# BERNSTEIN-SZEGŐ THEOREM ON ALGEBRAIC S-CONTOURS

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Abstract. Given function $f$ holomorphic at infinity, the $n$-th diagonal Padé approximant to $f$, say $[n/n]_f$, is a rational function of type $(n,n)$ that has the highest order of contact with $f$ at infinity. Equivalently, $[n/n]_f$ is the $n$-th convergent of the continued fraction representing $f$ at infinity. Bernstein-Szegő theorem provides an explicit non-asymptotic formula for $[n/n]_f$ and all $n$ large enough in the case where $f$ is the Cauchy integral of the reciprocal of a polynomial with respect to the arcsine distribution on $[-1,1]$. In this note, Bernstein-Szegő theorem is extended to Cauchy integrals on the so-called algebraic S-contours.

## 1. Introduction

Orthogonal polynomials is a classical subject [37, 13, 11, 14] with important connections to spectral theory [27, 28] and approximation theory [34]. Perhaps the simplest result in asymptotic theory of orthogonal polynomials is the Bernstein-Szegő theorem [37, Thms. 11.2 and 11.5], [7].

**Theorem (Bernstein-Szegő).** *Let $p$ be a positive polynomial on $[-1,1]$ and $q_n$ be the monic polynomial satisfying*

$$\int_{[-1,1]} t^j q_n(t)\frac{\mathrm{d}t}{p(t)\sqrt{1-t^2}} = 0, \qquad j\in\{0,\dots,n-1\}. \tag{1}$$

*Then for all $2n>\deg(p)$, it holds that $q_n=\gamma_n\left(\Psi_n^{(0)}+p\Psi_n^{(1)}\right)$, where $\gamma_n$ is a normalizing factor turning $q_n$ into a monic polynomial,*

$$\left\{\begin{array}{lcl}\Psi_n^{(0)}(z) &:=& \left(z+\sqrt{z^2-1}\right)^n S_p(z),\\ \Psi_n^{(1)}(z) &:=& \left(z-\sqrt{z^2-1}\right)^n/S_p(z),\end{array}\right. \qquad z\in\overline{\mathbb{C}}\setminus[-1,1], \tag{2}$$

*$\sqrt{z^2-1}\sim z$ as $z\to\infty$, and $S_p$ is the unique holomorphic non-vanishing function in $\overline{\mathbb{C}}\setminus[-1,1]$ such that $|S_p^\pm|^2=p$ on $[-1,1]$*[1].

This theorem serves as a precursor to the results on asymptotics of orthogonal polynomials for more general weights [37, Ch. XI and XII]. Its various generalizations for the case of several intervals (finite gap case in the terminology of [28]) appeared in [2, Sec. 53],[1, 3, 38], and [22, 23, 24]. The main goal of this note is to show that the Bernstein-Szegő theorem owes little both to the positivity of the weight and the specific geometry of the real line. In what follows, we shall show that $[-1,1]$ can be replaced by any *algebraic S-contour* and $p$ can be an arbitrary

[1] In fact, $S_p^2(z)=\prod_{j=1}^{\deg(p)}\frac{z-z_j}{\Phi(z)}\frac{1-\Phi(z)\overline{\Phi(z_j)}}{\Phi(z)-\Phi(z_j)}$, where $\Phi(z)=z+\sqrt{z^2-1}$ and $p(z)=\prod_{j=1}^{\deg(p)}(z-z_j)$.

polynomial non-vanishing on such a contour. Earlier results in this direction for some subclasses of S-contours can be found in [35, 36, 6].

The approach we take exploits the connection between orthogonal polynomials and the theory of rational interpolants with free poles [5, 34]. To explain it, set

$$\widehat{p}(z) := \frac{1}{\pi}\int_{[-1,1]}\frac{1}{z-t}\frac{\mathrm{d}t}{p(t)\sqrt{1-t^2}}, \quad z\in\overline{\mathbb{C}}\setminus[-1,1]. \tag{3}$$

Then $\widehat{p}$ is an algebraic functions with with two branch points $\pm1$[2]. Denote further by $p_n$ the second kind polynomial associated to $q_n$, that is,

$$p_n(z) := \frac{1}{\pi}\int_{[-1,1]}\frac{q_n(z)-q_n(t)}{z-t}\frac{\mathrm{d}t}{p(t)\sqrt{1-t^2}}. \tag{4}$$

Then (4) can be rewritten as

$$R_n(z) := (q_n\widehat{p}-p_n)(z) = \mathcal{O}\big(z^{-n-1}\big) \quad \text{as} \quad z\to\infty, \tag{5}$$

where the second equality easily follows from (1). Consequently, the rational function $p_n/q_n$ interpolates $\widehat{p}$ at infinity with order $2n+1$ and (5) can be used to show that $p_n/q_n$ is the unique rational function of type $(n,n)$ with this property. Moreover, the following result holds true.

**Corollary (Bernstein-Szegő).** *Let $p$, $\widehat{p}$, $q_n$, $R_n$, $\Psi_n^{(k)}$, $k\in\{0,1\}$, be as above. Then*

$$\left(\widehat{p}-\frac{p_n}{q_n}\right)(z) = \left(\frac{R_n}{q_n}\right)(z) = \frac{2}{(z^2-1)^{1/2}}\frac{\Psi_n^{(1)}(z)}{\big(\Psi_n^{(0)}+p\Psi_n^{(1)}\big)(z)}$$

*for $2n>\deg(p)$ and $z\in\overline{\mathbb{C}}\setminus[-1,1]$.*

To generalize (1) and (2) via rational interpolation approach, we first discuss in Section 2 the convergence theory of rational interpolants with free poles to algebraic functions, where the notion of an algebraic S-contour arises. This will allow us to define an appropriate generalization of (3), see (31). In Sections 4 and 5 we construct the equivalent of the functions $\Psi_n^{(k)}$ as a solution of a boundary value problem on a certain Riemann surface descried in Section 3. Main results, namely Theorem 3 and Corollary 4, are stated and proved in Section 6.

## 2. Algebraic S-Contours

For the notions of potential theory appearing below such as Green's function and logarithmic capacity the reader might consult an excellent monograph [26].

**Definition.** *A compact set $\Delta$ is called an algebraic S-contour if the complement of $\Delta$, say $D$, is connected,*

$$\Delta = E_0\cup E_1\cup\bigcup\Delta_j,$$

*where $\bigcup\Delta_j$ is a finite union of open analytic arcs, $E_0\cup E_1$ is a finite set of points such that each element of $E_0$ is an endpoint of exactly one arc $\Delta_j$ while each element*

[2]In fact, $\widehat{p}(z) = \Big((z+\sqrt{z^2-1})^{-1/2}-\ell(z)\Big)/p(z)$, where $\ell$ is a polynomial of minimal degree interpolating $(z+\sqrt{z^2-1})^{-1/2}$ at the zeros of $p$.

*of $E_1$ is an endpoint of at least three arcs, and*

$$\frac{\partial g_D}{\partial \mathbf{n}^+} = \frac{\partial g_D}{\partial \mathbf{n}^-} \quad on \quad \bigcup \Delta_j,$$

*where $\partial/\partial\mathbf{n}^\pm$ are the partial derivatives with respect to the one-sided normals on each $\Delta_j$ and $g_D$ is Green's function for $D$ with pole at infinity.*

The sets $\Delta$ as above are called S-contours (or symmetric contours) due to the equality of the normal derivatives of Green's function for $D$ at every smooth point of $\Delta$ (symmetry property). We further call them algebraic to signify finiteness of $E_0 \cup E_1$ and emphasize their importance in convergence theory of Padé approximants to algebraic functions. The latter was developed in a series of pathbreaking papers [29, 30, 31, 32] by H. Stahl following the initial study of J. Nuttall [15, 16, 20, 17, 18] and lies in the following[3].

Let $f(z) = \sum_{j=0}^\infty f_j z^{-j}$ be a convergent power series in some neighborhood of infinity. We say that $f$ belongs to the class $\mathcal{A}$ if it has meromorphic continuation along any arc originating at infinity that belongs to $\overline{\mathbb{C}} \setminus E_f$, where $E_f$ is a finite set of points and for each $e \in E_f$ there exists a meromorphic continuation of $f$ that has a branch point at $e$. Given $f \in \mathcal{A}$, a compact set $K$ is called *admissible* if $\overline{\mathbb{C}} \setminus K$ is connected and $f$ has a meromorphic and single-valued extension there.

**Theorem (Stahl).** *Given $f \in \mathcal{A}$, there exists the unique admissible compact $\Delta_f$ such that $\mathrm{cp}(\Delta_f) \leq \mathrm{cp}(K)$ for any admissible compact $K$ and $\Delta_f \subseteq K$ for any admissible $K$ satisfying $\mathrm{cp}(\Delta_f) = \mathrm{cp}(K)$, where $\mathrm{cp}(K)$ is the logarithmic capacity of $K$. Moreover, $\Delta_f$ is an algebraic S-contour with $E_0 \subseteq E_f$.*

A *diagonal Padé approximant* to $f \in \mathcal{A}$ is a rational function $[n/n]_f = p_n/q_n$ of type $(n,n)$ that has maximal order of contact with $f$ at infinity [21, 5]. It is obtained from the solutions of the linear system (5) with $\widehat{p}$ replaced by $f$. This system is always solvable and no solution of it can be such that $q_n \equiv 0$ (we may thus assume that $q_n$ is monic). In general, a solution is not unique, but yields exactly the same rational function $[n/n]_f$. Thus, each solution is of the form $(lp_n, lq_n)$, where $(p_n, q_n)$ is the unique solution of minimal degree. Hereafter, $(p_n, q_n)$ will always stand for this unique pair of polynomials.

**Theorem (Stahl).** *Let $f$ and $\Delta_f$ be as before and $\{[n/n]_f\}_n$ be the sequence of diagonal Padé approximants to $f$. Then*

$$\left|f - [n/n]_f\right|^{1/2n} \overset{\mathrm{cp}}{\to} \exp\left\{-g_{D_f}\right\} \quad in \quad D_f,$$

*where $D_f := \overline{\mathbb{C}} \setminus \Delta_f$ and $\overset{\mathrm{cp}}{\to}$ stands for the convergence in capacity. The domain $D_f$ is optimal in the sense that the convergence does not holds in any other domain $D$ such that $D \setminus D_f \neq \varnothing$.*

Any algebraic S-contour is a minimal capacity contour for some algebraic function $f$. Given $\Delta$, an eligible function $f_\Delta \in \mathcal{A}$ can be constructed in the following

[3] In fact, the work of Stahl deals with a larger class of functions, namely, those that are meromorphic and multi-valued in $\overline{\mathbb{C}} \setminus E_f$ where $E_f$ is a polar set. Such a generality goes beyond the scope of the present note.

way. Denote by $m$ the number of connected components of $\Delta$, by $E_{0j}$ the intersection of $E_0$ with the $j$-th connected component, and by $m_j$ the cardinality of $E_{0j}$. Then one can take $f_\Delta(z)=\sum_{j=1}^m\big(\prod_{e\in E_{0j}}(z-e)\big)^{-1/m_j}$.

Algebraic S-contours admit a description via critical trajectories of rational quadratic differentials. For such a contour $\Delta$ with the complement $D$, set

$$h_\Delta(z):=2\partial_z g_D(z), \tag{6}$$

where $2\partial_z:=\partial_x-\mathrm{i}\partial_y$. The function $h_\Delta$ is holomorphic in $D$ and vanishes at infinity. For each point $e\in E_0\cup E_1$ denote by $i(e)$ the *bifurcation index* of $e$, that is, the number of different arcs $\Delta_j$ incident with $e$. It follows immediately from the definition of an algebraic S-contour that $i(e)=1$ for $e\in E_0$ and $i(e)\geq 3$ for $e\in E_1$. Denote also by $E_2$ the set of *critical points* of $g_D$ with $j(e)$ standing for the *order* of $e\in E_2$, i.e., $\partial_z^j g_D(e)=0$ for $j\in\{1,\dots,j(e)\}$ and $\partial_z^{j(e)+1}g_D(e)\neq 0$. The set $E_2$ is necessarily finite.

**Theorem (Perevoznikova-Rakhmanov).** *Let $\Delta$ be an algebraic S-contour with complement $D$. Then the arcs $\Delta_j$ are negative critical trajectories of the quadratic differential $h_\Delta^2(z)\mathrm{d}z^2$. That is, for any smooth parametrization $z(t):(0,1)\to\Delta_j$ it holds that $h_\Delta^2(z(t))\big(z'(t)\big)^2<0$ for all $t\in(0,1)$. Moreover,*

$$h_\Delta^2(z)=\prod_{e\in E_0\cup E_1}(z-e)^{i(e)-2}\prod_{e\in E_2}(z-e)^{2j(e)} \tag{7}$$

*and $h_\Delta^2(z)=z^{-2}+\mathcal{O}\big(z^{-3}\big)$ as $z\to\infty$.*

This theorem appeared in an unpublished work [25] and recently was reproduced in [4]. The variational approach to S-contours from [25] was later extended in [9] to more general critical contours. The general case of (non-rational) quadratic differentials and their relation to (non-algebraic) S-contours is treated in [33].

## 3. Associated Riemann Surface

Fix an algebraic S-contour $\Delta$ with complement $D$. The goal of the forthcoming Sections 4 and 5 is to define analogs of the functions $\Psi_n^{(k)}$ set up in (2). It appears that they naturally live on a certain two-sheeted Riemann surface $\mathfrak{R}$ associated to $\Delta$, which we describe below. The necessary background information on Riemann surfaces can be found in monographs [10, 8].

Let $h_\Delta$ be given by (6). As follows from (7), it is a hyperelliptic algebraic function. Denote by $\mathfrak{R}$ the Riemann surface defined by $h_\Delta$. We represent $\mathfrak{R}$ as a two-sheeted ramified cover of $\overline{\mathbb{C}}$ constructed in the following manner. Two copies of $\overline{\mathbb{C}}$ are cut along each arc $\Delta_j$. These copies are clipped together at the elements of $E_\Delta\subseteq E_0\cup E_1$, which consists of those points that have odd bifurcation index (branch points of $h_\Delta$). These copies are further glued together along the cuts in such a manner that the right (resp. left) side of the arc $\Delta_j$ belonging to the first copy, say $\mathfrak{R}^{(0)}$, is joined with the left (resp. right) side of the same arc $\Delta_j$ only belonging to the second copy, $\mathfrak{R}^{(1)}$. The genus of $\mathfrak{R}$, which we denote by $g$, satisfies the equality $2(g+1)=|E_\Delta|$.

According to the above construction, each arc $\Delta_j$ together with its endpoints corresponds to a cycle, say $\boldsymbol{\Delta}_j$, on $\mathfrak{R}$. We set $\boldsymbol{\Delta}:=\bigcup_j\boldsymbol{\Delta}_j$, denote by $\pi$ the canonical projection $\pi : \mathfrak{R} \to \overline{\mathbb{C}}$, and define

$$D^{(k)} := \mathfrak{R}^{(k)} \cap \pi^{-1}(D) \quad \text{and} \quad z^{(k)} := D^{(k)} \cap \pi^{-1}(z)$$

for $k \in \{0,1\}$ and $z \in D$. We further set $\boldsymbol{E}_\Delta := \pi^{-1}(E_\Delta)$, which is comprised exactly of the ramification points of $\mathfrak{R}$. The cycles $\boldsymbol{\Delta}_j$ are oriented so that $D^{(0)}$ remains on the left when $\boldsymbol{\Delta}_j$ is traversed in the positive direction. We designate the symbol $\cdot^*$ to stand for the conformal involution acting on the points of $\mathfrak{R}$ that fixes the ramification points $\boldsymbol{E}_\Delta$ and sends $z^{(k)}$ into $z^{(1-k)}$, $k \in \{0,1\}$. We use bolds lower case letters such as $\boldsymbol{z}, \boldsymbol{t}$ to indicate points on $\mathfrak{R}$ with canonical projections $z, t$.

Since $h_\Delta$ has only square root branching, each connected component of $\Delta$ contains even number of branch points. This allows us to number these points, $E_\Delta = \{e_0, e_1, \cdots, e_{2g+1}\}$, in the following fashion. If we consider $\partial D$ as a positively oriented Jordan curve (this way it contains two copies of each $\Delta_j$) and traverse it in the positive direction starting at $e_{2k}$, the next encountered branch point should be $e_{2k+1}$, $k \in \{1, \dots, g\}$.

Denote by $\boldsymbol{\alpha}_k$, $k \in \{1, \dots, g\}$, a smooth involution-symmetric Jordan curve that passes through $\boldsymbol{e}_1$ and $\boldsymbol{e}_{2k}$, and no other point of $\boldsymbol{\Delta}$ (until the end of the section we assume that $g \geq 1$), which is oriented so that the positive direction in $D^{(0)}$ goes from $\boldsymbol{e}_1$ to $\boldsymbol{e}_{2k}$. We require that $\boldsymbol{\alpha}_k \cap \boldsymbol{\alpha}_j = \{\boldsymbol{e}_1\}$ for each pair $k \neq j$. We further denote by $\boldsymbol{\beta}_k$ a smooth involution-symmetric Jordan curve that passes through $\boldsymbol{e}_{2k}$ and $\boldsymbol{e}_{2k+1}$ and is oriented so that at the point of intersection the tangent vectors to $\boldsymbol{\alpha}_k, \boldsymbol{\beta}_k$ form the right pair. Again, we suppose that $\boldsymbol{\Delta} \cap \boldsymbol{\beta}_k = \{\boldsymbol{e}_{2k}, \boldsymbol{e}_{2k+1}\}$ and also assume that $\boldsymbol{\beta}_j$ has empty intersection with any cycle $\boldsymbol{\gamma} \in \{\boldsymbol{\alpha}_k, \boldsymbol{\beta}_k\}_{k=1}^g$ except for $\boldsymbol{\alpha}_j$ with which it has only one point in common, necessarily $\boldsymbol{e}_{2j}$. Set

$$\widetilde{\mathfrak{R}} := \mathfrak{R} \setminus \bigcup_{k=1}^{g} (\boldsymbol{\alpha}_k \cup \boldsymbol{\beta}_k) \quad \text{and} \quad \widehat{\mathfrak{R}} := \mathfrak{R} \setminus \bigcup_{k=1}^{g} \boldsymbol{\alpha}_k.$$

The constructed collection $\{\boldsymbol{\alpha}_k, \boldsymbol{\beta}_k\}_{k=1}^g$ forms a homology basis on $\mathfrak{R}$ and so defined $\widetilde{\mathfrak{R}}$ is simply connected. In the case $g = 0$ these definitions are void and the whole surface is conformally equivalent to the Riemann sphere $\overline{\mathbb{C}}$.

Finally, we denote by $\mathrm{d}\vec{\Omega} := (\mathrm{d}\Omega_1, \dots, \mathrm{d}\Omega_g)^T$ the column vector of $g$ linearly independent holomorphic differentials normalized so that $\oint_{\boldsymbol{\alpha}_k} \mathrm{d}\vec{\Omega} = \vec{e}_k$, $k \in \{1, \dots, g\}$, where $\{\vec{e}_k\}_{k=1}^g$ is the standard basis for $\mathbb{R}^g$ and $\vec{e}^T$ is the transpose of $\vec{e}$. Since the genus of $\mathfrak{R}$ is $g$, the differentials $\mathrm{d}\Omega_k$ form a basis for the space of holomorphic differentials on $\mathfrak{R}$. Further, we set

$$\mathbf{B} := \left[ \oint_{\boldsymbol{\beta}_j} \mathrm{d}\Omega_k \right]_{j,k=1}^g. \tag{8}$$

It is known that the matrix $\mathbf{B}$ is symmetric and has positive definite imaginary part.

## 4. Normalizing Function $\Phi$ and Szegő Function $S_p$

Since $h_\Delta$ has only square root branching at the points of $E_\Delta$, the function

$$h(z^{(k)}) := (-1)^k h_\Delta(z), \quad z \in D, \tag{9}$$

which is extended to $\boldsymbol{\Delta}$ by continuity, is rational on $\mathfrak{R}$. By setting $\mathrm{d}G(\boldsymbol{z}) = h(\boldsymbol{z})\mathrm{d}z$, we obtain the so-called *Green's differential* on $\mathfrak{R}$. That is, all the periods (integrals over cycles on $\mathfrak{R}$) of $\mathrm{d}G$ are purely imaginary, in particular, we can define two vectors of real constants $\vec{\omega} = (\omega_1, \dots, \omega_g)^T$ and $\vec{\tau} = (\tau_1, \dots, \tau_g)^T$ by

$$\omega_k := -\frac{1}{2\pi\mathrm{i}} \oint_{\boldsymbol{\beta}_k} \mathrm{d}G \quad \text{and} \quad \tau_k := \frac{1}{2\pi\mathrm{i}} \oint_{\boldsymbol{\alpha}_k} \mathrm{d}G,$$

and $\mathrm{d}G$ is meromorphic having two simple poles at $\infty^{(1)}$ and $\infty^{(0)}$ with respective residues 1 and $-1$ (it holds that $\mathrm{d}G(z^{(k)}) = ((-1)^{k+1}/\zeta + \text{holomorphic})\mathrm{d}\zeta$ in local coordinates $\zeta = 1/z^{(k)}$).

Define

$$\Phi(\boldsymbol{z}) := \exp\left\{\int_{\boldsymbol{e}_0}^{\boldsymbol{z}} \mathrm{d}G\right\}, \quad \boldsymbol{z} \in \widetilde{\mathfrak{R}}. \tag{10}$$

The function $\Phi$ is holomorphic and non-vanishing on $\widetilde{\mathfrak{R}}$ except for a simple pole at $\infty^{(0)}$ and a simple zero at $\infty^{(1)}$. Furthermore, it possesses continuous traces on both sides of each cycle of the canonical basis that satisfy

$$\Phi^+ = \Phi^- \left\{ \begin{array}{ll} \exp\left\{2\pi\mathrm{i}\omega_k\right\} & \text{on} \quad \boldsymbol{\alpha}_k, \\ \exp\left\{2\pi\mathrm{i}\tau_k\right\} & \text{on} \quad \boldsymbol{\beta}_k. \end{array} \right. \tag{11}$$

In the case $g = 0$, $\Phi$ is a rational function well-defined on the whole Riemann surface.

Observe that the path of integration in (9) always can be chosen so it completely belongs to either $\mathfrak{R}^{(0)}$ or $\mathfrak{R}^{(1)}$. Thus, it readily follows from (9) and (6) that

$$\Phi(z^{(k)}) = \exp\left\{(-1)^k \int_{e_0}^{z} h_\Delta(t)\mathrm{d}t\right\} \quad \text{and} \quad \left|\Phi(z^{(k)})\right| = \exp\left\{(-1)^k g_D(z)\right\} \tag{12}$$

for $z \in D$. This computation has a trivial but remarkably important consequence, namely,

$$\Phi(z^{(0)})\Phi(z^{(1)}) \equiv 1 \quad \text{and} \quad |\Phi(z^{(0)})| > |\Phi(z^{(1)})|, \quad z \in D. \tag{13}$$

The depth of this inequality lies in the following. In no point in the remaining part of this note nor in Section 3, the S-contour structure of $\Delta$ played a role. We could have picked other arcs joining points in $E_\Delta$ or an entirely different system of arcs that makes $f_\Delta$ (see the paragraph before (6)) single-valued. This choice would yield an associated Riemann surface in exactly the same way as described in Section 3. This surface would have Green's differential and we would define function $\Phi$ in exactly the same way. This function would have all the above described properties besides (13). In an implicit way (13) carries all the information about the S-contour structure of $\Delta$.

When $g > 0$, the function $\Phi$ has the correct behavior at $\infty^{(0)}$ and $\infty^{(1)}$ but also is discontinuous across the cycles of the homology basis. Let us first remove its jumps from the $\boldsymbol{\beta}$-cycles. This is tantamount to replacing Green's differential in (10) with the normalized abelian differential of the third kind having poles at $\infty^{(0)}$ and $\infty^{(1)}$. However, we present it in a different form using discontinuous Cauchy kernel, which we also use to construct the desired Szegő functions.

Let $\boldsymbol{\gamma}$ be an involution-symmetric piecewise-smooth oriented chain on $\mathfrak{R}$ and $\lambda$ be a Hölder continuous function on $\boldsymbol{\gamma}$. Set

$$\Lambda(\boldsymbol{z})=\frac{1}{4\pi\mathrm{i}}\oint_{\boldsymbol{\gamma}}\lambda\mathrm{d}\Omega_{\boldsymbol{z},\boldsymbol{z}^*},\quad \boldsymbol{z}\notin\boldsymbol{\gamma},\tag{14}$$

where $\mathrm{d}\Omega_{\boldsymbol{z},\boldsymbol{z}^*}$ is the normalized abelian differential of the third kind (i.e., it is a meromorphic differential with two simple poles at $\boldsymbol{z}$ and $\boldsymbol{z}^*$ with respective residues 1 and $-1$ normalized to have zero periods on the $\boldsymbol{\alpha}$-cycles). It is known [40, Eq. (2.7)–(2.9)] that $\Lambda$ is a holomorphic function in $\widehat{\mathfrak{R}}\setminus\boldsymbol{\gamma}$, $\Lambda(\boldsymbol{z})+\Lambda(\boldsymbol{z}^*)\equiv 0$ there, the traces $\Lambda^\pm$ are continuous and satisfy

$$\Lambda^+(\boldsymbol{z})-\Lambda^-(\boldsymbol{z})=\frac{1}{2}\left\{\begin{array}{rl}\lambda(\boldsymbol{z})+\lambda(\boldsymbol{z}^*), & \boldsymbol{z}\in\boldsymbol{\gamma},\\ -\displaystyle\oint_{\boldsymbol{\gamma}}\big(\lambda(\boldsymbol{t})+\lambda(\boldsymbol{t}^*)\big)\mathrm{d}\Omega_k(\boldsymbol{t}) & \boldsymbol{z}\in\boldsymbol{\alpha}_k.\end{array}\right.$$

That is, the differential $\mathrm{d}\Omega_{\boldsymbol{z},\boldsymbol{z}^*}$ plays the role of the Cauchy kernel on $\mathfrak{R}$.

To remove the jumps of $\Phi$ across the $\boldsymbol{\beta}$-cycles, define $\lambda_{\vec\tau}$ to be the function on $\boldsymbol{\gamma}=\cup\boldsymbol{\beta}_k$ such that $\lambda_{\vec\tau}\equiv-2\pi\mathrm{i}\tau_k$ on $\boldsymbol{\beta}_k$ and set

$$S_{\vec\tau}(\boldsymbol{z}):=\exp\big\{\Lambda_{\vec\tau}(\boldsymbol{z})\big\},\quad \boldsymbol{z}\in\widetilde{\mathfrak{R}}.\tag{15}$$

Then $S_{\vec\tau}$ is a holomorphic function in $\widetilde{\mathfrak{R}}$ with continuous traces that satisfy

$$S_{\vec\tau}^+=S_{\vec\tau}^-\left\{\begin{array}{ll}\exp\big\{2\pi\mathrm{i}\big(\mathbf{B}\vec\tau\,\big)_k\big\} & \text{on}\quad \boldsymbol{\alpha}_k,\\ \exp\big\{-2\pi\mathrm{i}\tau_k\big\} & \text{on}\quad \boldsymbol{\beta}_k,\end{array}\right.\tag{16}$$

where the upper equality follows straight from (8) and the convention $(\vec c)_k=c_k$ for $\vec c=(c_1,\dots,c_g)$ is adopted.

Let now $p(z)=\prod_{j=1}^d(z-z_j)$ be a polynomial non-vanishing on $\Delta$. Set $\lambda_p:=\log P$, where $P:=p\circ\pi$ and a continuous determination of the argument of $p$ is chosen. This way $\lambda_p$ is a smooth involution-symmetric function on $\boldsymbol{\Delta}$. Define

$$S_p(\boldsymbol{z}):=\exp\big\{\Lambda_p(\boldsymbol{z})\big\}\tag{17}$$

and set $\vec c_p:=-\frac{1}{2\pi\mathrm{i}}\oint_{\boldsymbol{\Delta}}\lambda_p\mathrm{d}\vec\Omega$. Then $S_p$ is a holomorphic and non-vanishing function in $\widehat{\mathfrak{R}}\setminus\boldsymbol{\Delta}$ with continuous traces that satisfy

$$S_p^+=S_p^-\left\{\begin{array}{ll}\exp\big\{2\pi\mathrm{i}\big(\vec c_p\big)_k\big\} & \text{on}\quad \boldsymbol{\alpha}_k,\\ P & \text{on}\quad \boldsymbol{\Delta}.\end{array}\right.\tag{18}$$

Density $1/p$ can be further modified by any polynomial with simple zeros at the elements of $E_\Delta$. To this end, fix $\sigma:E_\Delta\mapsto\{0,1\}$ and set

$$u_\sigma(z):=\prod_{e\in E_\Delta}(z-e)^{\sigma(e)}.\tag{19}$$

The Szegő function of $u_\sigma^{-1}$, denoted by $S_{u_\sigma^{-1}}$, could be defined exactly as in (17), but such a definition will require a substantial effort to describe the behavior of $S_{u_\sigma^{-1}}$ near $\boldsymbol{e}\in\boldsymbol{E}_\Delta$. Hence, we choose a simpler path of explicitly defining $S_{u_\sigma^{-1}}$.

Put $U_\sigma:=u_\sigma\circ\pi$ and $d_\sigma:=\sum_{e\in E_\Delta}\sigma(e)=\deg(u_\sigma)$ and consider the function $U_\sigma^{-1}\Phi^{d_\sigma}$. It is holomorphic in $\widetilde{\mathfrak{R}}$ and non-vanishing there except for a zero of order $2d_\sigma$ at $\infty^{(1)}$. As $\widetilde{\mathfrak{R}}$ is an open simply connected Riemann surface with boundary, it is conformally equivalent to the unit disk. This, in particular, implies that we can take a square root of $U_\sigma^{-1}\Phi^{d_\sigma}$, which we now fix. Hence, by setting

$$S_{u_\sigma^{-1}} := \sqrt{U_\sigma^{-1}\Phi^{d_\sigma}}\, S_{d_\sigma\vec\tau/2} U_\sigma^k \quad \text{in} \quad \mathfrak{R}^{(k)}, \quad k\in\{0,1\}, \tag{20}$$

where $S_{d_\sigma\vec\tau/2}$ is defined as in (15) and (14) with $\lambda_{\vec\tau}$ replaced by $(d_\sigma/2)\lambda_{\vec\tau}$, we obtain the desired function which is holomorphic and non-vanishing in $\widehat{\mathfrak{R}}\setminus\boldsymbol{\Delta}$ and satisfies

$$S^+_{u_\sigma^{-1}} = S^-_{u_\sigma^{-1}} \left\{ \begin{array}{ll} \exp\big\{2\pi\mathrm{i}(d_\sigma/2)\big(\vec\omega + \mathbf{B}\vec\tau\,\big)_k\big\} & \text{on} \quad \boldsymbol{\alpha}_k, \\ U_\sigma^{-1} & \text{on} \quad \boldsymbol{\Delta}. \end{array}\right. \tag{21}$$

by (11) and (16).

Finally, gathering together (11), (16), (18), and (21), we deduce that

$$(\Phi^n S_{pu_\sigma^{-1}} S_{n\vec\tau})^+ = (\Phi^n S_{pu_\sigma^{-1}} S_{n\vec\tau})^- \left\{ \begin{array}{ll} \exp\big\{2\pi\mathrm{i}\big(\vec c_p + (n+d_\sigma/2)\big(\vec\omega+\mathbf{B}\vec\tau\big)\big)_k\big\} & \text{on} \quad \boldsymbol{\alpha}_k, \\ PU_\sigma^{-1} & \text{on} \quad \boldsymbol{\Delta}, \end{array}\right. \tag{22}$$

where $S_{pu_\sigma^{-1}} := S_p S_{u_\sigma^{-1}}$ and $S_{n\vec\tau} := S_{\vec\tau}^n$.

## 5. Construction of $\Psi_n$

To remove the jump of $\Phi^n S_{pu_\sigma^{-1}} S_{n\vec\tau}$ from the $\boldsymbol{\alpha}$-cycles and finish the construction of $\Psi_n^{(k)}$, let us digress into explaining what a Jacobi inversion problem is.

An *integral divisor* is a formal symbol of the form $\mathcal{D} = \sum n_j \boldsymbol{z}_j$, where $\{\boldsymbol{z}_j\}$ is an arbitrary finite collection of distinct points on $\mathfrak{R}$ and $\{n_j\}$ is a collection of positive integers. The sum $\sum n_j$ is called the *degree* of the divisor $\mathcal{D}$. Let $\mathcal{D}_1 = \sum n_j \boldsymbol{z}_j$ and $\mathcal{D}_2 = \sum m_j \boldsymbol{w}_j$ be integral divisors. A divisor $\mathcal{D}_1 - \mathcal{D}_2$ is called *principal* if there exists a rational function on $\mathfrak{R}$ that has a zero at every $\boldsymbol{z}_j$ of multiplicity $n_j$, a pole at every $\boldsymbol{w}_j$ of order $m_j$, and otherwise is non-vanishing and finite. By Abel's theorem, $\mathcal{D}_1 - \mathcal{D}_2$ is principle if and only if the divisors $\mathcal{D}_1$ and $\mathcal{D}_2$ have the same degree and

$$\vec\Omega(\mathcal{D}_1) - \vec\Omega(\mathcal{D}_2) \equiv \vec 0 \quad \big(\text{mod periods } \mathrm{d}\vec\Omega\big),$$

where $\vec\Omega(\mathcal{D}_1) := \sum n_j \int_{\boldsymbol{e}_0}^{\boldsymbol{z}_j} \mathrm{d}\vec\Omega$ and the equivalence of two vectors $\vec c, \vec e \in \mathbb{C}^g$ is defined by $\vec c \equiv \vec e$ $\big(\text{mod periods } \mathrm{d}\vec\Omega\big)$ if and only if $\vec c - \vec e = \vec j + \mathbf{B}\vec m$ for some $\vec j, \vec m \in \mathbb{Z}^g$.

Set $\mathcal{D}_* = g\infty^{(1)}$. We are seeking a solution of the following Jacobi inversion problem: find an integral divisor $\mathcal{D}$ of degree $g$ such that

$$\vec\Omega(\mathcal{D}) - \vec\Omega(\mathcal{D}_*) \;\equiv\; \vec c_p + (n+d_\sigma/2)\big(\vec\omega + \mathbf{B}\vec\tau\,\big) \quad \Big(\text{mod periods } d\vec\Omega\Big). \tag{23}$$

This problem is always solvable and the solution is unique up to a principal divisor. That is, if $\mathcal{D} - \big\{$ principal divisor $\big\}$ is an integral divisor, then it also solves (23). Immediately one can see that the subtracted principal divisor should have an integral part of degree at most $g$. As $\mathfrak{R}$ is hyperelliptic, such divisors come solely from rational functions on $\overline{\mathbb{C}}$ lifted to $\mathfrak{R}$. In particular, such principal divisors are involution-symmetric. Hence, if a solution of (23) contains at least one pair of involution-symmetric points, then replacing this pair by another such pair produces a different solution of (23). However, if a solution does not contain such a pair, then it solves (23) uniquely.

In what follows, we denote by $\mathcal{D}_n$ either the unique solution of (23) or the solution where each conjugate-symmetric pair is replaced by $\infty^{(0)} + \infty^{(1)}$. We further set $\mathbb{N}_{JIP}$ to be the subsequence of all indices for which (23) is uniquely solvable and

$\mathcal{D}_n$ does not contain $\infty^{(0)}$. Non-unique solutions are related to unique solutions in the following manner:

$$\mathcal{D}_n = \sum_{i=1}^{g-l} \boldsymbol{t}_i + k\infty^{(0)} + (l-k)\infty^{(1)} \quad \Leftrightarrow \quad \mathcal{D}_{n+j} = \mathcal{D}_n + j\big(\infty^{(0)} - \infty^{(1)}\big), \tag{24}$$

for $j \in \{-k, \dots, l-k\}$, where $l > 0$, $k \in \{0, \dots, l\}$, and $|t_i| < \infty$. Indeed, Riemann's relations state that

$$\oint_{\boldsymbol{\beta}_k} \mathrm{d}\Omega_{\infty^{(1)},\infty^{(0)}} = 2\pi\mathrm{i} \int_{\infty^{(0)}}^{\infty^{(1)}} \mathrm{d}\Omega_k$$

for each $k \in \{1, \dots, g\}$, where the path of integration lies entirely in $\widetilde{\mathfrak{R}}$. Since the differentials $\mathrm{d}\Omega_{\infty^{(1)},\infty^{(0)}}$ and $\mathrm{d}G$ have the same poles with the same residues, they differ by a holomorphic differential. Their normalizations imply that

$$\mathrm{d}G = \mathrm{d}\Omega_{\infty^{(1)},\infty^{(0)}} + 2\pi\mathrm{i}\sum_{k=1}^{g} \tau_k \mathrm{d}\Omega_k.$$

Combining the last two equations we get that

$$\vec{\Omega}(\mathcal{D}_n) - \vec{\Omega}(\mathcal{D}_*) + j\left(\vec{\Omega}\big(\infty^{(0)}\big) - \vec{\Omega}\big(\infty^{(1)}\big)\right) \equiv \vec{c}_p + (n + j + d_\sigma/2)\big(\vec{\omega} + \mathbf{B}\vec{\tau}\,\big)$$

from which (24) easily follows. In particular, (24) implies the unique solvability of (23) for the indices $n-k$ and $n+l-k$.

The solution of the Jacobi inversion problem (23) helps to remove the jump from the $\boldsymbol{\alpha}$-cycles in (22) via *Riemann's theta function*. The theta function associated with $\mathbf{B}$ is an entire transcendental function of $g$ complex variables defined by

$$\theta\left(\vec{u}\right) := \sum_{\vec{n}\in\mathbb{Z}^g} \exp\left\{\pi\mathrm{i}\vec{n}^T\mathbf{B}\vec{n} + 2\pi\mathrm{i}\vec{n}^T\vec{u}\right\}, \quad \vec{u} \in \mathbb{C}^g.$$

As shown by Riemann, the symmetry of $\mathbf{B}$ and positive definiteness of its imaginary part ensures the convergence of the series for any $\vec{u}$. It can be directly checked that $\theta$ enjoys the following periodicity properties:

$$\theta\left(\vec{u} + \vec{j} + \mathbf{B}\vec{m}\right) = \exp\left\{ -\pi\mathrm{i}\vec{m}^T\mathbf{B}\vec{m} - 2\pi\mathrm{i}\vec{m}^T\vec{u}\right\}\theta\big(\vec{u}\big), \quad \vec{j}, \vec{m} \in \mathbb{Z}^g. \tag{25}$$

Specializing integral divisors to one point $\boldsymbol{z}$, we reduce $\vec{\Omega}(\boldsymbol{z})$ to a vector of holomorphic functions in $\widetilde{\mathfrak{R}}$ with continuous traces on the cycles of the homology basis that satisfy

$$\vec{\Omega}^+ - \vec{\Omega}^- = \left\{ \begin{array}{rll} -\mathbf{B}\vec{e}_k & \text{on} & \boldsymbol{\alpha}_k, \\ \vec{e}_k & \text{on} & \boldsymbol{\beta}_k, \end{array} \right. \tag{26}$$

$k \in \{1, \dots, g\}$. It readily follows from the relations above that each $\Omega_k$ is, in fact, holomorphic in $\widehat{\mathfrak{R}} \setminus \boldsymbol{\beta}_k$. It is known that

$$\theta\left(\vec{u}\right) = 0 \quad \Leftrightarrow \quad \vec{u} \equiv \vec{\Omega}\left(\mathcal{D}_{\vec{u}}\right) + \vec{K} \quad \left(\text{mod periods } d\vec{\Omega}\right)$$

for some integral divisor $\mathcal{D}_{\vec{u}}$ of degree $g-1$, where $\vec{K}$ is the vector of Riemann constants defined by $(\vec{K})_j := ((\mathbf{B})_{jj} - 1)/2 - \sum_{k\neq j}\oint_{\boldsymbol{\alpha}_k} \Omega_j^- \mathrm{d}\Omega_k$, $j \in \{1, \dots, g\}$.

For $n \in \mathbb{N}_{JIP}$, set

$$\Theta_n(\boldsymbol{z}) := \frac{\theta\left(\vec{\Omega}(\boldsymbol{z}) - \vec{\Omega}(\mathcal{D}_n) - \vec{K}\right)}{\theta\left(\vec{\Omega}(\boldsymbol{z}) - \vec{\Omega}(\mathcal{D}_*) - \vec{K}\right)}. \tag{27}$$

Since the divisors $\mathcal{D}_n$ and $\mathcal{D}_*$ do not contain involution-symmetric pairs, $\vec{\Omega}(\boldsymbol{z}) + \vec{\Omega}(\boldsymbol{z}^*) \equiv 0$, and $\theta(-\vec{u}) = \theta(\vec{u})$, $\Theta_n$ is a multiplicatively multi-valued meromorphic function on $\mathfrak{R}$ with zeros at the points of the divisor $\mathcal{D}_n$ of respective multiplicities, a pole of order $g$ at $\infty^{(1)}$, and otherwise non-vanishing and finite. In fact, it is meromorphic and single-valued in $\widehat{\mathfrak{R}}$ and

$$\begin{aligned} \Theta_n^+ &= \Theta_n^- \exp\left\{2\pi\mathrm{i}\big(\Omega_k(\mathcal{D}_*) - \Omega_k(\mathcal{D}_n)\big)\right\} \\ &= \Theta_n^- \exp\left\{-2\pi\mathrm{i}\left(\vec{c}_p + (n+d_\sigma/2)\big(\vec{\omega} + \mathbf{B}\vec{\tau}\,\big) + \mathbf{B}\vec{m}_n\right)_k\right\} \end{aligned} \tag{28}$$

on $\boldsymbol{\alpha}_k$ by (25) and (26), where $\vec{m}_n, \vec{j}_n \in \mathbb{Z}^g$ are such that $\vec{\Omega}(\mathcal{D}) - \vec{\Omega}(\mathcal{D}_*) = \vec{c}_p + (n+d_\sigma/2)\big(\vec{\omega} + \mathbf{B}\vec{\tau}\,\big) + \vec{j}_n + \mathbf{B}\vec{m}_n$.

Finally, let $\lambda_{\vec{m}_n}$ be the function on $\boldsymbol{\gamma} = \cup\boldsymbol{\beta}_k$ such that $\lambda_{\vec{m}_n} \equiv -2\pi\mathrm{i}(\vec{m}_n)_k$ on $\boldsymbol{\beta}_k$ and set

$$S_{\vec{m}_n}(\boldsymbol{z}) = \exp\left\{\Lambda_{\vec{m}_n}(\boldsymbol{z})\right\}, \quad \boldsymbol{z} \in \widetilde{\mathfrak{R}}. \tag{29}$$

Since $\vec{m}_n \in \mathbb{Z}$, $S_{\vec{m}_n}$ is holomorphic across the $\boldsymbol{\beta}$-cycles by the analytic continuation principle and therefore is holomorphic in $\widehat{\mathfrak{R}}$. It has continuous traces on the $\boldsymbol{\alpha}$-cycles that satisfy

$$S_{\vec{m}_n}^+ = S_{\vec{m}_n}^- \exp\left\{2\pi\mathrm{i}\big(\mathbf{B}\vec{m}_n\big)_k\right\}. \tag{30}$$

By combining the material of the last two sections, we obtain the following theorem.

**Theorem 1.** *For $n \in \mathbb{N}$, let $\tilde{n}$ be the largest integer in $\mathbb{N}_{JIP}$ smaller or equal to $n$. With (10), (15), (17), (20), (27), and (29) at hand, set*

$$\Psi_n := \Phi^{\tilde{n}} S_{pu_\sigma^{-1}} S_{\tilde{n}\vec{\tau}+\vec{m}_{\tilde{n}}} \Theta_{\tilde{n}}.$$

*Then $\Psi_n$ is a sectionally meromorphic function in $\mathfrak{R} \setminus \boldsymbol{\Delta}$ whose zeros and poles there*[4] *are described by the divisor $(n-g)\infty^{(1)} + \mathcal{D}_n - n\infty^{(0)}$. Moreover, it has continuous traces on $\boldsymbol{\Delta} \setminus \boldsymbol{E}_\Delta$ satisfying*

$$\Psi_n^+ = PU_\sigma^{-1}\Psi_n^-$$

*by (22), (28), and (30). Finally, $\Psi_n$ is bounded near those points in $\boldsymbol{E}_\Delta$ for which $\sigma(e) = 0$ and $|\Psi_n(z^{(k)})| \sim |z-e|^{(-1)^{k+1}/2}$ otherwise.*

Observe that $\tilde{n} = n$ for $n \in \mathbb{N}_{JIP}$. Otherwise, if $n \notin \mathbb{N}_{JIP}$, $\mathcal{D}_n$ has the form as on the left-hand side of (24) with $0 < k \leq l$. As explained after (24), this yields that $\tilde{n} = n-k$ and the gap $n - \tilde{n}$ is equal to $k \leq g$. The function $\Psi_n$ is unique in the following sense.

**Theorem 2.** *Let $\Psi$ be a function with the same properties as $\Psi_n$ except for the divisor $\mathcal{D}_n$ being replaced by some integral divisor $\mathcal{D}$. Then the degree of $\mathcal{D}$ is $g$ and*

[4] $\Psi_n$ is non-vanishing and finite in $D^{(0)} \cup D^{(1)}$ except at the elements of its divisor that stand for zeros (resp. poles) if preceded by the plus (resp. minus) sign and the integer coefficients in front of them indicate multiplicity.

*it solves (23) for the index $n$. In particular, $\Psi = (\ell\circ\pi)\Psi_n$, where $\ell$ is a polynomial with the principal divisor of $\ell\circ\pi$ given by $\mathcal{D}-\mathcal{D}_n$.*

*Proof.* Given $\Psi$ as described, $\Psi/\Psi_n$ is holomorphic across $\boldsymbol{\Delta}$ by the principle of meromorphic continuation and therefore is rational on $\mathfrak{R}$ with the principal divisor $\mathcal{D}-\mathcal{D}_n$. Thus, the degree of $\mathcal{D}$ must be $g$, which also implies that $\Psi/\Psi_n$ is the lift of a rational function on $\overline{\mathbb{C}}$ to $\mathfrak{R}$. Zeros and poles of such lifts are necessarily involution-symmetric. Hence, $\mathcal{D}-\mathcal{D}_n=\sum_{i=1}^j\big(z_i^{(0)}+z_i^{(1)}\big)-j\big(\infty^{(0)}-\infty^{(1)}\big)$ for some $j\in\{0,\dots,g\}$ by the very definition of $\mathcal{D}_n$. As explained before, this means that $\mathcal{D}$ solves (23) for the index $n$, and it holds that $(\Psi/\Psi_n)(\pi^{-1}(z)) = c\prod_{i=1}^j(z-z_i)$. $\square$

## 6. Bernstein-Szegő Theorem

As before, let $\Delta$ be an algebraic S-contour and $E_\Delta$ be the set of the branch points of $h_\Delta$, i.e., the set of points in $E_0\cup E_1$ with odd bifurcation index. Set

$$w_\Delta^2(z) := \prod_{e\in E_\Delta}(z-e)$$

and chose a branch of $w_\Delta$ normalized so $z^{-g-1}w_\Delta(z)\to 1$ as $z\to\infty$. Pick an orientation for each $\Delta_j$ comprising $\Delta$. According to this orientation, the $+$ (resp. $-$) side of $\Delta_j$ is the one that remains on the left (resp. right) as $\Delta_j$ is traversed in the positive direction. For a polynomial $p$ non-vanishing on $\Delta$, $\sigma: E_\Delta\mapsto\{0,1\}$, and $u_\sigma$ as in (19), define

$$\widehat{p}_\sigma(z) := \frac{1}{\pi\mathrm{i}}\int_\Delta \frac{1}{t-z}\frac{u_\sigma(t)\mathrm{d}t}{p(t)w_\Delta^+(t)}, \quad z\in D, \tag{31}$$

where integration is taken place according to the chosen orientation. Then the following theorem takes place.

**Theorem 3.** *Let $\Delta$, $p$, and $\widehat{p}_\sigma$ be as above. Further, let $[n/n]_{\widehat{p}_\sigma}=p_n/q_n$ be the $n$-th diagonal Padé approximants to $\widehat{p}_\sigma$ and $R_n$ be the associated function of the second kind defined by (5). Then*

$$\left\{\begin{array}{lcl} q_n &=& \gamma_n\left(\Psi_n^{(0)}+pu_\sigma^{-1}\Psi_n^{(1)}\right) \\ R_n &=& 2\gamma_n w_\Delta^{-1}\Psi_n^{(1)}\end{array}\right. \tag{32}$$

*for all $2n>3g+\deg(p)$, where $\Psi_n^{(k)}(z):=\Psi_n(z^{(k)})$, $k\in\{0,1\}$, $\Psi_n$ is the function granted by Theorem 1, and $\gamma_n:=\lim_{z\to\infty}z^{\tilde n}/\Psi^{(0)}(z)$, $\tilde n=\max\{m\in\mathbb{N}_{JIP}:\ m\leq n\}$.*

The strength of Theorem 3 is in non-asymptotic character of formulae (32). Using Theorem 3 as an intermediate step one can prove an analogous theorem where $p$ is replaced by a non-vanishing Dini-continuous function as it was done in [19] for $\Delta=[-1,1]$ (the genus of the corresponding Riemann surface is 0), in [6] for a connected $\Delta$ with the corresponding Riemann surface of genus 1, and in [35] for $\Delta$ consisting of disjoint arcs. This extension will be carried out in the forthcoming publication [39].

*Proof of Theorem 3.* Since $R_n$ is holomorphic off $\Delta$ and vanishes at infinity, it holds by Cauchy's theorem that

$$R_n(z) = \oint_\Gamma \frac{(q_n\widehat{p} - p_n)(x)}{z-x}\frac{\mathrm{d}x}{2\pi\mathrm{i}} = \oint_\Gamma \frac{(q_n\widehat{p})(x)}{z-x}\frac{\mathrm{d}x}{2\pi\mathrm{i}}$$

for $z$ exterior to $\Gamma$, where $\Gamma$ is any positively oriented rectifiable Jordan curve encompassing $\Delta$. Then

$$\begin{aligned} R_n(z) &= \oint_\Gamma \frac{q_n(x)}{z-x}\left[\frac{1}{\pi\mathrm{i}}\int_\Delta \frac{1}{t-x}\frac{u_\sigma(t)\mathrm{d}t}{p(t)w_\Delta^+(t)}\right]\frac{\mathrm{d}x}{2\pi\mathrm{i}} \\ &= \int_\Delta\left[\frac{1}{2\pi\mathrm{i}}\oint_\Gamma \frac{q_n(x)}{z-x}\frac{\mathrm{d}x}{t-x}\right]\frac{u_\sigma(t)}{p(t)w_\Delta^+(t)}\frac{\mathrm{d}t}{\pi\mathrm{i}} = \int_\Delta \frac{q_n(t)}{t-z}\frac{u_\sigma(t)}{p(t)w_\Delta^+(t)}\frac{\mathrm{d}t}{\pi\mathrm{i}} \end{aligned}$$

by Fubini-Tonelli's theorem. The Cauchy integral structure of $R_n$ allows us to apply Sokhotski-Plemelj formulae [12, Sec. 4.2], which yield that

$$(w_\Delta R_n)^+ + (w_\Delta R_n)^- = 2q_n u_\sigma p^{-1} \quad \text{on} \quad \Delta. \tag{33}$$

The function $w_\Delta R_n$ has continuous traces on both sides of $\Delta\setminus E_\Delta$. To discuss its behavior near the point in $E_\Delta$, observe that $R_n = \sum_j R_{nj}$, where $R_{nj}$ is the integral over $\Delta_j$. Set $u_j(z) := (z-e_{j1})^{\sigma(e_{j1})}(z-e_{j2})^{\sigma(e_{j2})}$ and $w_j(z) := \sqrt{(z-e_{j1})(z-e_{j2})}$, where $e_{j1}, e_{j2}$ are the endpoints of $\Delta_j$ and $z^{-1}w_j(z)\to 1$ as $z\to\infty$. Notice that the function $(u_\sigma/u_j)(w_\Delta/w_j)$ can be continued from $\infty$ to be non-vanishing on and holomorphic across $\Delta_j\cup\{e_{j1},e_{j2}\}$ with the values on the arc equal to $(u_\sigma/u_j)(w_\Delta^+/w_j^+)$. Thus, $R_{nj}$ is the Cauchy integral over $\Delta_j$ of an analytic density with respect to the weight $u_j/w_j^+$. It is known [12], see also [6, Sec. 3], that in this case $w_jR_{nj}$ is bounded near $e_{jk}$ if $\sigma(e_{jk})=0$ and $|(w_jR_{nj})(z)|\sim|z-e_{jk}|^{1/2}$ as $D\ni z\to e_{jk}$ if $\sigma(e_{jk})=1$.

Let us lift boundary value problem (33) to $\boldsymbol{\Delta}$. To this end, set

$$F_n(\boldsymbol{z}) := (w_\Delta R_n)(z), \quad z\in D.$$

It readily follows from the definition of $w_\Delta$ together with (5) that $F_n$ is a meromorphic function in $D^{(0)}\cup D^{(1)}$ with zeros of order at least $n-g$ at both $\infty^{(0)}$ and $\infty^{(1)}$. Moreover, $F_n$ has continuous traces on both sides of $\boldsymbol{\Delta}\setminus\boldsymbol{E}_\Delta$, it is bounded near those $\boldsymbol{e}\in\boldsymbol{E}_\Delta$, for which $\sigma(e)=0$ and $|F_n(z^{(k)})|\sim|z-e|^{1/2}$ for $z\to e$ if $\sigma(e)=1$.

Put, as before, $P = p\circ\pi$ and $U_\sigma = u_\sigma\circ\pi$. Further, set $Q_n := q_n\circ\pi$. It is easy to see that relation (33) remains valid on $\boldsymbol{\Delta}$ as well. That is,

$$F_n^+ + F_n^- = 2Q_nU_\sigma P^{-1}, \tag{34}$$

where this time the traces are taken on both sides of $\boldsymbol{\Delta}$ rather than $\Delta$. Let $\Psi_n$ be the function granted by Theorem 1 for $p$ and $\sigma$ as above. Assume for now that $n\in\mathbb{N}_{JIP}$. Dividing both sides of (34) by $\Psi_n^-$ and using the fact that $\Psi_n^+ = PU_\sigma^{-1}\Psi_n^-$, we get

$$\left(\frac{F_n}{\Psi_n}\right)^- = \frac{2Q_nU_\sigma}{P\Psi_n^-} - \frac{F_n^+}{\Psi_n^-} = \left(\frac{2Q_n - PU_\sigma^{-1}F_n}{\Psi_n}\right)^+. \tag{35}$$

Recall that $\Psi_n$ is meromorphic in $D^{(0)}\cup D^{(1)}$ whose divisor there is equal to $(n-g)\infty^{(1)} + \mathcal{D}_n - n\infty^{(0)}$. Thus, the left- and right-hand sides of (35) are meromorphic in $D^{(1)}$ and $D^{(0)}$, respectively. The principle of meromorphic continuation implies that there exists a rational function on $\mathfrak{R}$, say $M_n$, such that

$$M_n(z^{(0)}) = \left(\frac{2Q_n - PU_\sigma^{-1}F_n}{\Psi_n}\right)(z^{(0)}) \quad \text{and} \quad M_n(z^{(1)}) = \left(\frac{F_n}{\Psi_n}\right)(z^{(1)}).$$

Recall that $\Psi_n(z^{(1)}) \sim |z-e|^{1/2}$ if $\sigma(e)=1$ and is bounded near $e$ if $\sigma(e)=0$. As we just explained, $F_n$ exhibits exactly the same behavior near $e \in E_\Delta$. Thus, subtracting the divisor of $\Psi_n$ from the divisor of $F_n$ in $D^{(1)}$, we get that the poles of $M_n$ in $D^{(1)}$ are the elements of $\mathcal{D}_n$. Further, as $\deg(p) < 2n-g$ and $\mathcal{D}_n$ does not contain $\infty^{(0)}$, $M_n(\infty^{(0)})$ is finite. Moreover, since $\Psi_n(z^{(0)}) \sim |z-e|^{-1/2}$ near those $e \in E_\Delta$ for which $\sigma(e)=1$, the ratio $F_n/(U_\sigma\Psi_n)$ is bounded there. Thus, the only poles of $M_n$ in $D^{(0)}$ are again the elements of $\mathcal{D}_n$. Hence, $M_n$ is a rational function with at most $g$ poles. As established before, this means that all the poles of $M_n$ come in involution-symmetric pairs, which is impossible as $n \in \mathbb{N}_{JIP}$ unless $M_n$ is a constant. The theorem now follows from the normalization at $\infty^{(0)}$ and the fact that $M_n$ cannot be identically zero.

Finally, assume that $n \notin \mathbb{N}_{JIP}$, $2n > \deg(p)+3g$. Then it follows from (24) that $\mathcal{D}_n = \mathcal{D}_{\tilde n} + (n-\tilde n)\big(\infty^{(0)} - \infty^{(1)}\big)$ and that $\mathcal{D}_{\tilde n} = \sum_{i=1}^{g-k} \boldsymbol{t}_{\tilde n,i} + k\infty^{(1)}$ for some $n-\tilde n \le k \le g$, where $|t_{\tilde n,i}| < \infty$. Therefore, $2\tilde n > \deg(p)+g$ and we are in the preceding situation for which (32) has already been established. Hence, it holds that $\deg(q_{\tilde n}) = \tilde n$ and $R_{\tilde n}(z) \sim z^{-\tilde n-1-k}$ as $z \to \infty$. The so-called block structure of Padé approximants [5] then implies that $[\tilde n/\tilde n]_{\widehat{p}_\sigma} = [\tilde n + j/\tilde n + j]_{\widehat{p}_\sigma}$ for all $j \in \{1,\dots,k\}$ and therefore $q_n = q_{\tilde n}$ and $R_n = R_{\tilde n}$. This finishes the proof of the theorem as $\Psi_n = \Psi_{\tilde n}$. □

In the case where $\Delta = [-1,1]$, it can be easily verified that $\widehat{1}_{\sigma_0} = w_\Delta^{-1}$, $\sigma_0 \equiv 0$. Moreover, it can be readily observed that $[n/n]_{\widehat{1}_{\sigma_0}} = U_{n-1}/T_n$, where $T_n$ is the $n$-th degree Chebyshëv polynomials of the first kind and $U_{n-1}$ is the Chebyshëv polynomial of the second kind of degree $n-1$. It is also known that $U_n$ is the denominator of the $n$-th diagonal Padé approximant to $\widehat{1}_{\sigma_1}$, $\sigma_1 = 1-\sigma_0 \equiv 1$. These relations are preserved on an arbitrary algebraic $S$-contour.

**Corollary 4.** *Let $\Psi_n$ and $\hat\Psi_n$ be the function granted by Theorem 1 for $p \equiv 1$, $\sigma_0 \equiv 0$ and $\sigma_1 \equiv 1$, respectively*[5]*. Then $[n/n]_{\widehat{1}_{\sigma_0}} = c_n U_{n-1-g}/T_n$, where $c_n$ is a constant,*

$$\begin{cases} U_{n-1-g} &= \hat\gamma_{n-1-g}\left(\hat\Psi^{(0)}_{n-1-g} + w_\Delta^{-2}\hat\Psi^{(1)}_{n-1-g}\right) \\ T_n &= \gamma_n\left(\Psi_n^{(0)} + \Psi_n^{(1)}\right), \end{cases}$$

*and $\gamma_n$, $\hat\gamma_{n-1-g}$ are normalizing constants that make $T_n$, $U_{n-1-g}$ monic. In particular, $U_n$ is the denominator of $[n/n]_{\widehat{1}_{\sigma_1}}$.*

*Proof.* Clearly, formula for $T_n$ is simply (32) repeated. It follows from (5) and (32) that the numerator of $[n/n]_{\widehat{1}_{\sigma_0}}$ is equal to

$$\gamma_n\left(\Psi_n^{(0)} + \Psi_n^{(1)}\right)\widehat{1}_{\sigma_0} - 2\gamma_n w_\Delta^{-1}\Psi^{(1)} = \gamma_n w_\Delta^{-1}\left(\Psi_n^{(0)} - \Psi_n^{(1)}\right)$$

[5]Observe that $\Psi_n$ is a rational function on $\mathfrak{R}$ with the principal divisor $(n-g)\infty^{(1)} + \mathcal{D}_n - n\infty^{(0)}$.

as $\widehat{1}_{\sigma_0} = w_\Delta^{-1}$ (both functions are holomorphic in $D$ with the same jump over $\Delta$). To finish the proof just observe that

$$\hat{\Psi}_{n-g-1} = \begin{cases} w_\Delta^{-1}\Psi_n^{(0)} & \text{in} \quad D^{(0)} \\ w_\Delta \Psi_n^{(1)} & \text{in} \quad D^{(1)} \end{cases}$$

by Theorem 2 (to verify the behavior on $\boldsymbol{\Delta}$ notice that $w_\Delta^+ w_\Delta^- = w_\Delta^2$ and $w_\Delta^+ = -w_\Delta^-$ on $\Delta$). □

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