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Discrete Schrödinger Operator on a Tree, Angelesco Potentials, and Their Perturbations

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Abstract—We consider a class of discrete Schrödinger operators on an infinite homogeneous rooted tree. Potentials for these operators are given by the coefficients of recurrence relations satisfied on a multidimensional lattice by multiple orthogonal polynomials. For operators on a binary tree with potentials generated by multiple orthogonal polynomials with respect to systems of measures supported on disjoint intervals (Angelesco systems) and for compact perturbations of such operators, we show that the essential spectrum is equal to the union of the intervals supporting the orthogonality measures.

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1. JACOBI MATRICES ON TREES

1.1. Trees. Let $\mathcal{T}$ be an infinite $(d+1)$-homogeneous *tree* with *root* O, whose vertex set we denote by $\mathcal{V}$. That is, the root O is incident with exactly d edges $[O, Y_i]$ that connect it to vertices $\{Y_i\}_{i=1}^d \subset \mathcal{V}$, and each vertex Y_i is incident with $d+1$ edges that connect it to the root and another d vertices $\{Z_{i,j}\}_{j=1}^d$, and so on. Thus, the root O has d *neighbors*, while any other vertex $Y \in \mathcal{V}$ has $d+1$ neighbors. We will write $Z \sim Y$ if the vertices Z and Y are neighbors.

1.2. Discrete electromagnetic Schrödinger operator. Let V be a real-valued function defined on the vertices $\mathcal{V}$, while W be a positive function defined on edges $[Z, Y]\colon Z \sim Y$ of the tree $\mathcal{T}$. Assuming that

$$\sup_{Y\in\mathcal{V}} |V_Y| < \infty, \qquad 0 < W_{Z,Y}, \qquad \sup_{Z\sim Y,\ Y\in\mathcal{V}} W_{Z,Y} < \infty, \tag{1.1}$$

we can define a bounded self-adjoint operator $\mathcal{J}$ on vectors $f \in \ell^2(\mathcal{V})$:

$$(\mathcal{J}f)_Y := V_Y f_Y + \sum_{Z\sim Y} W_{Z,Y}^{1/2} f_Z, \qquad Y \in \mathcal{V} \cup O. \tag{1.2}$$

In analogy with operators defined via (1.2) on lattices (see [20, 24, 7]), we will call $\mathcal{J}$ a *discrete electromagnetic Schrödinger operator* on a tree. When $d = 1$, the operator (1.2), (1.1) reduces to a Jacobi matrix acting on $\ell^2(\mathbb{Z}_+)$ (see [1]); thus, for $d > 1$ we will also call it a *Jacobi matrix* on a tree.

In the last decade the spectral theory of Jacobi matrices on trees has been undergoing rapid development (see [2, 11, 12, 14, 16–18]). In [5] for operators $\mathcal{J}$ we defined a wide class of *potentials*

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$\{V_Y, \{W_{Z,Y}\}_{Z\sim Y}\}_{Y\in\mathcal{V}\cup O}$ given by the recurrence coefficients of *multiple orthogonal polynomials*. In this paper we present results about the spectral properties of such operators (obtained recently in [6]).

1.3. The lift of potentials from the lattice to a tree. Recall the definition of the considered class of potentials of operators $\mathcal{J}$. On the lattice $\mathbb{N}^d$, consider an infinite path $\{\vec{n}^{(1)}, \vec{n}^{(2)}, \dots\}$ originating at $\vec{1} := (1,1,\dots,1)$ and satisfying the relation $\vec{n}^{(j+1)} = \vec{n}^{(j)} + \vec{e}_{k_j}$, $k_j \in \{1,\dots,d\}$, for any $j = 0,1,\dots$ (we use the standard notation $\vec{e}_1 = (1,0,\dots,0),\dots,\vec{e}_d = (0,\dots,0,1)$). These are paths on the d-dimensional lattice for which, as we move from $\vec{1}$ to infinity, the multi-index of each next vertex is increasing by 1 at exactly one position. Each such path can be mapped bijectively to a non-self-intersecting path on $\mathcal{T}$ that starts at O.

This construction defines a projection $\Pi\colon \mathcal{V} \to \mathbb{N}^d$ as follows: given $X \in \mathcal{V}$ we consider a path from O to X (which is unique), map it to a path on $\mathbb{N}^d$, and let $\Pi(X) \in \mathbb{N}^d$ be the endpoint of the mapped path. Note that on paths starting at O every vertex $Y \in \mathcal{V}$, which is different from O, has a unique *parent*, which we denote by $Y_{(\mathrm{p})}$. This allows us to define the following index function:

$$\imath\colon\ \mathcal{V} \to \{1,\dots,d\}, \quad Y \mapsto \imath_Y \quad \text{such that} \quad \Pi(Y) = \Pi(Y_{(\mathrm{p})}) + \vec{e}_{\imath_Y}, \tag{1.3}$$

which, in turn, allows us to distinguish the *children* of $Y \in \mathcal{V}$ and write $Z = Y_{(\mathrm{ch}),\,\iota_Z}$ if $Y = Z_{(\mathrm{p})}$.

Let $\mathsf{P} := \{a_{\vec{n},i}, b_{\vec{n},i}\}_{\vec{n}\in\mathbb{Z}_+^d,\ i\in\{1,\dots,d\}}$ be a collection of real parameters such that

$$0 < a_{\vec{n},i}, \quad \vec{n} \in \mathbb{Z}_+^d, \quad i \in \{1,\dots,d\}, \qquad \sup_{\vec{n}\in\mathbb{Z}_+^d,\ i\in\{1,\dots,d\}} \max\{a_{\vec{n},i}, |b_{\vec{n},i}|\} < \infty. \tag{1.4}$$

Using P, we define a potential $\{V_Y, \{W_{Z,Y}\}_{Z\sim Y}\}_{Y\in\mathcal{V}\cup O}$ on $\mathcal{T}$:

$$V_Y := \begin{cases} b_{\Pi(Y_{(\mathrm{p})}),\imath_Y}, & Y \neq O, \\ \displaystyle\sum_{i=1}^{d} \kappa_i b_{\vec{1}-\vec{e}_i,i}, & Y = O, \end{cases} \qquad W_{Z,Y} := \begin{cases} a_{\Pi(Y_{(\mathrm{p})}),\imath_Y}, & Z = Y_{(\mathrm{p})}, \\ a_{\Pi(Y),i}, & Z = Y_{(\mathrm{ch}),\,i}, \end{cases} \tag{1.5}$$

where the parameter $\vec{\kappa} \in \mathbb{R}^d$, $|\vec{\kappa}| := \kappa_1 + \ldots + \kappa_d = 1$, in the definition of V_O compensates for the absence of a parent of the root. Clearly, (1.4) implies (1.1), and therefore (1.2) with potential (1.5) defines an operator $\mathcal{J}_{\vec{\kappa}}$, which is bounded and self-adjoint on $\ell^2(\mathcal{V})$.

Notice that to each vertex $\vec{n}$ of the lattice $\mathbb{N}^d$ there corresponded two d-dimensional vectors: $\mathbf{a}_{\vec{n}} := \{a_{\vec{n},i}\}_{i=1}^d$ and $\mathbf{b}_{\vec{n}} := \{b_{\vec{n},i}\}_{i=1}^d$, while after the lift to the tree $\mathcal{T}$ these vectors are rearranged so that they form a potential of an electromagnetic Schrödinger operator: *scalar potential* V_Y, constructed out of the coordinates of $\mathbf{b}$, and *vector potential* $W_{Z,Y}$, constructed out of the coordinates of $\mathbf{a}$.

1.4. Multiple orthogonal polynomials and potentials. In the one-dimensional case, parameters $\mathsf{P} := \{a_n, b_n\}_{n\in\mathbb{Z}_+}$ satisfying (1.4) can be interpreted as the recurrence coefficients of orthonormal polynomials. This connection provides powerful tools for the analysis of spectral properties of Jacobi matrices. Let us recall the main facts of the theory of multiple orthogonal polynomials (MOPs) that were used in [5] to define parameters P for $d > 1$.

Let $\vec{\mu} := (\mu_1,\dots,\mu_d)$, $d \in \mathbb{N}$, be a vector of positive finite Borel measures defined on $\mathbb{R}$ and $\vec{n}$ be a given multi-index in $\mathbb{Z}_{\ge 0}^d$, $|\vec{n}| \ge 1$.

Type I MOPs $\{A_{\vec{n}}^{(j)}\}_{j=1}^d$ are nontrivial (i.e., not identically zero) polynomial coefficients of the linear form

$$Q_{\vec{n}}(x) := \sum_{j=1}^{d} A_{\vec{n}}^{(j)}(x)\,\mathrm{d}\mu_j(x), \qquad \deg A_{\vec{n}}^{(i)} < n_i \quad \bigl(n_i = 0 \ \Rightarrow\ A_{\vec{n}}^{(i)} \equiv 0\bigr), \quad i \in \{1,\dots,d\},$$

defined by the orthogonality conditions

$$\int x^l Q_{\vec{n}}(x) = 0, \qquad l < |\vec{n}| - 1. \tag{1.6}$$

Type II MOPs $P_{\vec{n}}(x)$, $\deg P_{\vec{n}} \le |\vec{n}|$, are nontrivial polynomials defined by

$$\int P_{\vec{n}}(x) x^l \, \mathrm{d}\mu_i(x) = 0, \qquad l < n_i, \quad i \in \{1, \dots, d\}, \quad \vec{n} \in \mathbb{Z}_+^d. \tag{1.7}$$

The polynomials of both types always exist, but their uniqueness is not guaranteed. If $\deg P_{\vec{n}} = |\vec{n}|$ for every polynomial $P_{\vec{n}}(x)$ satisfying (1.7) that is not identically zero, then the multi-index $\vec{n}$ is said to be *normal*. In this case $P_{\vec{n}}(x)$ is unique up to a multiplicative factor and we normalize it to be monic, i.e., $P_{\vec{n}}(x) = x^{|\vec{n}|} + \dots$. It turns out that $\vec{n}$ is normal if and only if the linear form $Q_{\vec{n}}(x)$ is defined uniquely up to multiplication by a constant. In this case, we will normalize it by

$$\int x^{|\vec{n}|-1} Q_{\vec{n}}(x) = 1. \tag{1.8}$$

We will say that a vector $\vec{\mu}$ is *perfect* if all multi-indices $\vec{n} \in \mathbb{Z}_+^d$ are normal.

When $\vec{\mu}$ is perfect, it is known [25] that the polynomials $P_{\vec{n}}(x)$ and the forms $Q_{\vec{n}}(x)$ satisfy the following *nearest-neighbor recurrence relations* (NNRRs):

$$\begin{cases} zP_{\vec{n}}(z) = P_{\vec{n}+\vec{e}_j}(z) + b_{\vec{n},j} P_{\vec{n}}(z) + \displaystyle\sum_{i=1}^d a_{\vec{n},i} P_{\vec{n}-\vec{e}_i}(z), \\[2ex] zQ_{\vec{n}}(z) = Q_{\vec{n}-\vec{e}_j}(z) + b_{\vec{n}-\vec{e}_j,j} Q_{\vec{n}}(z) + \displaystyle\sum_{i=1}^d a_{\vec{n},i} Q_{\vec{n}+\vec{e}_i}(z), \end{cases} \qquad j \in \{1, \dots, d\}, \quad \vec{n} \in \mathbb{N}^d. \tag{1.9}$$

For the coefficients $\{a_{\vec{n},i}, b_{\vec{n},i}\}$, we have the representations (see, e.g., [5])

$$a_{\vec{n},j} = \frac{\int P_{\vec{n}}(x)\, x^{n_j} \, \mathrm{d}\mu_j(x)}{\int P_{\vec{n}-\vec{e}_j}(x)\, x^{n_j-1} \, \mathrm{d}\mu_j(x)}, \qquad b_{\vec{n}-\vec{e}_j,j} = \int x^{|\vec{n}|} Q_{\vec{n}}(x) - \int x^{|\vec{n}|-1} Q_{\vec{n}-\vec{e}_j}(x), \qquad \vec{n} \in \mathbb{N}^d. \tag{1.10}$$

If $d > 1$, unlike the one-dimensional case, we cannot prescribe $\{a_{\vec{n},j}\}$ and $\{b_{\vec{n},j}\}$ arbitrarily. In fact, these coefficients satisfy the so-called *consistency conditions*, which is a system of nonlinear difference equations. This *discrete integrable system* and the associated *Lax pair* were studied in [8, 25].

We can see that if the system of measures $\vec{\mu}$ is perfect and the NNRR coefficients for the MOPs (1.6) and (1.7) satisfy (1.4), then the coefficients $\{a_{\vec{n},i}, b_{\vec{n},i}\}_{i \in \{1,\dots,d\}}$ define the potential (1.5) of a self-adjoint Jacobi matrix $\mathcal{J}_{\vec{\kappa}}$ (1.2) on the tree $\mathcal{T}$.

2. ANGELESCO SYSTEMS AND ASYMPTOTICS OF POTENTIALS

2.1. Perfectness of Angelesco systems and zero distribution of MOPs. A well-known example of a perfect system $\vec{\mu}$ is the so-called *Angelesco systems* [3] defined by the condition

$$\operatorname{supp} \mu_j = \Delta_j := [\alpha_j, \beta_j]: \qquad \Delta_i \cap \Delta_j = \varnothing, \quad i \neq j, \quad i, j \in \{1, \dots, d\}. \tag{2.1}$$

It is easy to see using (1.4) that such $\vec{\mu}$ is always perfect and therefore the corresponding NNRR coefficients define a Jacobi matrix via (1.2) and (1.5).

Indeed, condition (1.4) follows directly from (1.10) (see [5]), while the perfectness of $\vec{\mu}$ can be established as follows. For any $\vec{n} \in \mathbb{Z}_+^d$ it follows from the orthogonality relations (1.7) that the polynomial $P_{\vec{n}}(x)$ has n_j sign changes on each interval Δ_j, while conditions (2.1) and $\deg P_{\vec{n}} \le |\vec{n}|$ imply that $\deg P_{\vec{n}} = |\vec{n}|$, which is equivalent to perfectness of $\vec{n}$. Thus, the zeros of $P_{\vec{n}}(x)$ are simple

(n_j zeros on each Δ_j) and the polynomial itself can be factored as

$$P_{\vec n}(x)=\prod_{j=1}^{d}P^{(j)}_{\vec n}(x),\qquad P^{(j)}_{\vec n}(x):=\prod_{k=1}^{n_j}\big(x-x^{(j)}_{k,\vec n}\big),\qquad \big\{x^{(j)}_{k,\vec n}\big\}_{k=1}^{n_j}\subset\Delta_j,\quad j\in\{1,\dots,d\}. \tag{2.2}$$

Notice that for marginal multi-indices a MOP reduces to a regular orthogonal polynomial:

$$\vec n=n\vec e_j\qquad\Rightarrow\qquad P_{\vec n}(x)=P^{(j)}_n(x),\qquad j\in\{1,\dots,d\}, \tag{2.3}$$

where $P^{(j)}_n$ is a usual *orthogonal polynomial* with respect to the measure μ_j on Δ_j.

Along *ray sequences*, i.e., sequences $\mathcal N_{\vec c}=\{\vec n\}$ such that

$$n_i=c_i|\vec n|+o(|\vec n|),\qquad i\in\{1,\dots,d\},\qquad \vec c=(c_1,\dots,c_d)\in(0,1)^d,\qquad |\vec c\,|:=\sum_{i=1}^{d}c_i=1, \tag{2.4}$$

the limiting zero distribution of the polynomials $P_{\vec n}$ was obtained by Gonchar and Rakhmanov [15]. Elaborating on $\{x^{(j)}_{k,\vec n}\}_{k=1}^{n_j}\subset\Delta_j$, they have obtained a complete description (in terms of the vector equilibrium potentials) of the limiting measures ν_j of the zero counting measures of the polynomials $P^{(j)}_{\vec n}$ and, in particular, of their supports

$$\operatorname{supp}\nu_j=\Delta_{\vec c,j}\subseteq\Delta_j,\qquad i\in\{1,\dots,d\}. \tag{2.5}$$

It is clear that for marginal rays with $n_j=0$ as $|\vec n|\to\infty$ it holds that $\Delta_{\vec c,j}=\varnothing$. However, as c_j increases (and therefore the ray sequence moves away from the marginal one) the interval $\Delta_{\vec c,j}$ increases as well, exhibiting *strict inclusion* in (2.5), until it coincides with the support of the measure of orthogonality (which indeed happens for the marginal ray $|\vec n|=n_j$). The possibility of strict inclusion in (2.5) is called a *pushing effect.*

2.2. Ray asymptotics of the recurrence coefficients. Motivated by the spectral properties of the operators $\mathcal J_{\vec\kappa}$ (see (1.2) and (1.5)), let us consider the limiting behavior of the potentials generated by Angelesco systems when the discrete space variables $\{n_j\}$ approach infinity, i.e., $|\vec n|\to\infty$. The limits of the recurrence coefficients for $d>1$ (as in the case $d=1$) can be conveniently described with the help of algebraic functions whose branch points are the endpoints of the intervals supporting the limiting measures of the zero distribution of MOPs.

We can define a $(d+1)$-sheeted compact Riemann surface $\mathfrak R_{\vec c}$ with branch points at the endpoints of the intervals $\Delta_{\vec c,i}$ in the following way. Take $d+1$ copies of $\overline{\mathbb C}$. Cut one of them along the union $\bigcup_{i=1}^{d}\Delta_{\vec c,i}$; this copy is henceforth denoted by $\mathfrak R^{(0)}_{\vec c}$. Cut each of the remaining copies along exactly one interval $\Delta_{\vec c,i}$, so that no two copies have the same cut, and denote this copy by $\mathfrak R^{(i)}_{\vec c}$. To form $\mathfrak R_{\vec c}$, take $\mathfrak R^{(i)}_{\vec c}$ and glue the banks of the cut $\Delta_{\vec c,i}$ crosswise to the banks of the corresponding cut on $\mathfrak R^{(0)}_{\vec c}$. It can easily be verified that the Riemann surface thus constructed has genus 0. Denote by $\pi_{\vec c}$ the natural projection from $\mathfrak R_{\vec c}$ to $\overline{\mathbb C}$. We will also employ the notation $z^{(i)}$ for a point on $\mathfrak R^{(i)}_{\vec c}$ with $\pi_{\vec c}(z^{(i)})=z$, $i=0,1,\dots,d$, and $\boldsymbol z$ for any point on $\mathfrak R_{\vec c}$ with $\pi_{\vec c}(\boldsymbol z)=z$.

Since $\mathfrak R_{\vec c}$ has genus zero, one can arbitrarily prescribe zero/pole multisets of rational functions on $\mathfrak R_{\vec c}$ as long as the multisets have the same cardinality. Hence, we define $\chi_{\vec c}(\boldsymbol z)$ to be the rational function on $\mathfrak R_{\vec c}$ with a single pole at $\infty^{(0)}$ and such that

$$\chi_{\vec c}(z^{(0)})=z+\mathcal O(z^{-1})\qquad\text{as}\quad z\to\infty. \tag{2.6}$$

This is in fact a conformal map of $\mathfrak R_{\vec c}$ onto the Riemann sphere (it is uniquely defined by (2.6)). Further, let us define constants $\{A_{\vec c,i},B_{\vec c,i}\}_{i=1}^{d}$ by

$$\chi_{\vec c}(z^{(i)})=B_{\vec c,i}+A_{\vec c,i}z^{-1}+\mathcal O(z^{-2})\qquad\text{as}\quad z\to\infty. \tag{2.7}$$

Then the following theorem holds.

Theorem 2.1 [5]. *Assume that the measure μ_i is absolutely continuous with respect to the Lebesgue measure on Δ_i and that the density $\mathrm{d}\mu_i(x)/\mathrm{d}x$ extends to a holomorphic and nonvanishing function in some neighborhood of Δ_i for each $i \in \{1, \dots, d\}$. Further, let $\mathcal{N}_{\vec{c}} = \{\vec{n}\}$ be a sequence of multi-indices for which* (2.4) *holds. Then the recurrence coefficients $\{a_{\vec{n},i}, b_{\vec{n},i}\}$ from* (1.9) *and* (1.10) *satisfy*

$$\lim_{\mathcal{N}_{\vec{c}}} a_{\vec{n},i} = A_{\vec{c},i}, \qquad \lim_{\mathcal{N}_{\vec{c}}} b_{\vec{n},i} = B_{\vec{c},i}, \qquad \vec{c} \in (0,1)^d, \quad i \in \{1, \dots, d\}. \tag{2.8}$$

When $d = 1$ and we denote the single interval of orthogonality by $[\alpha, \beta]$, the corresponding conformal map χ can be explicitly written as

$$\chi(z^{(k)}) = \frac{z + (\alpha+\beta)/2 + (-1)^k \sqrt{(z-\alpha)(z-\beta)}}{2} \tag{2.9}$$

for $k \in \{0, 1\}$, and therefore $A = (\beta - \alpha)^2/16$ and $B = (\beta + \alpha)/2$, as expected.

The existence of limits in (2.8) and expression (2.7) for $A_{\vec{c},i}$ and $B_{\vec{c},i}$ were obtained using the *strong asymptotics* of MOPs derived in [5] and [26]. Thus, we can say that the asymptotics of the potentials is found as a part of the solution of the scattering problem for the operator $\mathcal{J}_{\vec{\kappa}}$ (see [5] for details). Note that when $d = 2$ and $\vec{c} = (1/2, 1/2)$, the limits of the coefficients of the so-called step line recurrence relations (relations satisfied by MOPs with indices along the diagonal) were obtained in [9]. These limits were expressed with the help of conformal mappings of $\mathfrak{R}_{\vec{c}}$, defined similarly to (2.6), and were based on the formulae of the strong asymptotics for MOPs obtained in [4] in the case where the measures of orthogonality were absolutely continuous with densities $\mathrm{d}\mu_i(x)/\mathrm{d}x$, $i = 1, 2$, belonging to the Szegő class.

3. ESSENTIAL SPECTRUM OF THE SCHRÖDINGER–ANGELESCO OPERATORS ON A BINARY TREE

3.1. Main theorem. Let us now describe new results whose details and proofs can be found in [6]. They are concerned with binary rooted trees ($d = 2$) and the essential spectrum of the operators $\mathcal{J}$ (1.2) on them. The following theorem is the main result of our work.

Theorem 3.1 [6]. *The essential spectrum of the Jacobi matrix $\mathcal{J}_{\vec{\kappa}}$* (1.2) *generated by the potentials* (1.5) *of the Angelesco systems* (2.1) *on two intervals* ($d = 2$) *satisfying the conditions of Theorem* 2.1 *is given by*

$$\sigma_{\mathrm{ess}}(\mathcal{J}_{\vec{\kappa}}) = \Delta_1 \cup \Delta_2. \tag{3.1}$$

Thus, for any pair of disjoint intervals $\Delta_1 \cap \Delta_2 = \varnothing$ there exists an operator $\mathcal{J}$ on a binary rooted tree whose essential spectrum is equal to these two intervals.

3.2. About the proof. Asymptotics of the potentials along marginal rays. Recall that in the case $d = 1$ one can prove Theorem 3.1 as follows (see, for example, [19]). One begins with the fact that the coefficients $\mathsf{P} := \{a_n, b_n\}_{n \in \mathbb{Z}_+}$ of the recurrence relations for the orthogonal polynomials with respect to some measure μ supported on $\Delta := \operatorname{supp} \mu$ have limits as $n \to \infty$ when $\mathrm{d}\mu(x)/\mathrm{d}x > 0$ a.e. on Δ. This is Rakhmanov's theorem [21, 22] (see also [13]). It further follows from Weyl's criterion that compact perturbations of a self-adjoint operator do not change the essential spectrum. That is, the essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{J})$ of the Jacobi matrix $\mathcal{J}$ (when $d = 1$) that corresponds to the measure μ is equal to the essential spectrum of the Jacobi matrix $\mathcal{J}^\infty$ with constant coefficients (the limits of a_n and b_n as $n \to \infty$). Then on the last step one needs to find the spectrum $\sigma(\mathcal{J}^\infty)$.

In the multidimensional case ($d = 2$) the proof of Theorem 3.1 follows similar lines. We start by investigating the limits of $\mathsf{P} := \{a_{\vec{n},i}, b_{\vec{n},i}\}_{i=1}^2$ as $|\vec{n}| \to \infty$, the coefficients of the recurrence relations for MOPs of an Angelesco system (μ_1, μ_2) with $\operatorname{supp}\mu_i = \Delta_i := [\alpha_i, \beta_i]$, $i = 1, 2$. However, unlike the case $d = 1$, where there exists a unique limiting Jacobi matrix, in the multidimensional case (see (2.8), (2.7), and (2.5)) there exists a continuum of limit points of the potential P parametrized by $\vec{c} := (c, 1-c)$, $c \in [0,1]$. Notice that Theorem 2.1 does not describe all of them, as it omits the marginal cases $\mathcal{N}_{(c,1-c)}$, $c \in \{0,1\}$ (i.e., *boundary layer*):

$$\begin{gathered}\mathcal{N}_{i-1} := \mathcal{N}_{(i-1,2-i)}, \qquad n_1 = (i-1)|\vec{n}| + o(|\vec{n}|), \qquad n_2 = (2-i)|\vec{n}| + o(|\vec{n}|),\\ i = 1, 2, \qquad |\vec{n}| \to \infty.\end{gathered} \tag{3.2}$$

Our main technical achievement in [6] is the extension for $d = 2$ of the results in [5] about the strong asymptotics of MOPs of Angelesco systems to the whole region of $\vec{c}$: $\vec{c} \in [0,1]^2$. That is, the following theorem holds.

Theorem 3.2. *Let $\vec{\mu}$ satisfy the conditions of Theorem 2.1 for $d = 2$. Then the ray limits*

$$\lim_{\mathcal{N}_c} a_{\vec{n},i} = A_{c,i}, \qquad \lim_{\mathcal{N}_c} b_{\vec{n},i} = B_{c,i}, \qquad i = 1, 2, \quad \mathcal{N}_c := \mathcal{N}_{(c,1-c)}, \tag{3.3}$$

exist for any $c \in [0,1]$. Moreover, for $c \in (0,1)$ they are the same as in (2.8), *while for marginal rays* (3.2) *it holds that $A_{0,1} = 0$ and*

$$A_{0,2} = \frac{(\beta_2 - \alpha_2)^2}{16}, \qquad B_{0,1} = \frac{1}{2}\left(\alpha_1 + \frac{\alpha_2 + \beta_2}{2} - \sqrt{(\alpha_2 - \alpha_1)(\beta_2 - \alpha_1)}\right), \qquad B_{0,2} = \frac{\beta_2 + \alpha_2}{2}$$

and analogous formulae hold for $A_{1,1}$, $A_{1,2}$, $B_{1,1}$, and $B_{1,2}$.

Therefore, when $d = 2$, Theorem 3.2, together with (2.7) and (2.9), provides a full description of the limits of Angelesco potentials $\mathsf{P} := \{a_{\vec{n},i}, b_{\vec{n},i}\}_{\vec{n} \in \mathbb{Z}_+^2,\, i=1,2}$.

3.3. About the proof. Limiting Jacobi matrices on a tree and their spectrum. The next step of the proof of Theorem 3.1 consists in establishing the connection between the essential spectrum of $\mathcal{J}$ (1.2) and its limiting operators $\mathcal{J}'$. First, as in [11], let us clarify the definition of the limiting operator, the so-called *right limit*, on trees ($d = 2$).

Let $\mathcal{G}$ be a 3-homogeneous tree (without root) with a marked vertex $O' \in \mathcal{V}_{\mathcal{G}}$. An operator $\mathcal{J}'$ with potential $\{\widetilde{V}_Y, \{\widetilde{W}_{Z,Y}\}_{Z \sim Y}\}_{Y \in \mathcal{V}_G}$ is defined on $\mathcal{G}$ via (1.2). Let $\{Y_n\}$, $n \in \mathbb{N}$, be a path on $\mathcal{V}$: $Y_n \sim Y_{n+1}$ for all $n \in \mathbb{N}$ (i.e., it is an infinite branch of $\mathcal{T}$). Let $B_r(Y)$ stand for a ball in $\mathcal{V}$ of radius $r \in \mathbb{N}$ with center at Y. Following the structure of the tree, we fix the labeling of vertices in $B_r(Y_n)$ and $B_r(O')$ so that

$$Y \in B_r(Y_n) \;\longleftrightarrow\; Y' \in B_r(O') \qquad \text{with} \quad Y_n \;\longleftrightarrow\; O'.$$

The operator $\mathcal{J}'$ is called the *right limit* or *$\mathcal{R}$-limit* of the operator $\mathcal{J}$ along $\{Y_n\}$ (which will be denoted by $\mathcal{J} \xrightarrow{\mathcal{R}} \mathcal{J}'$) if

$$\exists \{n_j\} \subset \mathbb{N}\colon \qquad V_Y \to \widetilde{V}_{Y'}, \qquad W_{Z,Y} \to \widetilde{W}_{Z',Y'}, \qquad Y \in B_r(Y_{n_j}), \qquad j \to \infty, \tag{3.4}$$

for any fixed $r \in \mathbb{N}$ (recall that $Y' \in B_r(O')$ and $Z \sim Y$, which implies that $Z' \sim Y'$).

In [11, Theorem 4], it was shown that the essential spectrum of a self-adjoint operator on a graph coincides with the closure of the union of the spectra of all its $\mathcal{R}$-limits:

$$\sigma_{\mathrm{ess}}(\mathcal{J}) = \overline{\bigcup_{\mathcal{J}'} \sigma(\mathcal{J}')}, \qquad \mathcal{J} \xrightarrow{\mathcal{R}} \mathcal{J}'. \tag{3.5}$$

To apply (3.5) to the problem of finding $\sigma_{\mathrm{ess}}(\mathcal{J}_{\vec{\kappa}})$ for the operators $\mathcal{J}_{\vec{\kappa}}$ (1.2) generated by the potentials (1.5) of the Angelesco systems (2.1) supported on intervals Δ_1 and Δ_2, we introduce a family of operators that have the same totality of right limits as $\mathcal{J}_{\vec{\kappa}}$.

To this end, we fix a *type* for each edge of the rooted tree $\mathcal{T}$, assuming that two edges emanate from the root: one edge $\{O, Y'\}$ of the *first type* (we write $t(\{O, Y'\}) = 1$) and another edge $\{O, Y''\}$ of the *second type* (we write $t(\{O, Y''\}) = 2$). In this case, every vertex $Y \in \mathcal{V}$ is incident with three edges of types i_Y, 1, and 2. Accordingly, we say that a *vertex* $Y \in \mathcal{V}$ *has type* $i \in \{1, 2\}$ if it is incident with two edges of type i.

Define a family of operators $\{\mathcal{L}_c\}$, $c \in [0, 1]$, by

$$(\mathcal{L}_c\psi)_Y = \sum_{j\in\{1,2\},\ Y'\sim Y,\ t(Y,Y')=j} \sqrt{A_{c,j}}\,\psi_{Y'} + B_{c,\iota_Y}\psi_Y, \tag{3.6}$$

where the definition at the root is not essential for what is presented further below (due to the Weyl criterion [23]), just as the dependence of $\mathcal{J}_{\vec{\kappa}}$ on the parameter $\vec{\kappa}$, which we can also ignore: $\mathcal{J} := \mathcal{J}_{\vec{\kappa}}$.

Therefore, it follows from the definitions of the operator $\mathcal{L}_c$ (3.6) and the right limit (3.4), as well as from (3.3), that the totality of right limits of $\mathcal{J}$ coincides with the totality of right limits of the family $\{\mathcal{L}_c\}$:

$$\bigl\{\mathcal{J}'\colon\ \mathcal{J} \xrightarrow{\mathcal{R}} \mathcal{J}'\bigr\} = \bigcup_{c\in[0,1]} \bigl\{\mathcal{J}''\colon\ \mathcal{L}_c \xrightarrow{\mathcal{R}} \mathcal{J}''\bigr\}. \tag{3.7}$$

Further, applying (3.5) to the closures of both sides of (3.7) gives

$$\sigma_{\mathrm{ess}}(\mathcal{J}) = \bigcup_{c\in[0,1]} \overline{\bigcup_{\{\mathcal{J}''\colon\ \mathcal{L}_c \xrightarrow{\mathcal{R}} \mathcal{J}''\}} \sigma(\mathcal{J}'')}. \tag{3.8}$$

Now, applying (3.5) to $\mathcal{L}_c$, we get

$$\overline{\bigcup_{\{\mathcal{J}''\colon\ \mathcal{L}_c \xrightarrow{\mathcal{R}} \mathcal{J}''\}} \sigma(\mathcal{J}'')} = \sigma_{\mathrm{ess}}(\mathcal{L}_c).$$

The spectrum of $\mathcal{L}_c$ can be found directly by considering its action on vertices of the first and second types (see [6] for details), which gives

$$\sigma_{\mathrm{ess}}(\mathcal{L}_c) = \Delta_{c,1} \cup \Delta_{c,2}.$$

Finally, it follows from the already mentioned work of Gonchar and Rakhmanov [15] that

$$\bigcup_{c\in[0,1]} (\Delta_{c,1} \cup \Delta_{c,2}) = \Delta_1 \cup \Delta_2,$$

which, in view of (3.8), yields (3.1).

3.4. Consequences for compact perturbations. As we have already noted at the end of Subsection 1.4, far from all potentials satisfying (1.4) come from the coefficients of the recurrence relations (1.9) for MOPs (1.6), (1.7) when $d > 1$. This is, in particular, true for the operators $\{\mathcal{L}_c\}$ (see (3.6)). Nevertheless, the Angelesco potentials (1.5), (1.10), (2.1) can serve as a convenient "background" class of potentials for discrete electromagnetic Schrödinger operators on $(d+1)$-homogeneous rooted trees whose essential spectrum consists of d arbitrary disjoint intervals (which we have demonstrated in the case $d = 2$).

Let $d = 2$ and $\mathsf{P} := \{\widehat{a}_{\vec{n},i}, \widehat{b}_{\vec{n},i}\}_{\vec{n}\in\mathbb{N}^2,\ i=1,2}$ be a general class of potentials defined in (1.4). Further, let the constants $\{A_{c,1}, A_{c,2}, B_{c,1}, B_{c,2}\}_{c\in[0,1]}$ be the limits of the recurrence coefficients for some Angelesco system generated by the intervals Δ_1 and Δ_2, i.e., be defined via (2.7).

We will write $\mathsf{P} \in \mathsf{P}_{\mathrm{Ang}}(\Delta_1, \Delta_2)$ if P satisfies

$$\lim_{\mathcal{N}_c} \widehat{a}_{\vec{n},i} = A_{c,i}, \qquad \lim_{\mathcal{N}_c} \widehat{b}_{\vec{n},i} = B_{c,i}, \qquad c \in [0,1], \quad i \in \{1,2\} \tag{3.9}$$

(let us stress that P might not be a collection of the recurrence coefficients of an Angelesco system).

Let the Jacobi matrix $\widetilde{\mathcal{J}}$ be given by (1.2), (1.5) with coefficients $\mathsf{P} \in \mathsf{P}_{\mathrm{Ang}}(\Delta_1, \Delta_2)$. Then it follows from Theorem 3.1 that

$$\sigma_{\mathrm{ess}}(\widetilde{\mathcal{J}}) = \Delta_1 \cup \Delta_2.$$

Note that the constants $\{A_{c,1}, A_{c,2}, B_{c,1}, B_{c,2}\}$, considered as functions of the parameter $c \in [0,1]$, can be described via a system of differential equations and parametrizations of Riemann surfaces $\mathfrak{R}_c$, as was recently demonstrated in [10].

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