MEROMORPHIC APPROXIMATION: SYMMETRIC CONTOURS AND WANDERING POLES

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Abstract. This manuscript reviews the study of the asymptotic behavior of meromorphic approximants to classes of functions holomorphic at infinity. The asymptotic theory of meromorphic approximation is primarily concerned with establishing the types of convergence, describing the domains where this convergence takes place, and identifying its exact rates. As the first question is classical, it is the latter two topics that this survey is mostly focused on with the greater emphasis on the exact rates. Three groups of approximants are introduced: meromorphic (AAK-type) approximants, L^2-best rational approximants, and rational interpolants with free poles. Despite the groups being distinctively different, they share one common feature: much of the information on their asymptotic behavior is encoded in the non-Hermitian orthogonality relations satisfied by the polynomials vanishing at the poles of the approximants with the weight of orthogonality coming from the approximated function. The main goal of the study is extracting the generic asymptotic behavior of the zeros of these polynomials from the orthogonality relations and tracking down those zeros that do not conform to the general pattern (wandering poles of the approximants).

1. Introduction

This survey concerns functions meromorphic in a given domain that are the closest in some metric to a fixed function on the boundary of the domain. The first step in this direction was taken in [70] where the striking connection between spectral theory of Hankel operators and approximation by functions holomorphic in the unit disk was discovered. This result received an impressive development known as Adamyan-Arov-Krein Theory [1, 28, 90, 125]. The latter, in particular, provides the rate of approximation of a bounded function on the unit circle by functions meromorphic in the unit disk with an increasing number of poles in the uniform norm via singular values of a Hankel operator whose symbol is the approximated function.

The AAK theory had a considerable impact in rational approximation, since by retaining only the principal part of a best meromorphic approximant to a function analytic outside the disk, one obtains a near-best rational approximant to that function [47]. This is instrumental in Parfenov's solution to the Gonchar conjecture [48] on the degree of rational approximation to holomorphic functions on compact subsets of their domain of analyticity, and also in Peller's converse theorems on smoothness of functions from their error rates in rational approximation [87, 88].

The mathematical beauty of the AAK approach as well as its cross-area nature attracted a lot of attention that resulted in a deeper and more encompassing theory [91, 82, 53, 89, 35, 96, 92, 122]. In particular, the theory was extended to integral norms [69, 14, 97] and more general domains [97, 60, 11]. These generalizations turned out to be valuable in spectral theory and the modeling of signals and systems [4, 52, 73, 99]. Moreover, they found an application in inverse problems, namely, "crack detection" in homogeneous media from Neumann-to-Dirichlet data [9, 11, 55] and electroencephalography [10, 62, 56, 34].

One intriguing and extremely important feature of the L^2-best rational approximants is that they interpolate the approximated functions at the reflections of their poles across the boundary circle with order 2. This places them into the intersection between the theory of meromorphic approximation and the theory of rational interpolation with free poles. The latter is also known as multipoint Padé approximation when the interpolation points are scattered over the extended complex plane and as classical Padé approximation when interpolation is done at one point only (in the Newton sense, of course).

Padé approximants, a truncation of continued fractions in the field of Laurent series, are among the oldest and simplest constructions in function theory [57]. These are rational functions of type (m,n)[1] that interpolate a function element at a given point with order $m+n+1$. They were introduced for the exponential function by Hermite [54], who used them to prove the transcendency of e, and later expounded more systematically by his student Padé [81]. Ever since their introduction, Padé approximants have been an effective device in analytic number theory [54, 101, 103, 59], and over the last decades they became an important tool in physical modeling and numerical analysis [8, 29, 39, 95]. Padé approximants also provide an important link between theory of rational interpolation and the field of orthogonal polynomials [113].

As can be deduced from its definition, Padé approximant of type (m,n) is the best rational approximant to the function at one point. However, proving the convergence to the approximated function on a larger set is no small matter. Padé approximants with denominators of fixed degree converge uniformly to the approximated function in the disk of meromorphy of the latter granted the degree of the denominators matches the number of poles of the function [36]. In fact, even the converse statement takes place [49, 114, 115]. On the other hand, if the degree is greater than the number of poles, convergence still holds, but in a smaller domain as some poles of the Padé approximants cluster in the domain of meromorphy [27, 67]. This rather unpleasant feature (clustering of some poles in the domain of holomorphy of the approximated function) is in fact generic and has deep mathematical reasons behind it. The poles asymptotically accumulating in the domain of holomorphy received the name of *wandering* or *spurious poles* [110] and are one of the main objects of this review. Their behavior is so disruptive that the so-called Padé conjecture, actually raised in [7], which laid hope for the next best thing namely convergence of a subsequence, was eventually settled in the negative [66]. Shortly after, a weaker form of the conjecture [108], dealing with hyperelliptic functions, was disproved as well [26].

Even though, generically, uniform convergence cannot be achieved, it should be possible to track the dynamics of the wandering poles and prove nearly uniform convergence for a large class of functions, namely, Cauchy integrals on *symmetric contours*. Some work in this direction has already been done in [80, 77, 78, 116, 117,

[1] These are rational functions with the numerators of degree at most m and the denominators of degree at most n.

118, 119, 58, 42], including by the author [20, 5, 123]. However, the current state of knowledge is far from complete and concerns mostly classical Padé approximants.

2. Mathematical Framework

In this section we rigorously define all the main objects and describe their basic properties.

2.1. **Meromorphic Approximants.**

Denote by H^p, $p \in [1, \infty]$, the *Hardy space* of the unit disk consisting of holomorphic functions f such that

$$\|f\|_p^p := \sup_{0<r<1} \frac{1}{2\pi} \int_{\mathbb{T}} |f(r\xi)|^p |d\xi| < \infty \quad \text{if} \quad p \in [1, \infty),$$
$$\|f\|_\infty := \sup_{z \in \mathbb{D}} |f(z)| < \infty \quad \text{if} \quad p = \infty.$$

A function in H^p is uniquely determined by its non-tangential limit on the unit circle $\mathbb{T}$ and the L^p-norm of this trace is equal to the H^p-norm of the function, where L^p is the space of p-summable functions on $\mathbb{T}$. This way H^p can be regarded as a closed subspace of L^p.

For $p \in [1, \infty]$ and $n \in \mathbb{N}$, the class of *meromorphic functions of degree n in L^p* is defined by

$$H_n^p := H^p + \mathcal{R}_n, \tag{1}$$

which is a closed subset of L^p, where

$$\mathcal{R}_n := \left\{ \frac{p(z)}{q(z)} = \frac{p_{n-1}z^{n-1} + p_{n-2}z^{n-2} + \cdots + p_0}{z^n + q_{n-1}z^{n-1} + \cdots + q_0} : \ p \in \mathcal{P}_{n-1}, \ q \in \mathcal{M}_n \right\}, \tag{2}$$

$\mathcal{P}_n$ is the space of algebraic polynomials of degree at most n and $\mathcal{M}_n$ is its subset consisting of monic polynomials with n zeros in the unit disk $\mathbb{D}$.

Meromorphic approximation problem in L^p consists in the following:
Given $p \in [1, \infty]$, $f \in L^p$, and $n \in \mathbb{N}$, find $g_n \in H_n^p$ such that $\|f - g_n\|_p = \inf_{g \in H_n^p} \|f - g\|_p$.
The solution of this problem is known to be unique for $p = \infty$, provided that f belongs to the *Douglas algebra* $H^\infty + C(\mathbb{T})$, where $C(\mathbb{T})$ denotes the space of continuous functions on $\mathbb{T}$ [1]. When $p < \infty$, a solution needs not be unique even if f is very smooth [14].

2.2. **L^2-Best Rational Approximants.**

Denote by $\bar{H}_0^2$ the orthogonal complement of H^2 in L^2, $L^2 = H^2 \oplus \bar{H}_0^2$, with respect to the standard scalar product

$$\langle f, g \rangle := \int_{\mathbb{T}} f(\tau)\overline{g(\tau)}|d\tau|, \quad f, g \in L^2.$$

From the viewpoint of analytic function theory, $\bar{H}_0^2$ can be regarded as a space of traces of functions holomorphic in $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ and vanishing at infinity whose square-means on the concentric circles centered at zero (this time with radii greater then 1) are uniformly bounded above.

Let now $f \in L^2$ and $g_n \in H_n^2$ be an L^2-best meromorphic approximant to f. The orthogonal decomposition $L^2 = H^2 \oplus \bar{H}_0^2$ yields that

$$\|f - g_n\|_2^2 = \|f^+ - g_n^+\|_2^2 + \|f^- - g_n^-\|_2^2,$$

where $f = f^+ + f^-$, $g_n = g_n^+ + g_n^-$, and $f^+, g_n^+ \in H^2$, $f^-, g_n^- \in \bar{H}_0^2$. One can immediately see that in order for g_n to be a best approximant it is necessary that $g_n^+ = f^+$. Moreover, it follows from (1) that $g_n^- \in \mathcal{R}_n$. Thus, meromorphic approximation problem for $p = 2$ can be equivalently stated as the following rational approximation problem:

Given $f \in \bar{H}_0^2$ *and* $n \in \mathbb{N}$, *find* $r_n \in \mathcal{R}_n$ *such that* $\|f - r_n\|_2 = \inf_{r\in\mathcal{R}_n} \|f - r\|_2$.

2.3. **Irreducible Critical Points.** L^2-best rational approximants are a part of the larger class of *critical points* in rational $\bar{H}_0^2$-approximation. From the computational viewpoint, critical points are as important as best approximants since a numerical search is more likely to yield a locally best rather than a best approximant. For a fixed $f \in \bar{H}_0^2$, critical points can be defined as follows. Set

$$\begin{array}{rcl} \Sigma_{f,n} : \mathcal{P}_{n-1} \times \mathcal{M}_n & \to & [0,\infty) \\ (p,q) & \mapsto & \|f - p/q\|_2^2. \end{array} \tag{3}$$

In other words, $\Sigma_{f,n}$ is the squared error of approximation of f by $r = p/q$ in $\mathcal{R}_n$. The cross-product $\mathcal{P}_{n-1} \times \mathcal{M}_n$ is topologically identified with an open subset of $\mathbb{C}^{2n}$ with coordinates p_j and q_k, $j,k \in \{0,\dots,n-1\}$, see (2). Then a pair of polynomials $(p_c, q_c) \in \mathcal{P}_{n-1} \times \mathcal{M}_n$, identified with a vector in $\mathbb{C}^{2n}$, is said to be a *critical pair of order* n, if all the partial derivatives of $\Sigma_{f,n}$ do vanish at (p_c, q_c). Respectively, a rational function $r_c \in \mathcal{R}_n$ is a *critical point of order* n if it can be written as the ratio $r_c = p_c/q_c$ of a critical pair (p_c, q_c) in $\mathcal{P}_{n-1} \times \mathcal{M}_n$. A particular example of a critical point is a *locally best approximant*. That is, a rational function $r_l = p_l/q_l$ associated with a pair $(p_l, q_l) \in \mathcal{P}_{n-1} \times \mathcal{M}_n$ such that $\Sigma_{f,n}(p_l, q_l) \le \Sigma_{f,n}(p, q)$ for all pairs (p, q) in some neighborhood of (p_l, q_l) in $\mathcal{P}_{n-1} \times \mathcal{M}_n$. We call a critical point of order n *irreducible* if it belongs to $\mathcal{R}_n \setminus \mathcal{R}_{n-1}$. Best approximants, as well as local minima, are always irreducible critical points unless $f \in \mathcal{R}_{n-1}$. In general there may be other critical points, reducible or irreducible, which are saddles or maxima.

One of the most crucial features of the critical points is the fact that they are "maximal" rational interpolants. More precisely, if $f \in \bar{H}_0^2$ and r_n is an irreducible critical point of order n, then r_n *interpolates* f *at the reflection* ($z \mapsto 1/\bar{z}$) *of each pole of* r_n *with order twice the multiplicity that pole* [63], [24], which is the maximal number of interpolation conditions (*i.e.,* $2n$) that can be imposed in general on a rational function of type $(n-1, n)$.

2.4. **Padé Approximants.** Let f be a function holomorphic and vanishing at infinity (the second condition is there for convenience only as functions in $\bar{H}_0^2$ vanish at infinity by definition). Then f can be represented as a power series

$$f(z) = \sum_{k=1}^{\infty} \frac{f_k}{z^k}. \tag{4}$$

A *diagonal Padé approximant* to f is a rational function $[n/n]_f = p_n/q_n$ of type (n, n) that has maximal order of contact with f at infinity [81, 8]. It is obtained from the solutions of the linear system

$$R_n(z) := q_n(z)f(z) - p_n(z) = \mathcal{O}\left(1/z^{n+1}\right) \quad \text{as} \quad z \to \infty \tag{5}$$

whose coefficients are the moments f_k in (4). System (5) is always solvable and no solution of it can be such that $q_n \equiv 0$ (we may thus assume that q_n is monic). In general, a solution is not unique, but yields exactly the same rational function $[n/n]_f$. Thus, each solution of (5) is of the form (lp_n, lq_n), where (p_n, q_n) is the unique solution of minimal degree. Hereafter, (p_n, q_n) will always stand for this unique pair of polynomials.

The n-th diagonal Padé approximant $[n/n]_f$ as well as the index n are called *normal* if $\deg(q_n) = n$ [72]. The occurrence of non-normal indices is a consequence of overinterpolation. That is, if n is a normal index and[2]

$$f(z) - [n/n]_f(z) \sim z^{-(2n+l+1)} \quad \text{as} \quad z \to \infty$$

for some $l \geq 0$, then $[n/n]_f = [n+j/n+j]_f$ for $j \in \{0, \dots, l\}$, and $n + l + 1$ is normal.

2.5. **Multipoint Padé Approximants.** Let f be given by (4) and $\{E_n\}_{n\in\mathbb{N}}$ be a triangular scheme of points in the domain of holomorphy of f, i.e., each E_n consists of $2n$ not necessarily distinct nor finite points. Further, let v_n be the monic polynomial with zeros at the finite points of E_n. The *n-th diagonal Padé approximant to f associated with E_n* is the unique rational function $[n/n; E_n]_f = p_n/q_n$ satisfying:

- $\deg p_n \leq n$, $\deg q_n \leq n$, and $q_n \not\equiv 0$;
- $(q_n(z)f(z) - p_n(z))/v_n(z)$ is analytic in the domain of analyticity of f;
- $(q_n(z)f(z) - p_n(z))/v_n(z) = \mathcal{O}\left(1/z^{n+1}\right)$ as $z \to \infty$.

Multipoint Padé approximants always exist since the conditions for p_n and q_n amount to solving a system of $2n + 1$ homogeneous linear equations with $2n + 2$ unknown coefficients, no solution of which can be such that $q_n \equiv 0$ (we may thus assume that q_n is monic).

As mentioned above, irreducible critical points $r_n = p_n/q_n$ in rational $\bar{H}_0^2$-approximation turn out to be multipoint Padé approximants where $v_n := \tilde{q}_n^2$ and $\tilde{q}_n(z) := z^n\overline{q_n(1/\bar{z})}$ is the reciprocal polynomial of q_n.

2.6. **Approximated Functions.** To define the first class of functions we need several notions from potential theory [98, 100, 61]. For any compact set K in $\mathbb{C}$, the *logarithmic capacity* of K is defined by

$$\mathrm{cp}(K) := \exp\left\{-\inf_{\mathrm{supp}(\nu)\subseteq K}\int \log\frac{1}{|z-u|}d\nu(z)d\nu(u)\right\},$$

where the infimum is taken over all probability Borel measures supported on K. It is known that either $\mathrm{cp}(K) = 0$ (K is *polar*) or there exists the unique measure ω_K, the *logarithmic equilibrium distribution* on K, that realizes the infimum. The *Green's function* for the unbounded component of the complement of K, say D, is defined by

$$g_D(z) := I[\omega_K] + \int \log|z-t|d\omega_k(t)$$

and is the unique positive harmonic function in $D \setminus \{\infty\}$ that is equal to zero everywhere on ∂D perhaps with the exception of a polar set (*quasi everywhere*) and such that $g_D(z) \sim \log|z|$ as $z \to \infty$.

We denote by $\mathcal{C}$ the class of functions analytic at infinity that have meromorphic, possible multi-valued, continuation along any arc in $\overline{\mathbb{C}} \setminus E_f$ starting from infinity with E_f compact and polar. We further say that f belongs to the class $\mathcal{A} \subset \mathcal{C}$ if f admits analytic, possibly multi-valued, continuation along any arc in $\overline{\mathbb{C}}\setminus E_f$ starting from infinity, where E_f is non-empty, finite, and the meromorphic continuation of

[2]We say that $a(z) \sim b(z)$ if $0 < \liminf_{z\to\infty}|a(z)/b(z)| \leq \limsup_{z\to\infty}|a(z)/b(z)| < \infty$.

f from infinity has a branch point at each element of E_f. The primary example of functions in $\mathcal{A}$ is that of algebraic functions. Other examples include functions of the form $g\circ\log(l_1/l_2)+r$, where g is entire and $l_1,l_2\in\mathcal{P}_m$ while $r\in\mathcal{R}_k$ for some $m,k\in\mathbb{N}$. However, $\mathcal{A}$ is defined in such a way that it contains no function in $\mathcal{R}_n$, $n\in\mathbb{N}$, in order to avoid degenerate cases.

Together with the classes $\mathcal{A}$ and $\mathcal{C}$ we shall consider functions defined by a Cauchy integral for suitable classes of measures. Namely, we set

$$f_\mu(z) := \int\frac{d\mu(t)}{t-z}, \quad z\notin\operatorname{supp}(\mu). \tag{6}$$

Observe that by Cauchy integral theorem, each function given by (4) can equivalently written as f_μ. However, we shall be interested in (6) only for finite measures in the following classes:

- class of positive measures **Reg** of [113] (the definition of this class is rather complicated, but, in particular, it includes measures supported on an interval with almost everywhere positive Radon-Nikodym derivative);
- class **Bvt** consisting of complex-valued measures μ compactly supported on the real line with an argument of bounded variation and such that $|\mu|([x-\delta,x+\delta])\geq c\delta^L$ for all $x\in\operatorname{supp}(\mu)$ and δ small enough with some constants c,L;
- class $\mathbf{Sz}([a,b])$ of positive measures μ supported on $[a,b]$ satisfying the Szegő condition there: $\int_{[a,b]}\log\mu'(t)dt>-\infty$, $d\mu(t)=\mu'(t)dt$;
- class $\mathbf{Dini}(\Delta)$ of complex-valued measures supported on an analytic arc Δ with endpoints a,b satisfying $d\mu(t)=[\dot\mu(t)/w_\Delta^+(t)]dt$, where $\dot\mu$ is a Dini-continuous non-vanishing function on Δ and $w_\Delta(z):=\sqrt{(z-a)(z-b)}$ is a branch holomorphic outside of Δ such that $w_\Delta(z)\sim z$ as $z\to\infty$.

3. Convergence Theory

In this section we outline the current state of knowledge about convergence of meromorphic approximants and rational interpolants with free poles to the approximated functions.

3.1. Padé Approximants.

As briefly described in the introduction, uniform convergence of Padé approximants to the approximated function is rather rare. Few positive results include certain classes of entire functions [6, 64, 65] and some Markov functions as mentioned further below. However, it is still true that only a small number of poles ruins the uniform convergence and, in particular, convergence in measure holds [126].

3.1.1. *Convergence in Capacity.* In order to established positive results, the requirement of the uniform convergence should be weakened. The proper type, as turned out, is convergence in capacity. That is, $f_n\overset{\text{cp}}{\to}f$ in a bounded domain D if for every $\epsilon>0$ it holds that

$$\lim_{n\to\infty}\operatorname{cp}\big(\{z\in D:|\,|(f_n-f)(z)|>\epsilon\}\big)=0.$$

For unbounded domains this property is defined by first composing with the map $z\mapsto 1/(z-a)$ some, hence any, $a\notin\overline{D}$.

The first results in this direction belong to Nuttall [74] and Pommerenke [94] who showed convergence in capacity of the diagonal Padé approximants to functions

$f \in \mathcal{C}$ that are single-valued in $\overline{\mathbb{C}} \setminus E_f$ (in a way, such functions generalize entire functions). When $f \in \mathcal{C}$ is multi-valued, the whole new problem arrises: *if diagonal Padé approximants converge at all then where?* Indeed, Padé approximants are single-valued and so should be there limit. This issue was resolved in a series of pathbreaking papers [105, 106, 107, 109] by Stahl following the initial study of Nuttall [75, 76, 80, 77, 79].

Theorem 1. *Given a multi-valued $f \in \mathcal{C}$, there exists the unique admissible compact*[3] *Δ_f such that $\mathrm{cp}(\Delta_f) \leq \mathrm{cp}(K)$ for any admissible compact K and $\Delta_f \subseteq K$ for any admissible K satisfying $\mathrm{cp}(\Delta_f) = \mathrm{cp}(K)$. Moreover,*

$$\left| f - [n/n]_f \right|^{1/2n} \overset{\mathrm{cp}}{\to} \exp\left\{ - g_{D_f} \right\} \quad in \quad D_f,$$

where $D_f := \overline{\mathbb{C}} \setminus \Delta_f$ and g_{D_f} is the Green's function with pole at infinity for D_f. The domain D_f is optimal in the sense that the convergence does not holds in any other domain D such that $D \setminus D_f \neq \varnothing$.

Another fascinating part of Stahl's work is the geometrical description[4] of Δ_f.

Theorem 2. *It holds that*

$$\Delta_f = E_0 \cup E_1 \cup \bigcup \Delta_j, \tag{7}$$

where $\bigcup \Delta_j$ is a union of open analytic arcs, $E_0 \subseteq E_f$, E_1 is a set of points such that each element of E_1 is an endpoint of at least three arcs Δ_j, and

$$\frac{\partial g_D}{\partial \mathbf{n}^+} = \frac{\partial g_D}{\partial \mathbf{n}^-} \quad on \quad \bigcup \Delta_j, \tag{8}$$

where $\partial/\partial \mathbf{n}^\pm$ are the partial derivatives with respect to the one-sided normals on each Δ_j.

Due to the content of Theorems 1 and 2 contours Δ_f, $f \in \mathcal{C}$, received the names of *minimal capacity contours* and *S-contours (symmetric contours)*.

When $f \in \mathcal{A}$, the set E_f is finite and so are the number of arcs in $\bigcup \Delta_j$ and the cardinality of $E_0 \cup E_1$. For such functions f, a simpler proof of the existence of Δ_f has been given in [93, 41] using the variational argument. The minimal capacity contours associated with $f \in \mathcal{A}$ also admit a description via critical trajectories of rational quadratic differentials (for $f \in \mathcal{C}$ this is also true although differentials are no longer rational). More precisely, let us formally define:

Definition 3. *A compact set Δ is called an algebraic S-contour if the complement of Δ, say D, is connected, Δ can be decomposed as in (7), where $\bigcup \Delta_j$ is a finite union of open analytic arcs, $E_0 \cup E_1$ is a finite set of points such that each element of E_0 is an endpoint of exactly one arc Δ_j while each element of E_1 is an endpoint of at least three arcs, and (8) holds with g_{D_f} replaced by g_D.*

Any algebraic S-contour is a minimal capacity contour for some $f \in \mathcal{A}$. Indeed, given Δ, an eligible function $f_\Delta \in \mathcal{A}$ can be constructed in the following way. Denote by m the number of connected components of Δ, by E_{0j} the intersection

[3] Given $f \in \mathcal{C}$, a compact set K is called *admissible* if $\overline{\mathbb{C}} \setminus K$ is connected and f has a meromorphic and single-valued extension there.

[4] This theorem holds even without the requirement $\mathrm{cp}(E_f) = 0$ as shown in [112]. However, as of now, convergence theory in this case is not understood.

of E_0 with the j-th connected component, and by m_j the cardinality of E_{0j}. Then one can take $f_\Delta(z)=\sum_{j=1}^m\left(\prod_{e\in E_{0j}}(z-e)\right)^{-1/m_j}$.

To describe the connection to quadratic differentials, set

$$h_\Delta(z):=2\partial_z g_D(z), \tag{9}$$

where $2\partial_z:=\partial_x-i\partial_y$. The function h_Δ is holomorphic in D and vanishes at infinity. For each point $e\in E_0\cup E_1$ denote by $i(e)$ the *bifurcation index* of e, that is, the number of different arcs Δ_j incident with e. It follows immediately from the definition of an algebraic S-contour that $i(e)=1$ for $e\in E_0$ and $i(e)\geq 3$ for $e\in E_1$. Denote also by E_2 the set of *critical points* of g_D with $j(e)$ standing for the *order* of $e\in E_2$, i.e., $\partial_z^j g_D(e)=0$ for $j\in\{1,\dots,j(e)\}$ and $\partial_z^{j(e)+1}g_D(e)\neq 0$. The set E_2 is necessarily finite. We attribute the following theorem to Perevoznikova and Rakhmanov [93], where the variation approach to algebraic S-contours was introduced.

Theorem 4. *Let Δ be an algebraic S-contour with the complement D. Then the arcs Δ_j are negative critical trajectories of the quadratic differential $h_\Delta^2(z)\mathrm{d}z^2$. That is, for any smooth parametrization $z(t):(0,1)\to\Delta_j$ it holds that*

$$h_\Delta^2(z(t))\big(z'(t)\big)^2<0 \quad \textit{for all} \quad t\in(0,1).$$

Moreover,

$$h_\Delta^2(z)=\prod_{e\in E_0\cup E_1}(z-e)^{i(e)-2}\prod_{e\in E_2}(z-e)^{2j(e)} \tag{10}$$

and $h_\Delta^2(z)=z^{-2}+\mathcal{O}\big(z^{-3}\big)$ as $z\to\infty$.

3.1.2. *Markov Functions.* Let now f_μ be given by (6) and $[n/n]_{f_\mu}=p_n/q_n$ be the n-th diagonal Padé approximant to f_μ. Using (5), it is not hard to show that polynomials q_n satisfy

$$\int z^j q_n(z)d\mu(z)=0, \quad j\in\{0,\dots,n-1\}. \tag{11}$$

That is, the polynomials q_n are non-Hermitian orthogonal polynomials with respect to a generally complex-valued measure μ. However, when μ is a positive measure supported on $\mathbb{R}$, f_μ is called a *Markov function* and polynomials q_n become standard orthogonal polynomials on the real line, which is an extremely well-studied topic [120, 46, 45, 71, 102]. Initially, Markov [68] showed the uniform convergence outside of the convex hull of $\operatorname{supp}(\mu)$ without providing exact rates. By now these rates are well-understood: in the n-th root sense when $\mu\in\mathbf{Reg}$ and in the sense of strong asymptotics when $\mu\in\mathbf{Sz}$.

In any case, the uniform convergence holds only outside of the convex hull of $\operatorname{supp}(\mu)$. Indeed, it can be deduced from (11) that if $\operatorname{supp}(\mu)=\cup_{j=1}^J[a_j,b_j]$ than each gap (b_j,a_{j+1}) contains exactly one zero of q_n for all n large enough. That is, Padé approximant $[n/n]_{f_\mu}$ will have a pole in each interval (b_j,a_{j+1}) which is a subset of the domain of holomorphy of f_μ. The location of each zero in a gap changes with n and the dynamics of these zeros has been under thorough investigation in the last decade [104, 86, 31, 32, 33, 30, 102]. These zeros are the perfect example of wandering poles which turn uniform convergence into nearly uniform convergence (by clearing those poles the approximants will converge in the gaps).

3.1.3. *Nearly Uniform Convergence.* Relinquishing the condition of the positivity of μ greatly complicates the analysis. For starters, wandering poles, when present, are no longer confined to the real line. In fact, their union over n can be dense everywhere in the complex plane. To explain the situation, we need more definitions.

Let Δ be an algebraic S-contour and h_Δ be given by (9), see also (10). Denote by $\mathfrak{R}$ the Riemann surface defined by h_Δ. It can be represented as a two-sheeted ramified cover of $\overline{\mathbb{C}}$ constructed in the following manner. Two copies of $\overline{\mathbb{C}}$ are cut along each arc Δ_j, see (7). These copies are clipped together at the elements of $E_\Delta \subseteq E_0 \cup E_1$, which consists of those points that have odd bifurcation index (branch points of h_Δ). These copies are further glued together along the cuts in such a manner that the right (resp. left) side of the arc Δ_j belonging to the first copy, say $\mathfrak{R}^{(0)}$, is joined with the left (resp. right) side of the same arc Δ_j only belonging to the second copy, $\mathfrak{R}^{(1)}$. The genus of $\mathfrak{R}$, which we denote by g, satisfies the equality $2(g+1) = |E_\Delta|$. According to the above construction, each arc Δ_j together with its endpoints corresponds to a cycle, say $\boldsymbol{\Delta}_j$, on $\mathfrak{R}$. We set $\boldsymbol{\Delta} := \bigcup_j \boldsymbol{\Delta}_j$, denote by π the canonical projection $\pi : \mathfrak{R} \to \overline{\mathbb{C}}$, and define

$$D^{(k)} := \mathfrak{R}^{(k)} \cap \pi^{-1}(D) \quad \text{and} \quad z^{(k)} := D^{(k)} \cap \pi^{-1}(z)$$

for $k \in \{0,1\}$ and $z \in D$. We further set $\boldsymbol{E}_\Delta := \pi^{-1}(E_\Delta)$, which is comprised exactly of the ramification points of $\mathfrak{R}$. An *integral divisor* on $\mathfrak{R}$ is a formal symbol of the form $\mathcal{D} = \sum n_j \boldsymbol{z}_j$, where $\{\boldsymbol{z}_j\}$ is an arbitrary finite collection of distinct points on $\mathfrak{R}$ and $\{n_j\}$ is a collection of positive integers. The sum $\sum n_j$ is called the *degree* of the divisor $\mathcal{D}$. A general divisor is a difference of two integral divisors.

Theorem 5. *Let p be a non-vanishing polynomial on Δ and*

$$\widehat{p}(z) := \frac{1}{\pi \mathrm{i}} \int_\Delta \frac{1}{t-z} \frac{\mathrm{d}t}{p(t) w_\Delta^+(t)}, \quad z \in D, \tag{12}$$

where $w_\Delta^2(z) := \prod_{e \in E_\Delta}(z-e)$. There exists a sectionally meromorphic in $\mathfrak{R} \setminus \boldsymbol{\Delta}$ function Ψ_n whose zeros and poles there[5] *are described by the divisor $(n-g)\infty^{(1)} + \mathcal{D}_n - n\infty^{(0)}$, where $\mathcal{D}_n$ is an integral divisor of degree g that solves a special Jacobi inversion problem on $\mathfrak{R}$ with parameters depending on Δ, p, and n; and whose traces are continuous on $\boldsymbol{\Delta} \setminus \boldsymbol{E}_\Delta$ and satisfy*

$$\Psi_n^+ = (p \circ \pi) \Psi_n^-. \tag{13}$$

Given $[n/n]_{\widehat{p}} = p_n/q_n$, it holds that

$$\begin{cases} q_n &= \gamma_n \left(\Psi_n^{(0)} + p \Psi_n^{(1)} \right) \\ R_n &= 2\gamma_n w_\Delta^{-1} \Psi_n^{(1)} \end{cases} \tag{14}$$

for all $2n > 3g + \deg(p)$, where R_n is defined by (5), γ_n is a normalization constant turning q_n into a monic polynomial and $\Psi_n^{(k)}(z) := \Psi_n(z^{(k)})$, $z \in D$, $k \in \{0,1\}$.

More restrictive versions of this theorem appeared in [40, 2, 3, 121, 83, 84, 85, 20]. In its full generality it was proven by Nuttall and Singh in [80] and by the aurthor in [123], who was unaware of [80] at the moment. It is important to note that

[5] Ψ_n is non-vanishing and finite in $\mathfrak{R} \setminus \boldsymbol{\Delta}$ except at the elements of its divisor that stand for zeros (resp. poles) if preceded by the plus (resp. minus) sign and the integer coefficients in front of them indicate multiplicity.

in (14) the functions $\Psi_n^{(0)}$ nearly geometrically diverge to infinity[6] as $n\to\infty$ and the functions $\Psi_n^{(1)}$ geometrically converge to zero. Thus, near each $\boldsymbol t_{n,k}\in D^{(0)}$, $\mathcal D_n=\sum_{k=1}^g \boldsymbol t_{n,k}$, the Padé approximant $[n/n]_{\widehat p}$ has a wandering pole, and at each $\boldsymbol t_{n,k}\in D^{(1)}$ it has an additional interpolation point.

Using Theorem 5 as an intermediate step, one can prove a similar theorem where p in (12) is replaced by a non-vanishing Hölder continuous function as it was done by Suetin in [116, 117] for Δ consisting of disjoint arcs and by Baratchart and the author in [20] for Δ consisting of three arcs meeting at one point. In this case formulae (14) become asymptotic.

The dynamics of the wondering poles is quite complicated and is understood to a degree only when Δ consists of either two disjoint arcs or three arcs meeting at one point (in both cases $g=1$). The following theorem is due to Baratchart and the author and is contained in [20].

Theorem 6. *Let $\Delta=\{a_1,a_2,a_3\}\cup\{b\}\cup\bigcup_{k=1}^3\Delta_k$, see (7), and ω_Δ be the equilibrium measure on Δ. The set $\mathbf Z:=\overline{\cup_n\{\boldsymbol t_n\}}$ ($g=1$ and therefore $\mathcal D_n=\boldsymbol t_n$) is equal to $\mathfrak R$ when the numbers $\omega_\Delta(\Delta_k)$, $k\in\{1,2,3\}$, are rationally independent; it is the union of finitely many pairwise disjoint arcs when $\omega_\Delta(\Delta_k)$ are rationally dependent but at least one of them is irrational; $\mathbf Z$ is a finite set of points when $\omega_\Delta(\Delta_k)$ are all rational. All the points $\boldsymbol t_n$ are mutually distinct in the first two cases and $\{\boldsymbol t_n\}=\mathbf Z$ in the third one. The set of triples (a_1,a_2,a_3) for which the numbers $\omega_\Delta(\Delta_k)$ are rationally dependent form a dense subset of zero measure in $\mathbb C^3$. The triples (a_1,a_2,a_3) for which $\omega_\Delta(\Delta_k)$ are rational are also dense.*

To understand the second part of the theorem, recall that for a given points $\{a_1,a_2,a_3\}$, there exists exactly one choice of the arcs Δ_k (and necessarily of the meeting point b) such that $\{a_1,a_2,a_3\}\cup\{b\}\cup\bigcup_{k=1}^3\Delta_k$ is an algebraic S-contour.

The first part of this theorem was originally proven in for Δ consisting of two arcs by Suetin [117] where the constants $\omega_\Delta(\Delta_k)$, $k\in\{1,2,3\}$, are replaced by $\omega_\Delta(\Delta_k)$, $k\in\{1,2\}$, and the third number τ which essentially carries the information how far apart the two arcs are.

The most general (in terms of classes of algebraic S-contours) theorem on near convergence was obtained in [5] by Aptekarev and the author.

Theorem 7. *Let Δ be an algebraic S-contour such that no point of E_1 has the bifurcation index greater than 3. Let also $\widehat\rho$ be defined as in (12) with p replaced by a function ρ such that $\rho_{|\Delta_k}=w_k$, where w_k is holomorphic and non-vanishing in a neighborhood of Δ_k. Finally, let Ψ_n be as in Theorem 5 with p replaced by ρ/h_Δ^+ in (13). Then for all n such that no two distinct points in $\mathcal D_n$ have the same standard projection onto $\mathbb C$ and all the points in $\mathcal D_n$ are uniformly bounded away from $\infty^{(0)},\infty^{(1)}$, it holds that*

$$\left\{\begin{array}{lll} q_n &=& (1+\upsilon_{n1})\,\gamma_n\Psi_n^{(0)}+\upsilon_{n2}\gamma_n^*\Psi_{n-1}^{(0)},\\ R_n &=& (1+\upsilon_{n1})\,\gamma_n h_\Delta\Psi_n^{(1)}+\upsilon_{n2}\gamma_n^* h_\Delta\Psi_n^{(1)}, \end{array}\right.$$

locally uniformly in D, where $|\upsilon_{nj}|\leq c(\varepsilon)/n$ in $\overline{\mathbb C}$ while $\upsilon_{nj}(\infty)=0$, γ_n is a normalizing constant, and γ_n^* is an explicitly defined constant.

[6] That is, the functions $\Psi_n^{(0)}$ could have at most g zeros coming from $\mathcal D_n$. However, after removing those zeros (for example, multiplying by a polynomial), the remaining functions do diverge geometrically.

Let us note that the collection of the indices n satisfying the requirements of the theorem is of infinite cardinality. Another important distinction between Theorem 7 and Theorem 5 with its generalizations is the method of proof. The proof of Theorem 7 is based on a Riemann-Hilbert problem for 2×2 matrices [43, 44] to which the steepest descent analysis [38, 37] is applied.

3.2. **Multipoint Padé Approximants.**

3.2.1. *Convergence in Capacity.* In the case of multipoint Padé approximants the definition of an S-contour should be appropriately modified to reflect the fact that interpolation is done at many points. For that we shall need the following definitions.

Let ν be a probability Borel measure supported in $\overline{\mathbb{D}}$. We set

$$U^\nu(z) := -\int \log|1-z\bar{u}|d\nu(u).$$

U^ν is a harmonic function outside of $\operatorname{supp}(\nu^*)$, in particular, in $\mathbb{D}$. Considering $-U^\nu$ as an *external field* acting on non-polar compact subsets of $\mathbb{D}$, we define the ν-capacity of $K\subset\mathbb{D}$ (weighted capacity) by

$$\operatorname{cp}_\nu(K) := \exp\left\{-\min_\omega\left(\iint\log\frac{1}{|z-u|}d\omega(z)d\omega(u) - 2\int U^\nu(z)d\omega(z)\right)\right\},$$

where the minimum is taken over all probability Borel measures ω supported on K. Clearly, $U^{\delta_0}\equiv 0$ and therefore $\operatorname{cp}_{\delta_0}(\cdot)$ is simply the classical logarithmic capacity $\operatorname{cp}(\cdot)$, where δ_0 is the Dirac delta at the origin. As in the case of the classical logarithmic capacity, there exists a unique measure that realizes the infimum. Moreover, this measure can be characterized as the *balayage* of ν onto K relative to the unbounded component of the complement of K.

Let D be an open connected set with non-polar boundary. The Green's function for D with pole at $t\in D$, denoted by $g_D(\cdot,t)$ is the unique positive harmonic function in $D\setminus\{t\}$, which is zero quasi everywhere on ∂D, and behaves like $\log|z|$ near $t=\infty$ or like $-\log|z-t|$ near finite t. Notice that $g_D(\cdot,\infty)=g_D$ as defined before. For any positive measure ν, $\operatorname{supp}(\nu)\subset D$, the Green's potential of ν relative to D is defined by

$$V_D^\nu(z) := \int g_D(z,t)d\nu(t).$$

The following theorem generalizes Theorem 1 for functions in $\mathcal{A}$.

Theorem 8. *Given $f\in\mathcal{A}$ with $E_f\subset\mathbb{D}$ and a probability Borel measure ν supported in $\overline{\mathbb{D}}$, there exists the unique admissible compact Δ_ν such that $\operatorname{cp}_\nu(\Delta_\nu)\leq\operatorname{cp}(K)$ for any admissible compact K and $\Delta_\nu\subseteq K$ for any admissible K satisfying $\operatorname{cp}_\nu(\Delta_\nu)=\operatorname{cp}_\nu(K)$. Further, let $\{[n/n;E_n]\}$ be the sequence of multipoint Padé approximants to f associated to an interpolation scheme $\{E_n\}$ satisfying $\frac{1}{2n}\sum_{e\in E_n}\delta_e\stackrel{*}{\to}\nu^*$, where $\stackrel{*}{\to}$ denotes weak* convergence of measures and ν^* is the reflection of ν across $\mathbb{T}$ by the map $z\mapsto 1/\bar{z}$. Then*

$$\left|f-[n/n;E_n]\right|^{1/2n}\stackrel{\operatorname{cp}}{\to}\exp\left\{-V_{D_\nu}^{\nu^*}\right\}\quad in\quad D_\nu:=\overline{\mathbb{C}}\setminus\Delta_\nu.$$

This theorem was proved in [18] and is due to Baratchart, Stahl, and the author. The restriction $E_f\subset\mathbb{D}$ can be equivalently restated as requiring interpolating sets E_n to be contained in the complement of the smallest disk encompassing E_f.

The main contribution of [18] is in showing the existence of the weighted minimal capacity contour Δ_ν as the convergence in capacity of the multipoint Padé approximants was shown in [51] by Gonchar and Rakhmanov assuming the existence of such contours. The analog of Theorem 2 and 4 was also obtained in [18].

Theorem 9. *The minimal weighted capacity contour Δ_ν has the structure (7). Moreover, (8) holds with g_D replaced by $V_{D^\nu}^{\nu^*}$. The function $H_\nu := \partial_z V_{D_\nu}^{\nu^*}$ is holomorphic in $D_\nu \setminus \operatorname{supp}(\nu^*)$ and has continuous traces from each side of every Δ_j in Δ_ν that satisfy $H_\nu^+ = -H_\nu^-$ on each Δ_j. Moreover, H_ν^2 is a meromorphic function in $\overline{\mathbb{C}} \setminus supp(\nu^*)$ that has a simple pole at each element of E_0 and a zero at each element e of E_1 whose order is equal to the bifurcation index of e minus 2. The arcs Δ_j are negative critical trajectories of $H_\nu^2 dz^2$.*

In the case where $f = f_\mu$, $\mu \in \mathbf{Bvt}$, the analog of Theorem 8 was derived by Baratchart and the author in [23].

3.2.2. *Uniform Convergence.* At the present moment only the case of a single arc has been developed. We shall assume for simplicity that the endpoints of Δ are ± 1. The following theorem by Baratchart and the author is from [21].

Theorem 10. *Let Δ be a rectifiable Jordan arc such that for $x = \pm 1$ and all $t \in \Delta$ sufficiently close to x it holds that $|\Delta_{t,x}| \leq const.|x-t|^\beta$, $\beta > 1/2$, where $|\Delta_{t,x}|$ is the arclength of the subsarc of Δ joining t and x and const. is an absolute constant. Then the following are equivalent:*

(a) *there exists a positive compactly supported Borel measure ν, $\operatorname{supp}(\nu) \subset D := \overline{\mathbb{C}} \setminus \Delta$, such that (8) holds with g_D replaced by V_D^ν on $\Delta \setminus \{\pm 1\}$;*

(b) *there exists a triangular scheme of points $\{E_n\}$, $\cap_{n\in\mathbb{N}} \overline{\cup_{k\geq n} E_k} \subset D$, such that $|r_n^\pm| = \mathcal{O}(1)$ uniformly on Δ, where*

$$r_n(z) := \prod_{e\in E_n} \frac{\phi(z) - \phi(e)}{1 - \phi(e)\phi(z)}$$

and $\phi(z) := z + \sqrt{z^2 - 1}$ is a branch holomorphic in D;

(c) *Δ is an analytic Jordan arc, i.e., there exists a univalent function p holomorphic in some neighborhood of $[-1,1]$ such that $\Delta = p([-1,1])$.*

Effectively, this theorem says that any analytic Jordan arc can be seen as a weighted S-contour and that the symmetry property can be equivalently restated with the help of an appropriate triangular scheme in a way which more attuned to the strong asymptotics, see (15) below.

Theorem 11. *Let Δ be an analytic Jordan arc connecting ± 1 that is symmetric with respect to a triangular scheme $\{E_n\}$. Further, let $\{[n/n; E_n]\}$ be the sequence of multipoint Padé approximants for f_μ, $\mu \in \mathbf{Dini}(\Delta)$. Then*

$$(f_\mu - [n/n; E_n])w = [2G_{\dot\mu} + o(1)]S_{\dot\mu}^2 r_n \quad \text{locally uniformly in} \quad D, \tag{15}$$

where $G_{\dot\mu}$ is constant (geometric mean of $\dot\mu$), $S_{\dot\mu}$ is the Szegő function of $\dot\mu$ on Δ (holomorphic and non-vanishing in D satisfying $S_{\dot\mu}^+ S_{\dot\mu}^- = G_{\dot\mu}\dot\mu$ on Δ), and r_n is defined as in Theorem 10.

This theorem was also proved in [21]. An analog of this theorem for Jacobi-type weights using $\bar\partial$-extension of the matrix Riemann-Hilbert analysis was obtained by Baratchart and the author in [25] via $\bar\partial$-extension of the Riemann-Hilbert analysis.

The case $\mu \in \mathbf{Sz}([-1,1])$ was derived By Gonchar and López Lagomasino in [50], see also Stahl's work [111].

3.3. **Meromorphic Approximants.**

3.3.1. *Weak Convergence.* Let f be a function in $\mathcal{A}$ with $E_f \subset \mathbb{D}$. Then $f \in \bar{H}_0^2$ and we can construct a sequence of irreducible critical points, say $\{r_n\}$, in rational $\bar{H}_0^2$-approximation of f. As discussed before, each r_n is also a multipoint Padé approximant to f associated with the set $E_n := \{1/\bar{z}_j, 1/\bar{z}_j\}_{j=1}^n$ (it interpolates at each point $1/\bar{z}_j$ with order 2), where $\{z_j\}_{j=1}^n$ are the zeros of r_n. Theorem 8 does not directly apply as we do not know the weak* behavior of the probability counting measures of the points in E_n. It has not been stated in Theorem 8, but it is shown in [18] that the probability counting measures of the poles of $[n/n; E_n]$ converge to the *balayage* of ν onto Γ_ν with respect to D_ν. Thus, if the counting measures of the poles of r_n converge weakly to a probability measure ω, this measure should be the balayage of its own reflection across the unit circle with respect to the complement of its support and the support itself is the set of minimal ω-capacity for f. It turns out that there is only one such measure for a given f and its support minimizes the *condenser capacity* in $\mathbb{D}$ among all admissible sets K in $\mathbb{D}$. Recall that the condenser capacity of a compact set $K \subset \mathbb{D}$ with respect to $\mathbb{D}$ is defined as

$$\mathrm{cp}(K,\mathbb{T}) := \left(\inf_{\mathrm{supp}(\mu)\subseteq K} \iint \log\left|\frac{1-z\bar{u}}{z-u}\right| d\mu(u)d\mu(z)\right)^{-1}$$

and the infimum is taken over all probability Borel measures supported on K (it can be easily verified that the integrand is, in fact, $g_{\mathbb{D}}(z,u)$, the Green's function for $\mathbb{D}$ with pole at u). As in the case of logarithmic capacity, for each non-polar K there exists the unique measure, say $\omega_{(K,\mathbb{T})}$, that realizes the infimum.

Theorem 12. *Given $f \in \mathcal{A}$ with $E_f \subset \mathbb{D}$, there exists the unique admissible compact Γ such that $\mathrm{cp}(\Gamma,\mathbb{T}) \leq \mathrm{cp}(K,\mathbb{T})$ for any admissible compact K and $\Gamma \subseteq K$ for any admissible K satisfying $\mathrm{cp}(\Gamma,\mathbb{T}) = \mathrm{cp}(K,\mathbb{T})$. If $\{r_n\}$ is a sequence of irreducible critical points in rational $\bar{H}_0^2$-approximation of f, then*

$$\left|f - r_n\right|^{1/2n} \stackrel{\mathrm{cp}}{\to} \exp\left\{-V_{\overline{\mathbb{C}}\setminus\Gamma}^{\omega^*_{(\Gamma,\mathbb{T})}}\right\} \quad in \quad \overline{\mathbb{C}} \setminus \left(\Gamma \cup \Gamma^*\right),$$

where $\omega^*_{(\Gamma,\mathbb{T})}$ is the reflection of $\omega_{(\Gamma,\mathbb{T})}$ across $\mathbb{T}$ and $\Gamma^* = \mathrm{supp}(\omega^*_{(\Gamma,\mathbb{T})})$ is the reflection of Γ across $\mathbb{T}$.

This theorem is due to Baratchart, Stahl, and the author and is contained as well as the variant of Theorem 9 in [18]. Moreover, [18] contains a generalization of the above theorem to any domain G with rectifiable boundary T and L^2-best meromorphic approximants on T to any function $f \in \mathcal{A}$ with $E_f \subset G$. In the case where $f = f_\mu$ is a Markov function with $\mu \in \mathbf{Reg}$, the conclusion of Theorem 12 was obtained by Baratchart, Stahl, and Wielonsky in [16], and when $\mu \in \mathbf{Bvt}$ it was derived by Baratchart and the author in [22].

3.3.2. *Uniform Convergence.* As in the case of the multipoint Padé approximants, only the case of a single arc is developed as of now when it comes to the uniform convergence. The minimal condenser structure of this arc requires it to be a part of the hyperbolic geodesic in $\mathbb{D}$ connecting the endpoints, which, in the case of points on the real line, specializes to a segment. In this case, the following was shown by the author in [124].

Theorem 13. *Let f_μ be given by (6) with $\mu \in \mathbf{Dini}([a,b])$, $[a,b] \subset (-1,1)$. Further, let $\{g_n\}$ be a sequence of L^p-best meromorphic approximants to f_μ, $p \in [2,\infty]$. Then the error of approximation $f_\mu - g_n$ admits a formula similar to (15) in $\mathbb{D} \setminus [a,b]$*[7].

Note that the asymptotics holds in $\mathbb{D} \setminus [a,b]$ only simply because meromorphic approximants are meaningful only in $\mathbb{D}$. The analogous result for Markov functions with $\mu \in \mathbf{Sz}([a,b])$ was derived by Baratchart, Prokhorov, and Saff in [13, 12].

3.3.3. *Uniqueness.* Given $f \in \bar{H}_0^2$, the functional $\Sigma_{f,n}$, defined in (3), might have multiple critical points (local minima, saddle points, maxima). From the constructive viewpoint no algorithm is known to constructively obtain L^2-best rational approximants. So, from a computational perspective this is a typical non-convex minimization problem whose numerical solution is often hindered by the occurrence of local minima. It is therefore of major interest in practice to establish conditions on the function to be approximated that ensure uniqueness of a local minimum. The first step in this direction was taken by Baratchart and Wielonsky in [19].

Theorem 14. *Let μ be a positive measure supported on $[a,b] \subset (-1,1)$ where a and b satisfy $b - a \leq \sqrt{2}\left(1 - \max\left\{a^2, b^2\right\}\right)$. Assume further that μ has at least n points of increase, i.e., $f_\mu \notin \mathcal{R}_{n-1}$. Then there is a unique critical point in rational $\bar{H}_0^2$-approximation of degree n to f_μ.*

Removing the restriction on the size of the support makes the situation significantly more difficult. The following theorem was proved by Baratchart, Stahl, and Wielonsky in [15, 17].

Theorem 15. *If $\mu \in \mathbf{Sz}([a,b])$, $[a,b] \subset (-1,1)$, then there is a unique critical point in rational $\bar{H}_0^2$-approximation of degree n to f_μ for all n large enough. Moreover, for each $n_0 \in \mathbb{N}$ there exists a positive measure $\mu \in \mathbf{Sz}([a,b])$ such that for each odd n between 1 and n_0 there exist at least two different best rational approximants of degree n to f_μ.*

The first part of the above theorem has an extension to complex-valued measures [24] as shown by Baratchart and the author.

Theorem 16. *Let $f := f_\mu + r$, where $r \in \mathcal{R}_m$ has no poles on $\operatorname{supp}(\mu) = [a,b] \subset (-1,1)$ and $\mu \in \mathbf{Dini}([a,b])$ with an argument of bounded variation. Then there is a unique critical point in rational $\bar{H}_0^2$-approximation of degree n to f for all n large enough.*

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