ON SMOOTH PERTURBATIONS OF CHEBYSHËV POLYNOMIALS AND $\bar\partial$-RIEMANN-HILBERT METHOD

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Abstract. $\bar\partial$-extension of the matrix Riemann-Hilbert method is used to study asymptotics of the polynomials $P_n(z)$ satisfying orthogonality relations

$$\int_{-1}^1 x^l P_n(x)\frac{\rho(x)dx}{\sqrt{1-x^2}}=0,\quad l\in\{0,\dots,n-1\},$$

where $\rho(x)$ is a positive m times continuously differentiable function on $[-1,1]$, $m\geqslant 3$.

1. Main Results

In this note we are interested in the asymptotic behavior of monic polynomials $P_{n,i}(x)$, $\deg(P_{n,i})=n$, dependent on a parameter $i\in\{1,2,3,4\}$, satisfying orthogonality relations

$$\int_{-1}^1 x^l P_{n,i}(x)\frac{\rho(x)|v_i(x)|dx}{\sqrt{1-x^2}}=0,\quad l\in\{0,\dots,n-1\}, \tag{1}$$

where $\rho(x)$ is a positive and smooth function on $[-1,1]$ and

$$v_1(z)\equiv 1,\quad v_2(z)=z^2-1,\quad v_3(z)=z+1,\quad\text{and}\quad v_4(z)=z-1.$$

That is, $P_{n,i}(z)$ are smooth perturbations of the Chebyshëv polynomials of the i-th kind. Besides polynomials themselves, we are also interested in the asymptotic behavior of their recurrence coefficients. That is, numbers $a_{n,i}\in[0,\infty)$ and $b_{n,i}\in(-\infty,\infty)$ such that

$$xP_{n,i}(x)=P_{n+1,i}(x)+b_{n,i}P_{n,i}(x)+a_{n,i}^2P_{n-1,i}(x).$$

To describe the results, let $w(z):=\sqrt{z^2-1}$ be the branch analytic in $\mathbb{C}\backslash[-1,1]$ such that $w(z)/z\to 1$ as $z\to\infty$. The Szegő function of the weight $\rho(x)$ is defined by

$$S(z):=\exp\left\{\frac{w(z)}{2\pi\mathrm{i}}\int_{-1}^1\frac{\log\rho(x)}{z-x}\frac{dx}{w_+(x)}\right\},\quad z\in\overline{\mathbb{C}}\backslash[-1,1], \tag{2}$$

which is an analytic and non-vanishing function in the domain of its definition satisfying

$$S_+(x)S_-(x)=\rho^{-1}(x),\quad x\in[-1,1]. \tag{3}$$

Since $\rho(x)$ is positive, it holds that $S_+(x)=\overline{S_-(x)}$ for $x\in[-1,1]$, and, utilizing the full power of Plemelj-Sokhotski formulae, (3) can be strengthen to

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$$\sqrt{\rho(x)}S_\pm(x) = e^{\pm \mathrm{i}\theta(x)}, \quad \theta(x) := \frac{\sqrt{1-x^2}}{2\pi}\fint_{-1}^{1}\frac{\log\rho(t)}{t-x}\frac{dt}{\sqrt{1-t^2}}, \tag{4}$$

where $\fint$ is the integral in the sense of the principal value. Further, let

$$\varphi(z) := z + w(z) \tag{5}$$

be the conformal map of $\overline{\mathbb{C}}\setminus[-1,1]$ onto $\mathbb{C}\setminus\{z:|z|\geqslant 1\}$ such that $\varphi(z)/z \to 2$ as $z\to\infty$. One can readily verify that

$$\varphi_\pm(x) = x \pm \mathrm{i}\sqrt{1-x^2} = e^{\pm\mathrm{i}\arccos(x)}, \quad x\in[-1,1]. \tag{6}$$

Finally, we explicitly define the Szegő functions of the weights $|v_i(x)|$. Namely, set

$$\begin{cases} S_1(z) :\equiv 1, & S_3(z) := \big(\varphi(z)/(z+1)\big)^{1/2}, \\ S_2(z) := \varphi(z)/w(z), & S_4(z) := \big(\varphi(z)/(z-1)\big)^{1/2}, \end{cases} \tag{7}$$

where the square roots are principal and one needs to notice that the images of $\overline{\mathbb{C}}\setminus[-1,1]$ under $(z+1)/\varphi(z)$ and $(z-1)/\varphi(z)$ are domains symmetric with respect to conjugation whose intersections with the real line are equal to $(0,2)$ (so the square roots are indeed well defined). These functions satisfy

$$S_{i+}(x)S_{i-}(x) = |S_{i\pm}(x)|^2 = 1/|v_i(x)|, \quad x\in(-1,1). \tag{8}$$

Observe also that $S_1(\infty)=1$, $S_2(\infty)=2$, and $S_3(\infty)=S_4(\infty)=\sqrt{2}$. Moreover, one can readily deduce from (6) and (8) that

$$S_{i\pm}(x) = \frac{e^{\pm\mathrm{i}\theta_i(x)}}{\sqrt{|v_i(x)|}}, \quad \begin{cases} \theta_1(x) :\equiv 0, & \theta_2(x) := \arccos(x) - \frac{\pi}{2}, \\ \theta_3(x) := \frac12\arccos(x), & \theta_4(x) := \frac12\arccos(x) - \frac{\pi}{2}. \end{cases} \tag{9}$$

Recall that the modulus of continuity of a continuous function $f(x)$ on $[-1,1]$ is given by

$$\omega(f;h) := \max_{|x-y|\leqslant h,\ x,y\in[-1,1]} |f(x)-f(y)|.$$

Theorem 1

Assume that $\rho(x)$ is a strictly positive m times continuously differentiable function on $[-1,1]$ for some $m\geqslant 3$. Set

$$\varepsilon_n := \frac{\log n}{n^m}\omega\left((1/\rho)^{(m)};1/n\right).$$

Then it holds for any $i\in\{1,2,3,4\}$ that

$$P_{n,i}(z) = (1+O(\varepsilon_n))\frac{(S_iS)(z)}{(S_iS)(\infty)}\left(\frac{\varphi(z)}{2}\right)^n$$

uniformly on closed subsets of $\overline{\mathbb{C}}\setminus[-1,1]$ and

$$P_{n,i}(x) = \frac{\cos\big(n\arccos(x)+\theta(x)+\theta_i(x)\big)+O(\varepsilon_n)}{2^{n-1}(S_iS)(\infty)\sqrt{\rho(x)|v_i(x)|}}$$

uniformly on $[-1,1]$. Moreover, it also holds for any $i\in\{1,2,3,4\}$ that

$$a_{n,i}=1/2+O(\varepsilon_n) \quad \text{and} \quad b_{n,i}=O(\varepsilon_n).$$

The above results are not entirely new. It is well known [18, Theorem 11.5] that perturbed first and second kind Chebyshëv polynomials can be expressed via orthogonal polynomials on the unit circle with respect to the weight $\rho(\frac{1}{2}(\tau+1/\tau))$. Then using [17, Corollary 5.2.3], that in itself is an extension of ideas from [5], and Geronimus relations, see [17, Theorem 13.1.7], one can show that

$$\sum(n+1)^\gamma\left(|a_{n,1}-1/2|+|b_{n,1}|\right)<\infty$$

for any $\gamma\in(0,m-1)$ and $m\geqslant 2$, which is consistent with Theorem 1. What is novel in this note is the method of proof. While the Baxter-Simon argument relies on the machinery of Banach algebras, we follow the approach of Fokas, Its, and Kitaev [11, 12] connecting orthogonal polynomials to matrix Riemann-Hilbert problems and then utilizing the non-linear steepest descent method of Deift and Zhou [9]. The main advantages of this approach are the ability to get full asymptotic expansions for analytic weights of orthogonality [8, 15] and its indifference to positivity of such weights [1, 6, 2]. However, here we deal with non-analytic densities by elaborating on the idea of extensions with controlled $\bar\partial$-derivative introduced by Miller and McLaughlin [16] and adapted to the setting of Jacobi-type polynomials by Baratchart and the author [4].

2. Weight Extension

Given $r>1$, let $E_r:=\{z:|\varphi(z)|<r\}$. The boundary ∂E_r is an ellipse with foci ±1.

Proposition 1

Let $\rho(x)$ and ε_n be as in Theorem 1. For each $r>1$ and $n>2m$ there exists a continuous function $\ell_{n,r}(z)=l_n(z)+L_{n,r}(z)$, $z\in\mathbb{C}$, such that

$$\ell_{n,r}(x)=\rho^{-1}(x), \quad x\in[-1,1],$$

where $l_n(z)$ is a polynomial of degree at most n satisfying

$$\mathrm{supp}_{x\in[-1,1]}|l_n(x)|\leqslant C'_\rho$$

for some constant C'_ρ independent of n, while $L_{n,r}(z)$ and $\bar\partial L_{n,r}(z)$ are continuous functions in $\mathbb{C}$ supported by $\overline{E}_r$ (in particular, $L_{n,r}(z)=0$ for $z\notin E_r$) and

$$\frac{|\bar\partial L_{n,r}(z)|}{\sqrt{|1-z^2|}}\leqslant C''_\rho\frac{n\varepsilon_n}{\log n}, \quad z\in\overline{E}_r,$$

for some constant C''_ρ independent of n and r, where $\bar\partial:=(\partial_x+\mathrm{i}\partial_y)/2$, $z=x+\mathrm{i}y$.

Proof. It follows from [14, Theorem 9] that for each $n>2m$ there exists a polynomial $l_n(z)$ of degree at most n such that

$$\left|\left(\rho^{-1}(x)\right)^{(k)}-l_n^{(k)}(x)\right|\leqslant C_{m,k}(1-x^2)^{\frac{m-k}{2}}n^{k-m}E_{n-m}\left(\left(\rho^{-1}\right)^{(m)}\right)$$

for all $x\in[-1,1]$ and each $k\in\{0,\dots,m\}$, where $C_{m,k}$ is a constant that depends only m and k and $E_j(f)$ is the error of best uniform approximation on the interval $[-1,1]$ of a continuous function $f(x)$ by algebraic polynomials of degree at most j. Furthermore, it was shown by Timan, see [14, Equation (3)], that

$$\begin{aligned} E_{n-m}(f) &\leqslant C_1\omega\left(f;\frac{\sqrt{1-x^2}}{n-m}+\frac{1}{(n-m)^2}\right)\leqslant C_1\omega\left(f;\frac{2}{n-m}\right) \\ &\leqslant C_1\omega\left(f;\frac{4}{n}\right)\leqslant 4C_1\omega\left(f;\frac{1}{n}\right) \end{aligned}$$

for some absolute constant C_1, where we used that $n>2m$ and $\omega(f;2h)\leqslant 2\omega(f;h)$ (in what follows, we understand that all constants C_j might depend on $\rho(x)$, but are independent of n). Set

$$\lambda_n(x):=\frac{\rho^{-1}(x)-l_n(x)}{\sqrt{1-x^2}},\quad x\in[-1,1].$$

It then holds that $\lambda_n(x)$ is a continuous function on $[-1,1]$ that satisfies $\|\lambda_n\|\leqslant C_3\varepsilon_n/\log n$, where $\|\cdot\|$ is the uniform norm on $[-1,1]$. Since $m\geqslant 3$, it also holds that

$$\lambda_n'(x)=\frac{\big(\rho^{-1}(x)\big)'-l_n'(x)}{\sqrt{1-x^2}}+x\frac{\rho^{-1}(x)-l_n(x)}{\sqrt{(1-x^2)^3}}$$

is a continuous function on $[-1,1]$ that satisfies $\|\lambda_n'\|\leqslant C_4n\varepsilon_n/\log n$ (this is exactly the place where condition $m\geqslant 3$ is used). Extend $\lambda_n(x)$ by zero to the whole real line. As the numerator of $\lambda_n(x)$ together with its first and second derivatives vanishes at ±1, $\lambda_n'(x)$ also extends continuously by zero to the whole real line. The following construction is standard, see [10, Proof of Theorem 3.67]. Define

$$\Lambda_n(z):=\frac{1}{|y|}\int_0^{|y|}\lambda_n(x+t)dt,\quad z=x+\mathrm{i}y,$$

which, due to continuity of $\lambda_n(x)$, is a continuous function in $\mathbb{C}$ satisfying $\Lambda_n(x)=\lambda_n(x)$ on the real line and $|\Lambda_n(z)|\leqslant\|\lambda_n\|$ in the complex plane. Similarly,

$$\big|\partial_x\Lambda_n(z)\big|=\left|\frac{1}{|y|}\int_0^{|y|}\lambda_n'(x+t)dt\right|\leqslant\|\lambda_n'\|$$

and the function $\partial_x\Lambda_n(z)$, which is given by the integral within the absolute value in the above equation, is also continuous in $\mathbb{C}$. Furthermore, we have that

$$\begin{aligned} \big|\partial_y\Lambda_n(z)\big| &= \left|\frac{1}{y^2}\int_0^{|y|}\big(\lambda_n(x+t)-\lambda_n(x+|y|)\big)dt\right| \\ &\leqslant \|\lambda_n'\|\int_0^{|y|}\frac{|y|-t}{y^2}dt=\frac{\|\lambda_n'\|}{2} \end{aligned}$$

and is also a continuous function in $\mathbb{C}$. Altogether, since $\bar\partial=(\partial_x+\mathrm{i}\partial_y)/2$, it holds that $\bar\partial\Lambda_n(z)$ is a continuous function in $\mathbb{C}$ that satisfies $|\bar\partial\Lambda_n(z)|\leqslant\|\lambda_n'\|$ in the complex plane. Let $\psi_r(z)$ be any real-valued continuous function with continuous partial derivatives that is equal to one on $[-1,1]$ and is equal to zero in the complement of E_r. Define

$$L_{n,r}(z):=\mathrm{i}w(z)\Lambda_n(z)\psi_r(z)\begin{cases}-1, & \mathrm{Im}(z)\geqslant 0,\\ 1, & \mathrm{Im}(z)<0.\end{cases}$$

Since $w_\pm(x)=\pm\mathrm{i}\sqrt{1-x^2}$ for $x\in[-1,1]$ and $\Lambda_n(x)=0$ for $x\notin(-1,1)$, it holds that $L_{n,r}(z)$ is a continuous function in $\mathbb{C}$ that is supported by $\overline{E}_r$ and is equal to $\rho^{-1}(x)-l_n(x)$

for $x \in [-1,1]$. Furthermore, since $\bar\partial(\Lambda_n(z)\psi_n(z))$ is continuous in $\mathbb{C}$ and vanishes for $z = x \notin (-1,1)$ while $w_+(x) = -w_-(x)$ for $x \in (-1,1)$, $\bar\partial L_{n,r}(z)$ is also continuous in $\mathbb{C}$. Moreover, it holds that

$$\begin{aligned} |\bar\partial L_{n,r}(z)| &= \sqrt{|1-z^2|}\,\left|\bar\partial(\Lambda_n(z)\psi_r(z))\right| \\ &\leqslant C_5\sqrt{|1-z^2|}\,\left(|\Lambda_n(z)| + |\bar\partial\Lambda_n(z)|\right) \\ &\leqslant C_6\sqrt{|1-z^2|}\,\frac{n\varepsilon_n}{\log n}, \quad z \in \overline{E}_r. \end{aligned}$$

Finally, observe that polynomials $l_n(x)$ approximate $\rho^{-1}(x)$ on $[-1,1]$ and therefore have uniformly bounded above uniform norms. The claim of the proposition now follows by setting $\ell_{n,r}(z) := l_n(z) + L_{n,r}(z)$ for $l_n(z)$ and $L_{n,r}(z)$ as above. □

3. Proof of Theorem 1

3.1. **Initial Riemann-Hilbert Problem.** Notice that the functions $v_i(x)$ and $|v_i(x)|$ are either equal to each other or differ by a sign when $x \in [-1,1]$. So, we can equally use $v_i(x)$ in (1) without changing the polynomials $P_{n,i}(x)$.

Denote by $R_{n,i}(z)$ the function of the second kind associated with $P_{n,i}(z)$. That is,

$$R_{n,i}(z) := \frac{1}{2\pi \mathrm{i}}\int_{-1}^{1}\frac{P_{n,i}(x)}{x-z}\frac{\rho(x)v_i(x)dx}{w_+(x)}, \tag{10}$$

which is a holomorphic function in $\overline{\mathbb{C}}\backslash[-1,1]$. It follows from Plemelj-Sokhotski formulae, [13, Chapter I.4.2], that

$$R_{n,i+}(x) - R_{n,i-}(x) = P_{n,i}(x)\frac{\rho(x)v_i(x)}{w_+(x)}, \quad x \in (-1,1),$$

and, see [13, Chapter I.8.4], that

$$R_{n,i}(z) = O\left(|z-a|^{\alpha_{a,i}}\right) \quad \text{as} \quad \mathbb{C}\backslash[-1,1] \ni z \to a \in \{-1,1\},$$

where $\alpha_{a,i} = 0$ if $v_i(a) = 0$ and $\alpha_{a,i} = -1/2$ otherwise. Moreover, we get from (1) that

$$R_{n,i}(z) = \frac{1}{m_{n,i}z^n} + O\left(\frac{1}{z^{n+1}}\right) \quad \text{as} \quad z \to \infty$$

for some finite constant $m_{n,i}$. Consider the following Riemann-Hilbert problem for 2×2 matrix functions (RHP-$\boldsymbol{Y}$):

(a) $\boldsymbol{Y}(z)$ is analytic in $\mathbb{C}\backslash[-1,1]$ and $\lim_{z\to\infty}\boldsymbol{Y}(z)z^{-n\sigma_3} = \boldsymbol{I}$;

(b) $\boldsymbol{Y}(z)$ has continuous traces on $(-1,1)$ that satisfy
$$\boldsymbol{Y}_+(x) = \boldsymbol{Y}_-(x)\begin{pmatrix} 1 & \frac{\rho(x)v_i(x)}{w_+(x)} \\ 0 & 1\end{pmatrix};$$

(c) $\boldsymbol{Y}(z)$ behaves like
$$\boldsymbol{Y}(z) = O\begin{pmatrix} 1 & |z-a|^{\alpha_{a,i}} \\ 1 & |z-a|^{\alpha_{a,i}}\end{pmatrix} \quad \text{as} \quad \mathbb{C}\backslash[-1,1] \ni z \to a \in \{-1,1\}.$$

The following lemma is well known [15, Theorem 2.4].

Lemma 1

RHP-$\boldsymbol{Y}$ is uniquely solvable by

$$\boldsymbol{Y}(z)=\begin{pmatrix} P_{n,i}(z) & R_{n,i}(z) \\ m_{n-1,i}P_{n-1,i}(z) & m_{n-1,i}R_{n-1,i}(z)\end{pmatrix}. \tag{11}$$

3.2. **Opening of the Lens.** Fix $1<r<R$ and orient ∂E_R clockwise. Set

$$\boldsymbol{X}(z):=\begin{cases}\boldsymbol{Y}(z)\begin{pmatrix}1 & 0\\ -\frac{w(z)\ell_{n,r}(z)}{v_i(z)} & 1\end{pmatrix}, & \text{in} \quad E_R\setminus[-1,1],\\ \boldsymbol{Y}(z), & \text{in} \quad \mathbb{C}\setminus\overline{E}_R,\end{cases} \tag{12}$$

where $\ell_{n,r}(z)$ is the extension of $\rho^{-1}(x)$ constructed in Proposition 1. Observe that

$$\ell_{n,r}(s)=l_n(s), \quad s\in\partial E_R, \quad \text{and} \quad \bar\partial\ell_{n,r}(z)=\bar\partial L_{n,r}(z), \quad z\in\overline{E}_r,$$

since $L_{n,r}(z)$ is supported by $\overline{E}_r$ and $l_n(z)$ is analytic (in fact, is a polynomial). It is trivial to verify that $\boldsymbol{X}(z)$ solves the following $\bar\partial$-Riemann-Hilbert problem ($\bar\partial$RHP-$\boldsymbol{X}$):

(a) $\boldsymbol{X}(z)$ is continuous in $\mathbb{C}\setminus([-1,1]\cup\partial E_R)$ and $\lim_{z\to\infty}\boldsymbol{X}(z)z^{-n\sigma_3}=\boldsymbol{I}$;
(b) $\boldsymbol{X}(z)$ has continuous traces on $(-1,1)\cup\partial E_R$ that satisfy

$$\boldsymbol{X}_+(s)=\boldsymbol{X}_-(s)\begin{cases}\begin{pmatrix}0 & \frac{\rho(s)v_i(s)}{w_+(s)}\\ -\frac{w_+(s)}{\rho(s)v_i(s)} & 0\end{pmatrix} & \text{on} \quad s\in(-1,1),\\ \begin{pmatrix}1 & 0\\ \frac{w(s)l_n(s)}{v_i(s)} & 1\end{pmatrix} & \text{on} \quad s\in\partial E_R;\end{cases}$$

(c) $\boldsymbol{X}(z)$ has the same behavior near ±1 as $\boldsymbol{Y}(z)$, see RHP-$\boldsymbol{Y}$(c);
(d) $\boldsymbol{X}(z)$ deviates from an analytic matrix function according to

$$\bar\partial\boldsymbol{X}(z)=\boldsymbol{X}(z)\begin{pmatrix}0 & 0\\ -\frac{w(z)\bar\partial L_{n,r}(z)}{v_i(z)} & 0\end{pmatrix}.$$

One can readily verified that the following lemma holds, see [4, Lemma 6.4].

Lemma 2

$\bar\partial$RHP-$\boldsymbol{X}$ and RHP-$\boldsymbol{Y}$ are simultaneously solvable and the solutions are connected by (12).

3.3. **Model Riemann-Hilbert Problem.** In this subsection we present the solution of the following Riemann-Hilbert problem (RHP-$\boldsymbol{N}$):

(a) $\boldsymbol{N}(z)$ is analytic in $\mathbb{C}\setminus[-1,1]$ and $\lim_{z\to\infty}\boldsymbol{N}(z)z^{-n\sigma_3}=\boldsymbol{I}$;
(b) $\boldsymbol{N}(z)$ has continuous traces on $(-1,1)$ that satisfy

$$\boldsymbol{N}_+(x)=\boldsymbol{N}_-(s)\begin{pmatrix}0 & \frac{\rho(x)v_i(x)}{w_+(x)}\\ -\frac{w_+(x)}{\rho(x)v_i(x)} & 0\end{pmatrix};$$

(c) $\boldsymbol{N}(z)$ has the same behavior near ±1 as $\boldsymbol{Y}(z)$, see RHP-$\boldsymbol{Y}$(c).

Recall the definition of the functions $S_i(z)$ in (7). Define $S_*(z) := S_i(z)$ when $i \in \{1,3\}$ and $S_*(z) := \mathrm{i}S_i(z)$ when $i \in \{2,4\}$. Then it follows from (8) that

$$S_{*+}(x)S_{*-}(x) = 1/v_i(x), \quad x \in (-1,1).$$

Let $S(z)$ and $\varphi(z)$ be given by (2) and (5), respectively. It follows from (3) and (6) that

$$(S_*S\varphi^n)_-^{\sigma_3}(x)\begin{pmatrix} 0 & \frac{\rho(x)v_i(x)}{w_+(x)} \\ -\frac{w_+(x)}{\rho(x)v_i(x)} & 0 \end{pmatrix}(S_*S\varphi^n)_+^{-\sigma_3}(x) = \begin{pmatrix} 0 & 1/w_+(x) \\ -w_+(x) & 0 \end{pmatrix}$$

for $x \in (-1,1)$. It also can be readily verified with the help of (6) that

$$\begin{pmatrix} 1 & \frac{1}{w_+(x)} \\ \frac{1}{2\varphi_+(x)} & \frac{\varphi_+(x)}{2w_+(x)} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{w_-(x)} \\ \frac{1}{2\varphi_-(x)} & \frac{\varphi_-(x)}{2w_-(x)} \end{pmatrix}\begin{pmatrix} 0 & 1/w_+(x) \\ -w_+(x) & 0 \end{pmatrix}$$

for $x \in (-1,1)$. Therefore, RHP-$\boldsymbol{N}$ is solved by $\boldsymbol{N}(z) = \boldsymbol{C}\boldsymbol{M}(z)$, where

$$\boldsymbol{C} := (2^nS_*S)^{-\sigma_3}(\infty) \quad \text{and} \quad \boldsymbol{M}(z) := \begin{pmatrix} 1 & \frac{1}{w(z)} \\ \frac{1}{2\varphi(z)} & \frac{\varphi(z)}{2w(z)} \end{pmatrix}(S_*S\varphi^n)^{\sigma_3}(z). \tag{13}$$

3.4. **Analytic Approximation.** To solve $\bar\partial$RHP-$\boldsymbol{X}$, we first solve its analytic version. That is, consider the following Riemann-Hilbert problem (RHP-$\boldsymbol{A}$):

(a) $\boldsymbol{A}(z)$ is analytic in $\mathbb{C}\setminus([-1,1]\cup\partial E_R)$ and $\lim_{z\to\infty}\boldsymbol{A}(z)z^{-n\sigma_3} = \boldsymbol{I}$;
(b,c) $\boldsymbol{A}(z)$ satisfies $\bar\partial$RHP-$\boldsymbol{X}$(b,c).

Lemma 3

For all n large enough there exists a matrix $\boldsymbol{Z}(z)$, analytic in $\overline{\mathbb{C}}\setminus\partial E_R$ and satisfying

$$\boldsymbol{Z}(z) = \boldsymbol{I} + \boldsymbol{O}\left(R_*^{-n}\right)$$

uniformly in $\overline{\mathbb{C}}$ for any $r < R_* < R$, such that $\boldsymbol{A}(z) = \boldsymbol{C}\boldsymbol{Z}(z)\boldsymbol{M}(z)$ solves RHP-$\boldsymbol{A}$.

Proof. Assume that there exists a matrix $\boldsymbol{Z}(z)$ that is analytic in $\overline{\mathbb{C}}\setminus\partial E_R$, is equal to $\boldsymbol{I}$ at infinity, and satisfies

$$\boldsymbol{Z}_+(s) = \boldsymbol{Z}_-(s)\boldsymbol{M}(s)\begin{pmatrix} 1 & 0 \\ \frac{w(s)l_n(s)}{v_i(s)} & 1 \end{pmatrix}\boldsymbol{M}^{-1}(s), \quad s \in \partial E_R.$$

It can be readily verified that $\boldsymbol{A}(z) = \boldsymbol{C}\boldsymbol{Z}(z)\boldsymbol{M}(z)$ solves RHP-$\boldsymbol{A}$. To show that such $\boldsymbol{Z}(z)$ does indeed exist, observe that

$$\det\boldsymbol{M}(z) = \frac{\varphi(z)}{2w(z)} - \frac{1}{2\varphi(z)w(z)} \equiv 1$$

in the entire complex plane and that

$$v_i(z)S_*^2(z) = (-1)^{i-1}\varphi^{k_i}(z), \quad z \notin [-1,1],$$

straight by the definition of $S_i(z)$ in (7), where $k_1 = 0$, $k_2 = 2$, and $k_3 = k_4 = 1$. Thus,

$$\boldsymbol{M}(s)\begin{pmatrix} 1 & 0 \\ \frac{w(s)l_n(s)}{v_i(s)} & 1 \end{pmatrix}\boldsymbol{M}^{-1}(s) = \boldsymbol{I} + \frac{(-1)^{i-1}l_n(s)}{w(s)S^2(s)\varphi^{2n+k_i}(s)}\begin{pmatrix} \frac{1}{2}\varphi(s) & -1 \\ \frac{1}{4}\varphi^2(s) & -\frac{1}{2}\varphi(s) \end{pmatrix} \tag{14}$$

for $s \in \partial E_R$. It follows from the very definition of E_R that $|\varphi(s)| = R$ for $s \in \partial E_R$. Moreover, since $\deg(l_n) \leqslant n$ and the uniform norms on $[-1,1]$ of these polynomials are bounded by C'_ρ, see Proposition 1, it holds that

$$|l_n(s)| \leqslant C'_\rho |\varphi(s)|^n = C'_\rho R^n, \quad s \in \partial E_R,$$

by the Bernstein-Walsh inequality. Hence, we can conclude that the jump of $\boldsymbol{Z}(z)$ on ∂E_R can be estimated as $\boldsymbol{I}+\boldsymbol{O}(R^{-n})$. It now follows from [7, Theorem 7.18 and Corollary 7.108] that such $\boldsymbol{Z}(z)$ does exist, is unique, and has continuous traces on ∂E_R whose L^2-norms with respect to the arclength measure are of size $O(R^{-n})$. This yields the desired pointwise estimate of $\boldsymbol{Z}(z)$ locally uniformly in $\overline{\mathbb{C}}\backslash\partial E_R$. Next, observe that the jump of $\boldsymbol{Z}(s)$ is analytic around ∂E_R and therefore we can vary the value of R. Since the solutions corresponding to different values of R are necessarily analytic continuations of each other, the desired uniform estimate follows from the locally uniform ones for any fixed $R_* < R$ and $R' > R$. □

3.5. **An Auxiliary Estimate.** Denote by dA the area measure and by $\mathcal{K}$ the Cauchy area operator acting on integrable functions on $\mathbb{C}$, i.e.,

$$\mathcal{K}f(z) = \frac{1}{\pi}\iint \frac{f(s)}{z-s}dA. \tag{15}$$

Lemma 4

Let $u(z)$ be a bounded function supported on $\overline{E}_r$. Then

$$\left\|\mathcal{K}\left(u|\varphi|^{-2n}\right)\right\| \leqslant C_r\frac{\log n}{n}\|u\|,$$

where $\|\cdot\|$ is the essential supremum norm and the constant C_r is independent of n.

Proof. Observe that the integrand is a bounded compactly supported function and therefore its Cauchy area integral is Hölder continuous in $\mathbb{C}$ with any index $\alpha < 1$, see [3, Theorem 4.3.13]. Moreover, since the integral is analytic in $\overline{\mathbb{C}}\backslash\overline{E}_r$, the maximum of its modulus is achieved on $\overline{E}_r$. Notice also that it is enough to prove the claim of the lemma only for $u(z) = \chi_{E_r}(z)$, the indicator function of E_r.

Let $z \in \overline{E}_r$. Observe that $\varphi(s) = \tau$ when $s = \frac{1}{2}(\tau + 1/\tau)$. Write $z = \frac{1}{2}(\xi + 1/\xi)$. Then

$$\begin{aligned}\left|\mathcal{K}\left(\frac{\chi_{E_r}}{|\varphi|^{2n}}\right)(z)\right| &\leqslant \frac{1}{\pi}\iint_{E_r}\frac{1}{|z-s|}\frac{dA}{|\varphi(s)|^{2n}} \\ &= \frac{1}{\pi}\iint_{1<|\tau|<r}\frac{|\tau^2-1|^2}{|(\xi-\tau)(1-1/(\tau\xi))|}\frac{dA}{|\tau|^{2n+4}}.\end{aligned}$$

Partial fraction decomposition now yields

$$\begin{aligned}\left|\mathcal{K}\left(\frac{\chi_{E_r}}{|\varphi|^{2n}}\right)(z)\right| &\leqslant \frac{1}{\pi}\iint_{1<|\tau|<r}\left|\frac{\xi}{\tau-\xi}+\frac{\tau}{\tau-1/\xi}\right|\frac{|\tau^2-1|}{|\tau|^{2n+4}}dA \\ &\leqslant \frac{2r^3}{\pi}\iint_{1<|\tau|<r}\left(\frac{1}{|\tau-\xi|}+\frac{1}{|\tau-1/\xi|}\right)\frac{dA}{|\tau|^{2n+4}}.\end{aligned}$$

Write $\tau=\varrho e^{\mathrm{i}\theta}$ and $\xi=\varrho_* e^{\mathrm{i}\theta_*}$. Then

$$\begin{aligned}|\tau-\xi| &= \sqrt{(\varrho-\varrho_*)^2+4\varrho\varrho_*\sin^2\left(\frac{\theta-\theta_*}{2}\right)}\\ &\geqslant \frac{1}{\sqrt{2}}\left(|\varrho-\varrho_*|+\sqrt{\varrho\varrho_*}\left|2\sin\left(\frac{\theta-\theta_*}{2}\right)\right|\right)\\ &\geqslant C\big(|\varrho-\varrho_*|+|\theta-\theta_*|\big)\end{aligned}$$

for some constant $C<1/\sqrt{2}$, where on the last step we used inequalities $\varrho\varrho_*\geqslant 1$ and $\min_{[-\pi/2,\pi/2]}|\sin x/x|>0$. Since $\varrho/\varrho_*\geqslant 1/r$, the constant C can be adjusted so that

$$|\tau-1/\xi|\geqslant C\big(|\varrho-1/\varrho_*|+|\theta+\theta_*|\big)\geqslant C\big(|\varrho-\varrho_*|+|\theta+\theta_*|\big)$$

is true as well. By going to polar coordinates and applying the above estimates we get that

$$\begin{aligned}\left|\mathcal{K}\left(\frac{\chi_{E_r}}{|\varphi|^{2n}}\right)(z)\right| &\leqslant \frac{4r^3}{\pi C}\int_1^r\left(\int_0^\pi\frac{d\theta}{|\varrho-\varrho_*|+\theta}\right)\frac{d\varrho}{\varrho^{2n+3}}\\ &= \frac{4r^3}{\pi C}\left(\int_{I_1}+\int_{I_2}\right)\log\left(1+\frac{\pi}{|\varrho-\varrho_*|}\right)\frac{d\varrho}{\varrho^{2n+3}} =: S_1+S_2,\end{aligned}$$

where $I_1=(1,r)\cap\big\{\varrho:|\varrho-\varrho_*|<\pi/n\big\}$ and $I_2=(1,r)\backslash I_1$. Then

$$\begin{aligned}S_1 &\leqslant \frac{8r^3}{\pi C}\int_0^{\pi/n}\log\left(1+\frac{\pi}{\varrho}\right)d\varrho=\frac{8r^3}{C}\int_{n+1}^\infty\frac{\log t\,dt}{(t-1)^2}\\ &= \frac{8r^3}{C}\left(\frac{\log(n+1)}{n}+\int_{n+1}^\infty\frac{dt}{t(t-1)}\right)\leqslant\frac{8r^3}{C}\frac{\log(n+1)+1}{n}.\end{aligned}$$

Finally, it holds that

$$S_2\leqslant\frac{8r^3\log(n+1)}{\pi C}\int_1^\infty\frac{d\varrho}{\varrho^{2n+3}}=\frac{4r^3}{\pi C}\frac{\log(n+1)}{n+1},$$

which finishes the proof of the lemma. □

3.6. **$\bar\partial$-Problem.** Consider the following $\bar\partial$-problem ($\bar\partial$P-$\boldsymbol{D}$):

(a) $\boldsymbol{D}(z)$ is a continuous matrix function on $\overline{\mathbb{C}}$ and $\boldsymbol{D}(\infty)=\boldsymbol{I}$;
(b) $\boldsymbol{D}(z)$ satisfies $\bar\partial\boldsymbol{D}(z)=\boldsymbol{D}(z)\boldsymbol{W}(z)$, where

$$\boldsymbol{W}(z):=\boldsymbol{Z}(z)\boldsymbol{M}(z)\begin{pmatrix}0 & 0\\ -w(z)\bar\partial L_{n,r}(z)/v_i(z) & 0\end{pmatrix}\boldsymbol{M}^{-1}(z)\boldsymbol{Z}^{-1}(z).$$

Notice that $\boldsymbol{W}(z)$ is supported by $\overline{E}_r$ and therefore $\boldsymbol{D}(z)$ is necessarily analytic in the complement of $\overline{E}_r$.

Lemma 5

The solution of $\bar\partial$P-$\boldsymbol{D}$ exists for all n large enough and it holds uniformly in $\overline{\mathbb{C}}$ that

$$\boldsymbol{D}(z)=\boldsymbol{I}+\boldsymbol{O}(\varepsilon_n).$$

Proof. As explained in [4, Lemma 8.1], solving $\bar\partial$P-$\boldsymbol{D}$ is equivalent to solving an integral equation

$$\boldsymbol{I}=(\mathcal{I}-\mathcal{K}_{\boldsymbol{W}})\boldsymbol{D}(z)$$

in the space of bounded matrix functions on $\mathbb{C}$, where $\mathcal{I}$ is the identity operator and $\mathcal{K}_{\boldsymbol{W}}$ is the Cauchy area operator (15) acting component-wise on the product $\boldsymbol{m}(s)\boldsymbol{W}(s)$ for a bounded matrix function $\boldsymbol{m}(z)$. If $|||\mathcal{K}_{\boldsymbol{W}}|||$, the operator norm of $\mathcal{K}_{\boldsymbol{W}}$, is less than $1-\epsilon$, $\epsilon\in(0,1)$, then $(\mathcal{I}-\mathcal{K}_{\boldsymbol{W}})^{-1}$ exists as a Neumann series and

$$\boldsymbol{D}(z) = (\mathcal{I}-\mathcal{K}_{\boldsymbol{W}})^{-1}\boldsymbol{I} = \boldsymbol{I} + \boldsymbol{O}_\epsilon(|||\mathcal{K}_{\boldsymbol{W}}|||)$$

uniformly in $\overline{\mathbb{C}}$ (it also holds that $\boldsymbol{D}(z)$ is Hölder continuous in $\mathbb{C}$). It follows from Lemma 4 that to estimate $|||\mathcal{K}_{\boldsymbol{W}}|||$, we need to estimate L^∞-norms of the entries of $\boldsymbol{W}(z)$. To this end, similarly to (14), we get that

$$\boldsymbol{W}(z) = \frac{(-1)^i\bar\partial L_{n,r}(z)}{w(z)S^2(z)\varphi^{2n+k_i}(z)}\boldsymbol{Z}(z)\begin{pmatrix}\frac12\varphi(z) & -1\\ \frac14\varphi^2(z) & -\frac12\varphi(z)\end{pmatrix}\boldsymbol{Z}^{-1}(z), \quad z\in\overline{E}_r.$$

Using Proposition 1 and Lemma 3 we can conclude that entries of $\boldsymbol{W}(z)$ are continuous functions on $\mathbb{C}$ supported by $\overline{E}_r$ with absolute values bounded above by $C_\rho|\varphi(z)|^{-2n}n\varepsilon_n/\log n$ for some constant C_ρ independent of n. Hence, $|||\mathcal{K}_{\boldsymbol{W}}||| = O(\varepsilon_n)$ as claimed. □

3.7. **Asymptotic Formulae.** It readily follows from RHP-$\boldsymbol{A}$ and $\bar\partial$P-$\boldsymbol{D}$ as well as Lemmas 3 and 5 that $\bar\partial$RHP-$\boldsymbol{X}$ is solved by

$$\boldsymbol{X}(z) = \boldsymbol{C}\boldsymbol{D}(z)\boldsymbol{Z}(z)\boldsymbol{M}(z).$$

Given a closed set $B\subset\overline{\mathbb{C}}\setminus[-1,1]$, we can choose r amd R so that $\overline{E}_R\cap B=\varnothing$. Then it holds that $\boldsymbol{Y}(z)=\boldsymbol{X}(z)$ for $z\in B$ by (12). Write

$$\boldsymbol{D}(z)\boldsymbol{Z}(z) = \boldsymbol{I} + \begin{pmatrix}\upsilon_{n1}(z) & \upsilon_{n2}(z)\\ \upsilon_{n3}(z) & \upsilon_{n4}(z)\end{pmatrix}.$$

It follows from Lemmas 3 and 5 that $|\upsilon_{nj}(z)| = O(\varepsilon_n)$ uniformly in $\overline{\mathbb{C}}$ and that $\upsilon_{nj}(\infty)=0$. Then we get from (11) and (13) that

$$P_n(z) = \left(1+\upsilon_{n1}(z)+\frac{\upsilon_{n2}(z)}{2\varphi(z)}\right)\frac{(S_*S)(z)}{(S_*S)(\infty)}\left(\frac{\varphi(z)}{2}\right)^n, \quad z\in B.$$

Since $S_*(z)/S_*(\infty) = S_i(z)/S_i(\infty)$, the first claim of the theorem follows. Next, notice that the first column of $\boldsymbol{Y}(z)$ is entire and is equal to the first column of

$$\boldsymbol{X}_+(x)\begin{pmatrix}1 & 0\\ w_+(x)/(\rho(x)v_i(x)) & 1\end{pmatrix}$$

for $x\in[-1,1]$ by (12) and Proposition 1. Since the functions $\upsilon_{ni}(z)$ are continuous across $[-1,1]$ and $S_{*\pm}(x)/S_*(\infty) = S_{i\pm}(x)/S_i(\infty)$, we deuce from (3), (6), (8), and (13) that

$$P_n(x) = (1+\upsilon_{n1}(x))\frac{(S_iS\varphi^n)_+(x)+(S_iS\varphi^n)_-(x)}{2^n(S_iS)(\infty)} + \upsilon_{n2}(x)\frac{(S_iS\varphi^{n-1})_+(x)+(S_iS\varphi^{n-1})_-(x)}{2^{n+1}(S_iS)(\infty)}$$

for any $x\in[-1,1]$. It now follows from (4), (6), and (8) that

$$(S_iS\varphi^k)_+(x)+(S_iS\varphi^k)_-(x) = \frac{2\cos\left(k\arccos(x)+\theta(x)+\theta_i(x)\right)}{\sqrt{\rho(x)|v_i(x)|}}, \quad x\in[-1,1].$$

The last two formulae now yield the second claim of the theorem. Finally, it is known, see [15, Equations (9.6) and (9.7)], that

$$\begin{cases} a_{n,i}^2 &= \lim_{z\to\infty} z^2[\boldsymbol{Y}(z)]_{12}[\boldsymbol{Y}(z)]_{21}, \\ b_{n,i} &= \lim_{z\to\infty} \big(z - P_{n+1,i}(z)[\boldsymbol{Y}(z)]_{22}\big), \end{cases}$$

where $\boldsymbol{Y}(z)$ corresponds to the index n. As in the first part of the proof, we get that

$$[\boldsymbol{Y}(z)]_{12} = [\boldsymbol{X}(z)]_{12} = \frac{1}{w(z)}\frac{1+\upsilon_{n1}(z)+\upsilon_{n2}(z)\varphi(z)/2}{2^n(S_*S)(\infty)(S_*S)(z)\varphi^n(z)}$$

and

$$[\boldsymbol{Y}(z)]_{21} = [\boldsymbol{X}(z)]_{21} = \left(\upsilon_{n3}(z) + \frac{1+\upsilon_{n4}(z)}{2\varphi(z)}\right)2^n(S_*S)(\infty)(S_*S)(z)\varphi^n(z)$$

for all z large. Since $\upsilon_{nj}(\infty)=0$, it holds that

$$a_{n,i}^2 = \frac{1}{4} + \lim_{z\to\infty} z\upsilon_{n3}(z)(1+z\upsilon_{n2}(z)) = \frac{1}{4} + O(\varepsilon_n)$$

by the maximum modulus principle for holomorphic functions. Similarly, we have that

$$[\boldsymbol{Y}(z)]_{22} = [\boldsymbol{X}(z)]_{22} = \left(\upsilon_{n3}(z) + \frac{1}{2}(1+\upsilon_{n4}(z))\varphi(z)\right)\frac{1}{w(z)}\frac{2^n(S_*S)(\infty)}{(S_*S)(z)\varphi^n(z)}$$

for all z large. Hence,

$$P_{n+1,i}(z)[\boldsymbol{Y}(z)]_{22} = \frac{\varphi^2(z)}{4w(z)}\left(1+\upsilon_{n+11}(z)+\frac{\upsilon_{n+12}(z)}{2\varphi(z)}\right)\left(1+\upsilon_{n4}(z)+2\frac{\upsilon_{n3}(z)}{\varphi(z)}\right)$$

in this case. It can be readily verified that

$$\frac{\varphi^2(z)}{4w(z)} = z + \frac{z}{2w(z)(z+w(z))} - \frac{1}{4w(z)} = z + O\left(\frac{1}{z}\right)$$

as $z\to\infty$. Therefore,

$$b_{n,i} = -\lim_{z\to\infty} z\big(\upsilon_{n+11}(z)+\upsilon_{n4}(z)\big) = O(\varepsilon_n)$$

again, by the maximum modulus principle for holomorphic functions. This finishes the proof of the theorem.

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