is the start of the block of size } \geqslant k.$$

4.2. **Block theorem and special divisors.** In this section, we will discuss how the leading-order term of the asymptotics of the rational approximants (see Theorem 3.1 in the neighborhood of the point at ∞) for large n and z

$$q_n \approx \gamma_n S_n \Phi^n, \quad R_n \approx \gamma_n\, h\, S_n^{(1)} \Phi^{-n} \tag{4.3}$$

is consistent with the theory of normality of Padé approximants, which we recalled in the previous section. Now we examine the zeros in the neighborhood of the point at infinity on the right-hand side of the approximate equalities (4.3).

We recall (see (2.2), (2.7)) that the function Φ has a pole of first order at infinity, and the function h has a zero of first order. The Szegő function S_n is holomorphic along the cut $\widetilde{\mathfrak{R}}$ (by convention, on the sheet $\widetilde{\mathfrak{R}}^{(0)}$, we keep the notation S_n, whereas we denote the values on $\widetilde{\mathfrak{R}}^{(1)}$ by $S_n^{(1)}$). On $\mathfrak{R}$ the Szegő function vanishes at g points $\{\mathbf{t}_{n,j}\}_{j=1}^g$ of the solution to the Jacobi problem (2.15) (except for solution points, which fall onto branch points of $\mathfrak{R}$):

$$\sum_{j=1}^{g} \int_{b_j^{(1)}}^{\mathbf{t}_{n,j}} d\Omega \equiv \vec{c}_\phi + n(\vec{\omega} + B_\Omega \vec{\tau}) \qquad (\text{mod periods } d\vec{\Omega}). \tag{4.4}$$

We define the sequence $\mathcal{N}_0$ of strictly normal indices of the solutions $\mathbf{t}_n$ of the Jacobi problem (4.4) (cf. (3.5)):

$$(\mathcal{N}_0) \quad n \in \mathcal{N}_0\colon \quad \infty^{(0)} \notin \{\mathbf{t}_{n,j}\}_{j=1}^g. \tag{4.5}$$

Turning to (4.3), we see that, for a strictly normal n, the zeros of the function S_n do not reduce the order of the pole of the leading-order term of the asymptotics q_n, which is equal to n, which agrees with (4.2).

Let us note one more property of strictly normal indices

$$n \in \mathcal{N}_0 \quad \Rightarrow \quad \exists!\{\mathbf{t}_{n,j}\}_{j=1}^{g} \tag{4.6}$$

ensures the uniqueness of the solution of (4.4). Indeed, involution points (special divisors) violate the uniqueness of the solution of (4.4), i.e.,

$$\mathbf{t}_{n,j_0} \in \mathfrak{R}^{(0)}, \quad \mathbf{t}_{n,j_1} \in \mathfrak{R}^{(1)} \quad \text{and} \quad \pi(\mathbf{t}_{n,j_0}) = \pi(\mathbf{t}_{n,j_1}),$$

since, in the solution of (4.4), a pair in involution can be replaced by another pair in involution but with any other projection, which is prohibited by (4.5). For the same reasons, for the indices

$$n\colon \quad \infty^{(1)} \notin \{\mathbf{t}_{n,j}\}_{j=1}^{g} \quad \Rightarrow \quad \exists!\{\mathbf{t}_{n,j}\}_{j=1}^{g}$$

the solution of (4.4) is unique.

Let us understand the block structure of non-unique solutions $\{\mathbf{t}_{n,j}\}$. The key role here is played by the Riemann relations, which express the b-periods of the Abelian differential of the third kind $d\Omega_{w_1,w_2}$ with simple poles and residues ± 1 at the points w_1, w_2, normalized

$$\oint_{a_j} d\Omega_{w_1,w_2} = 0, \quad j = 1, \dots, g, \tag{4.7}$$

via Abelian integrals of the first kind (2.10), (2.11)

$$\oint_{b_j} d\Omega_{w_1,w_2} = -2\pi i \int_{w_1}^{w_2} d\Omega_j, \quad j = 1, \dots, g. \tag{4.8}$$

Taking into account the connection of Green's differential (2.6) with purely imaginary periods and the differential with normalization (4.7)

$$dG = d\Omega_{\infty^{(1)},\infty^{(0)}} + 2\pi i \sum_{j=1}^{g} \tau_j \, d\Omega_j$$

relations (4.8) give:

$$\int_{\infty^{(1)}}^{\infty^{(0)}} d\vec{\Omega} = \vec{\omega} + B_\Omega \vec{\tau}. \tag{4.9}$$

This immediately leads to the remarkable implication

$$\infty^{(1)} \in \{\mathbf{t}_{n-1,j}\}_{j=1}^{g} \quad \Rightarrow \quad \infty^{(0)} \in \{\mathbf{t}_{n,j}\}_{j=1}^{g}. \tag{4.10}$$

Referring to (4.3), the resulting property can be interpreted as follows. If the zero of the Szegő function increases the order of the zero at the point ∞ in the leading-order term of the asymptotics of R_{n-1}, then in the leading-order term of the asymptotics of q_n this leads to a reduction in the order of the pole at the point ∞, which agrees with the theory of normality for the approximants (R_{n-1}, q_n).

Let us consider the scenario of the emergence of a block of involution points in the solutions of the Jacobi problem (4.4) and its structure. Let n and $n+k, k > 1$ be two consecutive strictly normal indices

$$n, n+k \in \mathcal{N}_0, \quad n+l \notin \mathcal{N}_0, \quad l = 1, \dots, k-1,$$

then

$$\mathbf{t}_n = \{\mathbf{t}_{n,j}\}_{j=1}^{g-k+1} \cup \{(\infty^{(1)})^{k-1}\}, \quad \pi(\mathbf{t}_{n,j}) < \infty,$$

and

$$\mathbf{t}_{n+l} = \{\mathbf{t}_{n,j}\}_{j=1}^{g-k+1} \cup \{(\infty^{(1)})^{k-1-l}\} \cup \{(\infty^{(0)})^l\}, \quad l = 1, \dots, k-1,$$

and

$$\infty^{(0)} \notin \mathbf{t}_{n+k}.$$

Since $\mathbf{t}_{n+k-1}$, $\mathbf{t}_{n+k}$ are the only solutions of (4.4), we obtain $\mathbf{t}_{n+k-1} \cap \mathbf{t}_{n+k} = \varnothing$.

The size of the block of non-unique solutions to the Jacobi problem is bounded by the number $g-1$. The maximal block is formed after a strictly normal index, for which $\mathbf{t}_n = \{(\infty^{(1)})^g\}$. It ends with the index $n+g-1$. Then comes the unique solution $\mathbf{t}_{n+g} = \{(\infty^{(0)})^g\}$ with the non-strictly normal index $n+g$, followed by the unique solution $\mathbf{t}_{n+g+1} \cap \mathbf{t}_{n+g} = \varnothing$ with the strictly normal index $n+g+1$. Thus, the block of non-strictly normal indices is bounded by g.

Now, we will make sure that the size of the blocks of rational approximants under the conditions of Theorem 3.1 is also bounded by g. To do so, we use the fact (see [4, Proposition 2, (2.23)–(2.24)]) that the limit points of the solutions $\mathbf{t}_n$ have the same block structure as the solutions $\mathbf{t}_n$. Then, for each limit point of $\mathbf{t}_n$, containing $\infty^{(0)}$, we can choose $\varepsilon > 0$, so that the strictly normal indices framing the corresponding block have become ε-normal for sufficiently large n, and, therefore, by Theorem 3.1, these ε-normal indices are normal indices for rational approximants, and the size of the block between them does not exceed g.

So, we have obtained Corollary 4.1, which is important for applications.

Corollary 4.1. *Under the conditions of Theorem* 3.1*, the size of the blocks of the Padé approximants for the function $f \in \mathcal{A}(\bar{\mathbb{C}} \setminus A)$ does not exceed the value g–genus of the Stahl Riemann surface* (2.4) *for f (i.e., of the double extremal domain D^* of the holomorphic continuation f).*

5. Theorems on the rate of Diophantine approximations and their functional analogues

In this section, we will discuss the main consequences of Theorem 3.1 concerning some well-known conjectures and problems on rational approximations of algebraic functions. For algebraic functions f, condition (**AL**) is obviously satisfied (see (3.1)), but, for the correct application of Theorem 3.1, we must assume that the branch points of f are in general position, i.e., conditions (**GP**) and ($\boldsymbol{f\Delta}$) (see (3.2) and (3.4)) are satisfied. Earlier in §3 (see Remark 3.2), we noted the technical nature and methods for removing these conditions.

5.1. **Rate of approximation of algebraic numbers by rationals.** First,we recall the basic known facts about the rate of Diophantine approximations of irrational numbers. Functional analogues will be considered below precisely for these theorems.

The rate of approximation of real numbers by rationals is measured on a scale, determined by the value of the denominator of the approximant. It is on this scale that the convergents of (1.1) are the best approximations, i.e., all rational numbers, which have a smaller denominator than that of the convergent, cannot be closer than this convergent to the number it approximates.

An upper bound on the rate of approximation is given by the following.

Theorem 5.1 (Markov–Hurwitz)**.** *For any irrational number $\alpha \notin \mathbb{Q}$, the inequality*

$$\left|\alpha - \frac{p}{q}\right| \leqslant \frac{1}{\sqrt{5}} q^{-2} \tag{5.1}$$

has infinitely many solutions $p/q \in \mathbb{Q}$, all of which are convergents for the number α.

If we do not specify the constant on the right-hand side of (5.1) and do not consider refinements of this theorem (Markov chains), then the proof of this statement (with constant 1) is very simple (it follows from the fact that among n natural numbers smaller than n, there are at least two identical ones).

Lower bounds for the rate of approximation of algebraic numbers $\alpha \in \mathbb{A}$ are much more non-trivial and have a long history. We begin with a simple statement showing that, for quadratic irrationalities, the exponent in the rate of approximations in (5.1) is exact. We have the following.

Theorem 5.2 (Liouville, 1844). *For any algebraic number of the k^{th} order*[2] $\alpha \in \mathbb{A}_k$, $k \geqslant 2$, *there exists a constant $C(\alpha)$ (effectively defined by α) such that*

$$\left|\alpha - \frac{p}{q}\right| \geqslant C(\alpha) q^{-k} \quad \forall p/q \in \mathbb{Q}. \tag{5.2}$$

For arbitrary $\alpha \in \mathbb{A}$, the fact that the exponent of the rate of approximation in the lower-bound estimate coincides with the exponent in the upper-bound estimate (5.1) up to an arbitrarily small $\varepsilon > 0$ was deduced by K. Roth. This result was preceded by several strong intermediate results, which must be recalled to complete the picture.

Theorem 5.3 (Thue, 1909). *Let $\alpha \in \mathbb{A}_k$, $k \geqslant 2$ be an arbitrary algebraic number. Then, $\forall \varepsilon > 0 \quad \exists C(\varepsilon)$ (not effective), such that*

$$\left|\alpha - \frac{p}{q}\right| \geqslant \frac{C(\varepsilon)}{q^{k/2+1+\varepsilon}} \quad \forall p/q \in \mathbb{Q}.$$

Then, the estimate of the Norwegian mathematician A. Thue was strengthened by C. L. Siegel.

Theorem 5.4 (Siegel, 1921). *Let $\alpha \in \mathbb{A}_k$, $k \geqslant 2$ be an arbitrary algebraic number. Then, $\exists C(\alpha)$ (not effective), such that*

$$\left|\alpha - \frac{p}{q}\right| \geqslant \frac{C(\alpha)}{q^{2\sqrt{k}}} \quad \forall p/q \in \mathbb{Q}. \tag{5.3}$$

Later, in 1947–1948, F. Dyson and A. O. Gelfand (independently) improved the constant factor 2 in the exponent on the right-hand side of (5.3), bringing it to $\sqrt{2}$. Finally, it is true that

Theorem 5.5 (Roth, 1955). *Let $\alpha \in \mathbb{A}$ be an arbitrary algebraic number. Then, $\forall \varepsilon > 0$ $\exists C(\varepsilon)$ (not effective), such that*

$$\left|\alpha - \frac{p}{q}\right| \geqslant \frac{C(\varepsilon)}{q^{2+\varepsilon}} \quad \forall p/q \in \mathbb{Q}. \tag{5.4}$$

Theorem 5.5 is often called the Thue–Siegel–Roth theorem. The original references and the proofs of the theorems given here can be seen in [18, 11].

Algebraic numbers, for which the Diophantine inequality (5.4) holds with $\varepsilon = 0$, are called *badly approximable irrationalities.* We will denote the set of such numbers by $\widetilde{\mathbb{A}}$. It is easy to show (see, for example, [18]) that the property $\alpha \in \widetilde{\mathbb{A}}$ is equivalent to boundedness of the partial quotients (coefficients) of the continued fraction of the number α:

$$\alpha \in \widetilde{\mathbb{A}} \quad \Leftrightarrow \quad \exists C(\alpha): \quad a_j < C, \quad j \in \mathbb{N}, \quad \text{for} \quad \alpha = a_0 + \frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+\dots}.$$

It is clear that $\mathbb{A}_2 \subseteq \widetilde{\mathbb{A}}$. This follows both from Liouville's theorem 5.2 and from the Euler–Lagrange theorem on periodicity of continued fractions for $\alpha \in \mathbb{A}_2$.

[2]That is, a root of an algebraic equation of the k^{th} degree with rational coefficients.

At the same time, to disprove or prove the same for

$$\mathbb{A}_k \subseteq \widetilde{\mathbb{A}}, \quad k > 2 \quad ??? \tag{5.5}$$

is an open and very difficult problem.

Numerical calculations of the continued fraction for $\alpha := \sqrt[3]{2}$ made in [7, 16, 2] leave little hope for (5.5) to hold:

$$\sqrt[3]{2} = 1 + \frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{5+}\,\frac{1}{1+\dots},$$

$a_{10} = 14, \dots, a_{36} = 543, \dots, a_{572} = 7451, \dots, a_{620} = 4941, \dots$.

5.2. **The non-Archemedean norm and approximations by rational functions.** As we have already noted in §1.1, the convergents (1.5) for (1.4) are the diagonal Padé approximants π_n of the power series $f(z)$ (see (1.3)). We recall the definition of Padé approximants equivalent to (1.6), which will clarify the analogy between the theories of normality of Padé approximants and the rate of Diophantine approximations. For any $n \in \mathbb{N}$, we let

$$\nu_n(f) = \sup\{\nu(f - r) \colon r \in \mathcal{R}_n\}, \quad \nu(f(z)) := \operatorname*{ord}_{z=\infty} f, \tag{5.6}$$

where $\mathcal{R}_n$ is the set of all rational functions of order no higher than n. Then,

$$\forall n \quad \exists! \pi_n \in \mathcal{R}_n \colon \nu_n(f) = \nu(f - \pi_n) \tag{5.7}$$

and the function π_n is called the n^{th} *diagonal Padé approximant of the series* $f(z)$. If $a, a > 1$ is fixed (arbitrarily), then the functional

$$\|f\|_a = a^{-\nu(f)}, \quad f(z) = \sum_{k>-\infty}^{\infty} \frac{f_k}{z^k} \in \mathbb{C}((z))_{z=\infty} \tag{5.8}$$

defines the non-Archemedean[3] norm $\|\cdot\|_a$ over the field of formal power series $\mathbb{C}((z))_{z=\infty}$ at the point $z = \infty$. Then, π_n from (5.7) is the rational function that best approximates f in the class $\mathcal{R}_n$ with respect to this norm:

$$\|f - \pi_n\|_a = \inf\{\|f - r\|_a \colon r \in \mathcal{R}_n\}. \tag{5.9}$$

Now a simple (linear algebra) formal property of Padé approximants:

$$\exists \Lambda(f) \colon \Lambda \subset \mathbb{N}, \quad \sharp\Lambda = \infty, \quad \nu_n(f) > 2n, \quad n \in \Lambda, \tag{5.10}$$

can be reformulated in the form of a functional analogue of Theorem 5.1.

Theorem 5.6 (Kronecker). *For any series $f \in \{\mathbb{C}((z))_{z=\infty} \setminus \mathcal{R}(z)\}$, the inequality*

$$\|f - r\|_a \leqslant (a^n)^{-2}, \quad r \in \mathcal{R}_n,$$

has infinitely many solutions $r \in \mathcal{R}(z) := \cup_n \mathcal{R}_n$, all of which are convergents of the series f (i.e., Padé approximants).

Thus, there are the following analogies between Diophantine approximations of numbers and rational approximations of power series:

$$\mathbb{R}_+, \quad \mathbb{Q}, \quad \mathbb{N}, \quad \mathbb{A} \;\leftrightarrow\; \mathbb{C}((z))_{z=\infty}, \quad \mathcal{R}(z), \quad \mathcal{P}(z), \quad \mathbb{A}(z);$$
$$|n| \quad \text{for} \quad n \in \mathbb{N}, \quad \|\cdot\| := |\cdot| \;\leftrightarrow\; \deg p \quad \text{for} \quad p \in \mathcal{P}(z), \quad \|\cdot\| := \|\cdot\|_a.$$

It will be more convenient below for us to formulate the results in terms of tangency of power series (5.6), than in terms of the non-Archemedean norm (5.8), where the usual signs of the inequalities in the theory of Diophantine approximations will be replaced by the opposite ones.

[3] I.e., on the right-hand side of the triangle inequality is the maximum term instead of a sum.

5.3. **Kolchin's conjecture and the functional theorem of Thue–Siegel–Roth.** In 1959, Kolchin (see [15]) put forward the hypothesis that, for the solutions of algebraic and differential equations, given by formal power series, over the field of rational functions $\mathcal{R}(z)$, for any $\varepsilon > 0$ $\exists C(f)$:

$$\nu_n(f) < (2+\varepsilon)n + C(f), \quad n \in \mathbb{N}. \tag{5.11}$$

This statement, rewritten in terms of the non-Archemedean norm (5.8), is a functional analogue of the Thue–Siegel–Roth theorem (Theorem 5.5). There exist several proofs of Kolchin's conjecture (5.11): both with a non-effective constant $C(f)$ (see [29]) and with effective constants [9, 8]. For more details about Kolchin's conjecture and its connection[4] with the modern development of the theory of Padé approximants, see the review by S. P. Suetin [27].

We note that the "ineffective" version of the functional analogue of the Thue–Siegel–Roth theorem (Theorem 5.5)

$$\lim_{n\to\infty} \frac{\nu_n(f)}{n} = 2 \tag{5.12}$$

was formulated (as an open problem) by A. A. Gonchar in [14] in a more general setting: for f – a multi-valued analytic function having a finite set of singular points, i.e., in a wider class than algebraic functions (1.7). The validity of this conjecture by Gonchar follows from Stahl's results [23, 24, 25].

5.4. **The Gonchar–Chudnovskies '$\varepsilon = 0$' conjecture.** In the same article of 1978 [14], A. A. Gonchar suggested that, for algebraic functions, the statement of the conjecture (5.12) can be significantly strengthened: *if f is an algebraic function, different from rational, then $\{\nu_n(f) - 2n\}$ is a bounded sequence.* That is, in the functional analogue of the Thue–Siegel–Roth theorem (Theorem 5.5) we can let $\varepsilon = 0$:

$$f \in \mathbb{A}(z) \quad \Rightarrow \quad \exists C(f)\colon \nu_n(f) < 2n + C(f), \quad n \in \mathbb{N}. \tag{5.13}$$

A recent (more accessible than [14]) review [5] presents interesting arguments by A. A. Gonchar motivating conjecture (5.12) and (5.13) in relation to the single-valuedness (multi-valuedness) of analytical functions, normality of Padé approximants, and lacunary power series.

Later (in 1984), the Chudnovskies [8] also assumed the validity of the '$\varepsilon = 0$'-strengthening of the Kolchin conjecture (5.13). In that work, they considered normality of special classes of Hermite–Padé approximants (functional analogue of Schmidt's theorem [22] for simultaneous approximations of a set of functions). In particular (see [8, c. 43]), they announced a proof of this conjecture for a set of solutions of a differential equation with constant coefficients, the hypergeometric equation. Also, in [8, Section 5], the '$\varepsilon = 0$'-conjecture is proven for a set of $d-1$ meromorphic functions on the Riemann surface of an algebraic function of order $d-1$ (it is appropriate to compare this result with a theorem by Nuttall: see [20]).

For algebraic functions of the second order, the validity of the '$\varepsilon = 0$'-conjecture (5.13) follows from Liouville's functional theorem: see [29].

The validity of the '$\varepsilon = 0$'-conjecture (5.13) (under the conditions of general position in Theorem 3.1) for an algebraic function of an arbitrary order follows from Corollary 4.1. Moreover, we have an effective constant (5.13):

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[4]This connection was noted earlier; see the quote from [8]: "In the functional case, the solution of Kolchin's problem is equivalent to the normality and almost normality of Padé approximants. This point will be expanded upon in further papers related to the conjecture that we can set $\varepsilon = 0$ in Kolchin's problem".

Corollary 5.1 (of Theorem 3.1)**.** *Let $f \in \mathbb{A}(z)$ satisfy the conditions of general position* (**GP**) *and* ($\boldsymbol{f\Delta}$) *(see* (3.1) *and* (3.4)*). Then* (5.13) *holds with a constant $C(f) := g$, where g is the genus of the Stahl Riemann surface* (2.4) *for f (i.e., a double holomorphic domain D^* of the holomorphic continuation f).*

As we have already noted (see Remark 3.2), the conditions of general position (**GP**) and ($\boldsymbol{f\Delta}$) are of a technical nature, and there are approaches for their possible elimination. This circumstance allows us to make an assumption about the effective constant in the most general case.

Hypothesis 5.1. *For an arbitrary $f \in \mathbb{A}(z)$,* (5.13) *holds, and*

$$C(f) := g + d, \quad \text{where} \quad g := \operatorname{gen}(\mathfrak{R}), \quad d := \deg(\operatorname{Dis}(f(z))). \tag{5.14}$$

Here, $\operatorname{Dis}(f(z))$ *denotes a polynomial (in z), which is a discriminant of the polynomial (in f), defining the algebraic function $f(z)$.*

Removing the (**GP**) constraint following (5.14) entails an extension of the class of Stahl's Riemann surfaces $\mathfrak{R}$, which is reflected by the first term in (5.14). The second term is related to condition ($\boldsymbol{f\Delta}$), it dominates the number of zeros of the jump of f on the extremal cut Δ. Each change in the sign of the jump on Δ can generate a "spurious" pole (see §6), which can end up at infinity and violate normality (this can be understood by cutting neighbourhoods of simple zeros of the jump out of Δ with subsequent passage to the limit).

In conclusion of this point, we note that the validity of the '$\varepsilon = 0$'-conjecture (5.13) implies boundedness of the (degree of) partial quotients of the continued fraction (1.4) for $f \in \mathbb{A}(z)$. To imagine something similar for the numbers $\alpha \in \mathbb{A}$ (see §5.1 or the quote[5] from [8]) is simply impossible.

6. "Spurious" poles, doublets and special divisors

There is probably no strict definition of a *wandering* or *spurious* pole of a rational approximation, but it is intuitively clear—this is a pole, which does not model the approximated function (i.e., the pole of the rational approximant is located "far" from the pole of the meromorphic one or from the extremal cut for multi-valued functions). In this section, we will discuss this phenomenon from the point of view of Theorem 3.1.

6.1. **Stahl's conjecture.** In 1998, Stahl formulated the following.

Hypothesis ([26]*Conjecture 6)**.** *Let f be an algebraic function, holomorphic at the point z_0. Then, there is a number $N = N(f) \in \mathbb{N}$ such that, for any $n \in \mathbb{N}$, the total number (taking into account multiplicities) of spurious poles of the Padé approximant π_n does not exceed $N = N(f)$.*

For us, the loss of normality means the appearance of a spurious pole at infinity, so Stahl's conjecture generalizes the Gonchar–Chudnovskies conjecture. We have the following.

Corollary 6.1 (of Theorem 3.1)**.** *Let $f \in \mathbb{A}(z)$ satisfy the conditions of general position* (**GP**) *and* ($\boldsymbol{f\Delta}$) *(see* (3.1) *and* (3.4)*). Then, there exists a subsequence of indices $\{n_j\}_{j=1}^{\infty} \subseteq \mathbb{N}\colon n_{j+1} - n_j < g \ \forall j \in \mathbb{N}$, for which Stahl's conjecture holds with a constant $N(f) := g$, where g is the genus of the Stahl's Riemann surface* (2.4) *for f (i.e., double extremal domain D^* of the holomorphic continuation f).*

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[5] "We would like to note that our conjecture that $\varepsilon = 0$ in Kolchin's problem refers to the problem of rational approximations in the functional case only. For numbers, it seems highly improbable that $\varepsilon = 0$."

As in the cases of Corollary 5.1 and Hypothesis 5.1, we may assume that the following is true in the general case.

Hypothesis 6.1. *Stahl's conjecture holds* $\forall f \in \mathbb{A}(z)$ *with a constant*

$$N(f) := g + d, \quad \text{where } g := \operatorname{gen}(\mathfrak{R}), \quad d := \deg(\operatorname{Dis}(f(z))).$$

6.2. **Classification of spurious poles and Froissart doublets.** The structure of the solutions to the Jacobi problem (2.15) allows us to classify them into so called *wandering* or *spurious* poles. We have

$$\mathbf{t}_n = \{\mathbf{t}_{n,j}\}_{j=1}^{g-2l} \cup \{z_j^{(0)}, z_j^{(1)}\}_{j=1}^{l}, \tag{6.1}$$

where, among the points $\{\mathbf{t}_{n,j}\}_{j=1}^{g-2l}$, there are no involution points. As we have already noted, involution points violate the uniqueness of the solution to the Jacobi problem and can be chosen anywhere. However, small perturbations of the coefficients of the function's germ transform these points into points with close projections in $\mathbb{C}$, thereby the solution acquires uniqueness, and the position of these points is fixed. Since

$$R_n(z) = q_n(z)\, f(z) - p_n(z),$$

then, taking into account (4.3), in the neighbourhood of the projections of points close to involution, there will be a zero of q_n, a zero of R_n, and therefore, a zero of p_n. This corresponds to the effect of the doublet [12]: zero and pole close to each other. However, doublets arise not only in the neighbourhood of additional interpolation points and the spurious pole. We consider the situation when in the neighbourhood of the projections of the points $\{\mathbf{t}_{n,j}\}_{j=1}^{g-2l} \cap \mathfrak{R}^{(0)}$ (i.e., in the neighbourhoods of spurious poles) there are no additional interpolation points. However, the uniform convergence (3.7) of the approximations on the boundary of the neighbourhood of the spurious pole and Rouchet's theorem (the version for meromorphic functions) imply the presence of a zero P_n in this neighbourhood.

In the next subsection, we consider an example of a spurious pole, in the neighbourhood of which there is no additional interpolation point, but there is a doublet.

6.3. **Example.** We fix the parameters a, b, c, $0 < c < a < b$. We construct a function f with branch points

$$A = \{-b, -a, a, b\}$$

such that its odd diagonal Padé approximants have a pole at the point c, which belongs to the domain of holomorphy $\bar{\mathbb{C}} \setminus \{[-b,-a] \cup [a,b]\}$. Following [13], we consider the periodic continued fraction:

$$f := \cfrac{l_1}{z - c_1 - \cfrac{l_2}{z - c_2 - f}}, \quad c_1 := c, \quad c_2 := -c, \quad l_1,\, l_2 > 0. \tag{6.2}$$

The numerators $p_n(z)$ and the denominators $q_n(z)$ of the convergents $\frac{p_n(z)}{q_n(z)}$ of the continued fraction (6.2) have the form for the initial indices

$$\begin{array}{llll} q_{-1} = 0, & q_0 = 1, & q_1 = z - c_1, & q_2 = (z - c_1)(z - c_2) - l_2, \\ p_{-1} = 1, & p_0 = 0, & p_1 = l_1, & p_2 = l_1(z - c_2). \end{array}$$

We find the algebraic function $f(z)$ directly from (6.2):

$$q_1(z) f^2 - (q_2(z) + p_1) f + p_2(z) = 0, \quad f = \frac{q_2(z) + p_1 - \Phi(z)}{2 q_1(z)}, \tag{6.3}$$

where, taking into account that $q_1 p_2 - q_2 p_1 = l_1 l_2$, we have

$$\Phi(z) := \sqrt{(q_2 + p_1)^2 - 4 q_1 p_2} = \sqrt{(q_2 - p_1)^2 - 4 l_1 l_2}.$$

Let

$$(6.4)\qquad a^2 := c^2 + (\sqrt{l_1} - \sqrt{l_2})^2, \quad b^2 := c^2 + (\sqrt{l_1} + \sqrt{l_2})^2,$$

then

$$(6.5)\qquad q_2 - p_1 = z^2 - (c^2 + l_1 + l_2) = z^2 - \frac{a^2+b^2}{2}, \quad -2\sqrt{l_1 l_2} = \frac{a^2-b^2}{2}.$$

From this, we have

$$\Phi(z) = \sqrt{(z^2-a^2)(z^2-b^2)}.$$

Also, from the relation (6.5), follows an equation for determining the coefficients $l_{1,2}$ (based on the branch points a, b and the parameter c):

$$l^2 - \Big(\frac{b^2+a^2}{2} - c^2\Big) l + \Big(\frac{b^2-a^2}{4}\Big)^2 = 0.$$

The choice of the root $l_1 > l_2$ ensures the cancellation of the zero in the denominator of the representation (6.3) of f (when choosing the corresponding branch $\Phi(z)$). Thus, the function f, defined by the continued fraction (6.2), whose coefficients are expressed by the parameters $0 < c < a < b$, is holomorphic in the domain $\bar{\mathbb{C}} \setminus \{[-b,-a] \cup [a,b]\}$. Moreover, from the formula for the numerators and denominators of convergents

$$y_{2r+\nu} = \frac{\rho_1^{2r}(y_{\nu+2} - \rho_2^2 y_\nu) - \rho_2^{2r}(y_{\nu+2} - \rho_1^2 y_\nu)}{\rho_1^2 - \rho_2^2}, \quad \nu = -1, 0, \quad r \in \mathbb{N},$$

where $\rho_{1,2}^2 = \frac{q_2 - p_1 + \Phi}{2}$, it follows that the odd convergents have a pole at the point c. Indeed, the initial conditions

$$y_1(z)|_{z=c} = q_1(c) = 0, \quad y_{-1} = q_{-1} = 0$$

guarantee that all denominators $q_{2n-1}(c)$ of the convergents are equal to zero at the point c. Similarly, it is verified that all numerators $p_{2n-1} \neq 0$, but in this case the point c is a limit point of the zeros of $p_{2n-1}(z)$. Indeed, we have the following numerical values (we let $c = 1, a = 2, b = 3$):

$$\begin{aligned} &q_5(1) = 0, \quad p_5(1) = 1, \quad p_5(1,00778\ldots) = 0,\\ &q_7(1) = 0, \quad p_7(1) = 1, \quad p_7(1,00045\ldots) = 0. \end{aligned}$$

Since the genus of the elliptic Riemann surface of the function f is equal to one, the presence of a false pole of the approximant at a point of the extremal domain of holomorphy guarantees the absence of an additional interpolation point not only in the neighbourhood of this pole, but also everywhere in $\bar{\mathbb{C}} \setminus \{[-b,-a] \cup [a,b]\}$. Thus, an example of a doublet without a close additional interpolation point is constructed.

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